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ONE SIDED $(\sigma, \tau)-$ LIE IDEALS AND GENERALIZED DERIVATIONS IN PRIME RINGS

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Abstract. Let R be a prime ring with characteristic not 2 and $\sigma, \tau, \lambda, \mu$, automorphisms of R. Let V be a nonzero left $(\sigma, \tau)-$Lie ideal ,U a nonzero right $(\sigma, \tau)-$Lie ideal of R and $a, b \in R$. The main object in this article is to study the situations. (1) $aVb \subset C_{\lambda,\mu}$. (2) (i) $bV \subset C_{\lambda,\mu}$ (or $Vb \subset C_{\lambda,\mu}$), (ii) $bU \subset C_{\lambda,\mu}$ (or $Ub \subset C_{\lambda,\mu}$), (iii) $(V, b)_{\lambda,\mu} = 0$ (or $(b, V)_{\lambda,\mu} = 0$), (iv) $(U, b)_{\lambda,\mu} = 0$, (3) (i) $(h(I), a)_{\lambda,\mu} = 0$, (ii) $ah(I) \subset C_{\lambda,\mu}(J)$, (iii) $ah(R)b \subset C_{\lambda,\mu}$, (iv) $h[I, a]_{\lambda,\mu} = 0$, (v) $h(V) = 0$ where I, J are ideals and h is a left (or right)-generalized $(\sigma, \tau)-$derivation of R.

1 Introduction

Let R be a ring and σ, τ two mappings of R. We write $[x, y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$ for $x, y \in R$ and so $[x, y]_{1,1} = [x, y] = xy - yx$, where $1 : R \longrightarrow R$ is an identity mapping. Let U be an additive subgroup of R. If $[U, R] \subset U$ then U is called a Lie ideal of R. The definition of $(\sigma, \tau)-$Lie ideal of R is introduced in [9] as follows: (i) U is called a right $(\sigma, \tau)-$Lie ideal of R if $[U, R]_{\sigma,\tau} \subset U$, (ii) U is called a left $(\sigma, \tau)-$Lie ideal if $[R, U]_{\sigma,\tau} \subset U$. (iii) U is called a $(\sigma, \tau)-$Lie ideal if U is both a right and left $(\sigma, \tau)-$Lie ideal of R. Every Lie ideal of R is a $(1, 1)-$Lie ideal of R. If $R = \{ \left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right) \mid x$ and y are integers$\}$, $U = \{ \left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right) \mid x$ is integer$\}$, $\sigma\left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right)$ and $\tau\left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & -y \\ 0 & 0 \end{smallmatrix}\right)$ then U is a right $(\sigma, \tau)-$Lie ideal but not a Lie ideal of R.

Let $d : R \longrightarrow R$ be an additive mapping of R. If $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$ then d is called a derivation. An additive mapping $h : R \longrightarrow R$ is said to be right-generalized derivation associated with derivation d if $h(xy) = h(x)y + xd(y), \forall x, y \in R$ and left-generalized derivation associated with derivation d_1, if $h(xy) = d_1(x)y + xh(y), \forall x, y \in R$. Every derivation $d : R \to R$ is a right (and left)-generalized derivation with d.

The mapping defined by $h(r) = [r, a]_{\sigma,\tau}, \forall r \in R$ is a right-generalized derivation associated with derivation $d(r) = [r, \sigma(a)], \forall r \in R$ and left-generalized derivation associated with derivation $d_1(r) = [r, \tau(a)], \forall r \in R$.

Let $d : R \longrightarrow R$ be an additive mapping of R. If $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ for all $x, y \in R$ then d is called a $(\sigma, \tau)-$derivation of R. If there exist a $(\sigma, \tau)-$derivation d of R such that $h(xy) = h(x)\sigma(y) + \tau(x)d(y)$ for all $x, y \in R$ then h is called a right-generalized $(\sigma, \tau)-$derivation associated with d. If there exist a $(\sigma, \tau)-$derivation d_1 such that $h(xy) = d_1(x)\sigma(y) + \tau(x)h(y)$ for all $x, y \in R$ then h is called a left-generalized $(\sigma, \tau)-$derivation associated with d_1 (see [5]). Every $(\sigma, \tau)-$derivation $d : R \to R$ is a right (and left)-generalized $(\sigma, \tau)-$derivation associated with d.

The mapping $h(r) = (a, r)_{\sigma,\tau}, \forall r \in R$ is a left-generalized $(\sigma, \tau)-$derivation associated with $(\sigma, \tau)-$derivation $d_1(r) = [a, r]_{\sigma,\tau}, \forall r \in R$ and right-generalized $(\sigma, \tau)-$derivation associated with $(\sigma, \tau)-$derivation $d(r) = -[a, r]_{\sigma,\tau}, \forall r \in R$.

In this paper we have given some results on one sided $(\sigma, \tau)-$Lie ideals and left (or right)-generalized $(\sigma, \tau)-$derivation in prime rings. Some algebraic properties of $(\sigma, \tau)-$Lie ideal are discussed in [1], [2], [3], [6], [8], [9], [11] and [13] where further references can be found.

Throughout, R will be a prime ring with characteristic not 2 and $\sigma, \tau, \alpha, \beta, \lambda, \mu$ automorphisms of R. We write $C_{\sigma,\tau} = \{c \in R \mid c\sigma(r) = \tau(r)c, \forall\ r \in R\}$, and will make extensive use of the following basic commutator identities:

$[xy,z]_{\sigma,\tau} = x[y,z]_{\sigma,\tau} + [x,\tau(z)]y = x[y,\sigma(z)] + [x,z]_{\sigma,\tau}y$
$[x,yz]_{\sigma,\tau} = \tau(y)[x,z]_{\sigma,\tau} + [x,y]_{\sigma,\tau}\sigma(z)$
$(x,yz)_{\sigma,\tau} = \tau(y)(x,z)_{\sigma,\tau} + [x,y]_{\sigma,\tau}\sigma(z) = -\tau(y)[x,z]_{\sigma,\tau} + (x,y)_{\sigma,\tau}\sigma(z)$
$(xy,z)_{\sigma,\tau} = x(y,z)_{\sigma,\tau} - [x,\tau(z)]y = x[y,\sigma(z)] + (x,z)_{\sigma,\tau}y.$

2 Results

Lemma 2.1. *[11, Theorem2] Let V be a noncentral left $(\sigma,\tau)-$Lie ideal of R. Then there exist a nonzero ideal M of R such that ($[R,M]_{\sigma,\tau} \subset V$ and $[R,M]_{\sigma,\tau} \nsubseteq C_{\sigma,\tau}$) or $\sigma(u)+\tau(u) \in Z, \forall u \in V$.*

Lemma 2.2. *[4, Lemma3] Let d be a nonzero $(\sigma,\tau)-$ derivation on R ,$a \in R$ and $U \neq 0$ an ideal of R. If $ad(U) = 0$ (or $d(U)a = 0$) then $a = 0$.*

Lemma 2.3. *[4, Lemma1] Let R be a prime ring and $d : R \longrightarrow R$ a $(\sigma,\tau)-$derivation. If U is a right ideal of R and $d(U) = 0$ then $d = 0$.*

Lemma 2.4. *[12, Lemma4] If a prime ring contains a nonzero commutative right ideal then R is commutative.*

Lemma 2.5. *[7, Theorem1] Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\sigma,\tau)-$ derivation associated with a nonzero $(\lambda,\mu)-$derivation d and I, J nonzero ideals of R. If $h(I) \subset C_{\alpha,\beta}(J)$ then R is commutative.*

Lemma 2.6. *Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\sigma,\tau)-$derivation associated with a nonzero $(\sigma,\tau)-$derivation d and I a nonzero ideal of R. If $a, b \in R$ such that $[ah(I), b]_{\lambda,\mu} = 0$ then $[a,\mu(b)]a = 0$ or $d\sigma^{-1}\lambda(b) = 0$.*

Proof. Let $[ah(I), b]_{\lambda,\mu} = 0$. Then we have,

$$\begin{aligned} 0 &= [ah(x\sigma^{-1}\lambda(b)), b]_{\lambda,\mu} = [ah(x)\lambda(b) + a\tau(x)d\sigma^{-1}\lambda(b), b]_{\lambda,\mu} \\ &= ah(x)[\lambda(b),\lambda(b)] + [ah(x), b]_{\lambda,\mu}\lambda(b) + a\tau(x)[d\sigma^{-1}\lambda(b), b]_{\lambda,\mu} \\ &+ [a\tau(x),\mu(b)]d\sigma^{-1}\lambda(b), \forall x \in I \end{aligned}$$

and so

$$a\tau(x)[k,b]_{\lambda,\mu} + [a\tau(x),\mu(b)]k = 0, \forall x \in I\ ,\ k = d\sigma^{-1}\lambda(b). \tag{2.1}$$

Let us replace x by $\tau^{-1}(a)x$ in (2.1). Then using (2.1) we get,

$$\begin{aligned} 0 &= aa\tau(x)[k,b]_{\lambda,\mu} + [aa\tau(x),\mu(b)]k \\ &= aa\tau(x)[k,b]_{\lambda,\mu} + a[a\tau(x),\mu(b)]k + [a,\mu(b)]a\tau(x)k \\ &= [a,\mu(b)]a\tau(x)k, \forall x \in I \end{aligned}$$

and so $[a,\mu(b)]a\tau(I)d\sigma^{-1}\lambda(b) = 0$. Since $\tau(I)$ is a nonzero ideal of R then we obtain that $[a,\mu(b)]a = 0$ or $d\sigma^{-1}\lambda(b) = 0$. □

Theorem 2.7. *Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\sigma,\tau)-$derivation associated with $(\sigma,\tau)-$derivation d and I, J nonzero ideals of R. If $a \in R$ such that $ah(I) \subset C_{\lambda,\mu}(J)$ then $a \in Z$ or $d = 0$.*

Proof. Let $ah(I) \subset C_{\lambda,\mu}(J)$. This means that $[ah(I), x]_{\lambda,\mu} = 0, \forall x \in J$. Using Lemma (2.6) we obtain that, for any $x \in J$,

$$[a,\mu(x)]a = 0 \text{ or } d\sigma^{-1}\lambda(x) = 0.$$

Let $K = \{x \in J \mid [a,\mu(x)]a = 0\}$ and $L = \left\{x \in J \mid d\sigma^{-1}\lambda(x) = 0\right\}$. Then K and L are subgroups of J and $J = K \cup L$. Hence we have $J = K$ or $J = L$. That is

$$[a,\mu(J)]a = 0 \text{ or } d\sigma^{-1}\lambda(J) = 0.$$

Since $\sigma^{-1}\lambda(J)$ is a nonzero ideal of R then $d\sigma^{-1}\lambda(J)=0$ implies that $d=0$ by Lemma (2.3). If $[a,\mu(J)]a=0$ then using this relation we get,

$$0=[a,\mu(rx)]a=\mu(r)[a,\mu(x)]a+[a,\mu(r)]\mu(x)a=[a,\mu(r)]\mu(x)a, \forall x\in J, r\in R.$$

That is $[a,R]\mu(J)a=0$. Since $\mu(J)\neq 0$ is an ideal of R then we have $[a,R]=0$ or $a=0$. This means that $a\in Z$ for two case. □

Theorem 2.8. *Let J be a nonzero ideal of R and $a,b\in R$.*

(i) If $b(a,R)_{\alpha,\beta}\subset C_{\lambda,\mu}(J)$ then $a\in C_{\alpha,\beta}$ or $b\in Z$.
(ii) If $b(R,a)_{\alpha,\beta}\subset C_{\lambda,\mu}(J)$ then $a\in Z$ or $b\in Z$.

Proof. (i) Let $h(r)=(a,r)_{\alpha,\beta}, \forall r\in R$ and $d(r)=-[a,r]_{\alpha,\beta}, \forall r\in R$. Since ,

$$\begin{aligned} h(rs) &= (a,rs)_{\alpha,\beta}=-\beta(r)[a,s]_{\alpha,\beta}+(a,r)_{\alpha,\beta}\alpha(s)\\ &= h(r)\alpha(s)+\beta(r)d(s), \forall r,s\in R\end{aligned}$$

and

$$\begin{aligned} d(rs) &= -[a,rs]_{\alpha,\beta}=-\beta(r)[a,s]_{\alpha,\beta}-[a,r]_{\alpha,\beta}\alpha(s)\\ &= d(r)\alpha(s)+\beta(r)d(s), \forall r,s\in R\end{aligned}$$

then d is a $(\alpha,\beta)-$derivation and h is a right-generalized $(\alpha,\beta)-$derivation associated with d.

If $h=0$ then we have $Rd(R)=0$ by the above relation. This gives that $d=0$ and so $a\in C_{\alpha,\beta}$. Let $h\neq 0$.

If $b(a,R)_{\alpha,\beta}\subset C_{\lambda,\mu}(J)$ then we have $bh(R)\subset C_{\lambda,\mu}(J)$. This gives that $b\in Z$ or $d=0$ by Theorem (2.7). Finally we obtain that $b\in Z$ or $a\in C_{\alpha,\beta}$.

(ii) Consider the mappings defined by $g(r)=(r,a)_{\alpha,\beta}, \forall r\in R$ and $d(r)=[r,\alpha(a)], \forall r\in R$. Since,

$$d(rs)=[rs,\alpha(a)]=r[s,\alpha(a)]+[r,\alpha(a)]s=d(r)s+rd(s), \forall r,s\in R$$

and

$$g(rs)=(rs,a)_{\alpha,\beta}=r[s,\alpha(a)]+(r,a)_{\alpha,\beta}s=g(r)s+rd(s), \forall r,s\in R,$$

then d is a derivation and g is a right-generalized derivation associated with d. If $g=0$ then we obtain that $d(R)=0$ and so $a\in Z$ by the above relation.

If $b(R,a)_{\alpha,\beta}\subset C_{\lambda,\mu}(J)$ then we have $bg(R)\subset C_{\lambda,\mu}(J)$. This means that $b\in Z$ or $d=0$ by Theorem (2.7). That is $b\in Z$ or $a\in Z$. □

Using Theorem (2.8) we can prove the following Corollary immediately.

Corollary 2.9. *Let V be a nonzero left $(\sigma,\tau)-$lie ideal and U a nonzero right $(\sigma,\tau)-$lie ideal of R. Let $J\neq(0)$ be an ideal of R and $b\in R$.*

(i) If $b(V,R)_{\alpha,\beta}\subset C_{\lambda,\mu}(J)$ then $V\subset C_{\alpha,\beta}$ or $b\in Z$.
(ii) If $b(R,V)_{\alpha,\beta}\subset C_{\lambda,\mu}(J)$ then $V\subset Z$ or $b\in Z$.
(iii) If $b(U,R)_{\alpha,\beta}\subset C_{\lambda,\mu}(J)$ then $U\subset C_{\alpha,\beta}$ or $b\in Z$.
(iv) If $b(R,U)_{\alpha,\beta}\subset C_{\lambda,\mu}(J)$ then $U\subset Z$ or $b\in Z$.

Theorem 2.10. *Let $d:R\longrightarrow R$ be a nonzero $(\sigma,\tau)-$derivation and $b\in R$. If $d(R)b\subset C_{\lambda,\mu}(R)$ then $b\in Z$.*

Proof. If $d(R)b\subset C_{\lambda,\mu}(R)$ then we have

$$\begin{aligned} 0 &= [d(r\sigma^{-1}(b))b,\mu^{-1}\tau(r)]_{\lambda,\mu}=[d(r)bb+\tau(r)d\sigma^{-1}(b)b,\mu^{-1}\tau(r)]_{\lambda,\mu}\\ &= d(r)b[b,\lambda\mu^{-1}\tau(r)]+[d(r)b,\mu^{-1}\tau(r)]_{\lambda,\mu}b+\tau(r)[d\sigma^{-1}(b)b,\mu^{-1}\tau(r)]_{\lambda,\mu}\\ &\quad +[\tau(r),\tau(r)]d\sigma^{-1}(b)b\\ &= d(r)b[b,\lambda\mu^{-1}\tau(r)], \forall r\in R.\end{aligned}$$

That is

$$d(r)b[b, \lambda\mu^{-1}\tau(r)] = 0, \forall r \in R. \tag{2.2}$$

Since $d(r)b \in C_{\lambda,\mu}(R), \forall r \in R$ then, for any $r \in R$, we obtain that

$$d(r)b = 0 \text{ or } [b, \lambda\mu^{-1}\tau(r)] = 0$$

by (2.2). Let $K = \{r \in R \mid d(r)b = 0\}$ and $L = \{r \in R \mid [b, \lambda\mu^{-1}\tau(r)] = 0\}$. Considering as in the proof of Theorem (2.7) we get

$$d(R)b = 0 \text{ or } [b, R] = 0.$$

If $[b, R] = 0$ then we have $b \in Z$. Since $d \neq 0$ then $d(R)b = 0$ implies that $b = 0$ by Lemma (2.2) and so $b \in Z$. □

Remark 2.11. [10, Lemma3] Let R be a prime ring and $a, b \in R$. If $b, ab \in C_{\sigma,\tau}$ then $b = 0$ or $a \in Z$.

Lemma 2.12. *Let I be a nonzero ideal of R and $a, b \in R$. If $[I, a]_{\sigma,\tau}b \subset C_{\lambda,\mu}$ then $b = 0$ or $a \in Z$.*

Proof. Let $[I, a]_{\sigma,\tau}b \subset C_{\lambda,\mu}$. Then we have

$$C_{\lambda,\mu} \ni [\tau(a)x, a]_{\sigma,\tau}b = \tau(a)[x, a]_{\sigma,\tau}b + [\tau(a), \tau(a)]xb = \tau(a)[x, a]_{\sigma,\tau}b, \forall x \in I$$

and so $\tau(a)[x, a]_{\sigma,\tau}b \in C_{\lambda,\mu}, \forall x \in I$. Considering the last relation and hypothesis we obtain that, for any $x \in I$,

$$[x, a]_{\sigma,\tau}b \in C_{\lambda,\mu} \text{ and } \tau(a)[x, a]_{\sigma,\tau}b \in C_{\lambda,\mu}.$$

Using Remark (2.11) we get $\tau(a) \in Z$ or $[x, a]_{\sigma,\tau}b = 0$. Applying this argument for all $x \in I$ we have $a \in Z$ or $[I, a]_{\sigma,\tau}b = 0$.

On the other hand $[I, a]_{\sigma,\tau}b = 0$ gives that,

$$0 = [rx, a]_{\sigma,\tau}b = r[x, a]_{\sigma,\tau}b + [r, \tau(a)]xb = [r, \tau(a)]xb, \forall x \in I, r \in R$$

and so $[R, \tau(a)]Ib = 0$. Since R is prime ring and $I \neq (0)$ an ideal of R then we have $b = 0$ or $a \in Z$. □

Theorem 2.13. *Let V be a nonzero left $(\sigma, \tau)-$Lie ideal $b \in R$ and U a nonzero right $(\sigma, \tau)-$Lie ideal of R.*

(i) If $bV \subset C_{\lambda,\mu}$ then $b \in Z$ or $V \subset Z$.
(ii) If $Vb \subset C_{\lambda,\mu}$ then $b = 0$ or $V \subset Z$.
(iii) If $bU \subset C_{\lambda,\mu}$ (or $Ub \subset C_{\lambda,\mu}$) then $b \in Z$ or $U \subset C_{\sigma,\tau}$.

Proof. For any $v \in V$, let us consider the mapping defined by $h(r) = [r, v]_{\sigma,\tau}, \forall\, r \in R$. Since,

$$\begin{aligned} h(rs) &= [rs, v]_{\sigma,\tau} = r[s, \sigma(v)] + [r, v]_{\sigma,\tau}s \\ &= h(r)s + rd_1(s), \forall r, s \in R, \text{ where } d_1(s) = [s, \sigma(v)], \forall s \in R \end{aligned}$$

then h is a right-generalized derivation with derivation d_1.

(i) If $bV \subset C_{\lambda,\mu}$ then, for any $v \in V$, we have $b[R, v]_{\sigma,\tau} \subset bV \subset C_{\lambda,\mu}$ and so $bh(R) \subset C_{\lambda,\mu}$. This means that $b \in Z$ or $d_1 = 0$ by Theorem (2.7). If $d_1 = 0$ then we have $v \in Z$.

If we consider for all $v \in V$ the same argument we get $b \in Z$ or $V \subset Z$.

(ii) If $Vb \subset C_{\lambda,\mu}$ then $[R, V]_{\sigma,\tau}b \subset C_{\lambda,\mu}$. This gives that $b = 0$ or $V \subset Z$ by Lemma (2.12).

(iii) For any $u \in U$, define the mapping $d(r) = [u, r]_{\sigma,\tau}, \forall\, r \in R$. It is clear that, d is a $(\sigma, \tau)-$derivation and so right (and left)-generalized $(\sigma, \tau)-$derivation associated with d.

If $bU \subset C_{\lambda,\mu}$ then, for any $u \in U$, we have $b[u, R]_{\sigma,\tau} \subset bU \subset C_{\lambda,\mu}$ and so $bd(R) \subset C_{\lambda,\mu}$. Using Theorem (2.7) we get $b \in Z$ or $d = 0$. On the other hand $d = 0$ implies that $u \in C_{\sigma,\tau}$.

Considering as in the proof of (i) we get $b \in Z$ or $U \subset C_{\sigma,\tau}$.

If $Ub \subset C_{\lambda,\mu}$ then, for any $u \in U$, we have $[u, R]_{\sigma,\tau}b \subset Ub \subset C_{\lambda,\mu}$ gives that $d(R)b \subset C_{\lambda,\mu}$. Using Theorem (2.10) we have $b \in Z$ or $d = 0$. That is $b \in Z$ or $u \in C_{\sigma,\tau}$.

If we consider for all $u \in U$ the same thing we get $b \in Z$ or $U \subset C_{\sigma,\tau}$. □

Lemma 2.14. *Let $h : R \longrightarrow R$ be a nonzero **left-generalized** $(\sigma,\tau)-$ derivation with $(\sigma,\tau)-$ derivation d and $a \in R$. If I is a nonzero ideal of R such that $h[I,a]_{\lambda,\mu} = 0$ then $a \in Z$ or $d\mu(a) = 0$.*

Proof. Let $h[x,a]_{\lambda,\mu} = 0, \forall x \in I$. Then we have,

$$\begin{aligned} 0 &= h[\mu(a)x,a]_{\lambda,\mu} = h\{\mu(a)[x,a]_{\lambda,\mu} + [\mu(a),\mu(a)]x\} \\ &= h\{\mu(a)[x,a]_{\lambda,\mu}\} = d\mu(a)\sigma[x,a]_{\lambda,\mu} + \tau\mu(a)h[x,a]_{\lambda,\mu} = d\mu(a)\sigma[x,a]_{\lambda,\mu}, \forall x \in I \end{aligned}$$

and so $k[x,a]_{\lambda,\mu} = 0, \forall x \in I$, where $k = \sigma^{-1}d\mu(a)$. Replacing x by $xr, r \in R$ we get

$$\begin{aligned} 0 &= k[xr,a]_{\lambda,\mu} = kx[r,\lambda(a)] + k[x,a]_{\lambda,\mu}r \\ &= kx[r,\lambda(a)], \forall\, x \in I, r \in R. \end{aligned}$$

That is $kI[R,\lambda(a)] = 0$. Since R is prime and $I \neq 0$ an ideal of R then we obtain that $a \in Z$ or $d\mu(a) = 0$ by the last relation. □

Lemma 2.15. *Let $h : R \longrightarrow R$ be a nonzero l**eft-generalized** $(\alpha,\beta)-$ derivation with $(\alpha,\beta)-$ derivation d. If V is a nonzero left $(\sigma,\tau)-$Lie ideal of R such that $h(V) = 0$ then $d = 0$ or $\sigma(v) + \tau(v) \in Z$ for all $v \in V$.*

Proof. If $V \subset Z$ then $\sigma(v) + \tau(v) \in Z$ for all $v \in V$. If $V \nsubseteq Z$ then there exist a nonzero ideal M of R such that

$$([R,M]_{\sigma,\tau} \subset V \text{ and } [R,M]_{\sigma,\tau} \nsubseteq C_{\sigma,\tau}) \text{ or } \sigma(v) + \tau(v) \in Z\ , \forall\, v \in V$$

by Lemma (2.1). Let $[R,M]_{\sigma,\tau} \subset V$ and $[R,M]_{\sigma,\tau} \nsubseteq C_{\sigma,\tau}$. If $h(V) = 0$ then we have $h[R,M]_{\sigma,\tau} \subset h(V) = 0$ and so $h[R,M]_{\sigma,\tau} = 0$. This gives that, for any $m \in M$,

$$m \in Z \text{ or } d\tau(m) = 0$$

by Lemma (2.14). Let $K = \{m \in M \mid m \in Z\}$ and $L = \{m \in M \mid d\tau(m) = 0\}$. Then K and L are subgroups of M and $M = K \cup L$. Hence, we have $M = K$ or $M = L$. That is $M \subset Z$ or $d\tau(M) = 0$. Since $\tau(M) \neq 0$ is an ideal of R then $d\tau(M) = 0$ implies that $d = 0$ by Lemma (2.3). On the other hand, if $M \subset Z$ then we obtain that R is commutative by Lemma (2.4) and so $\sigma(v) + \tau(v) \in Z, \forall v \in V$. □

Lemma 2.16. *Let $h : R \longrightarrow R$ be a nonzero **left-generalized** $(\sigma,\tau)-$ derivation associated with a nonzero $(\sigma,\tau)-$derivation d. If I is a nonzero ideal of R and $a \in R$ such that $(h\,(I)\,, a)_{\lambda,\mu} = 0$ then $a \in Z$ or $d\tau^{-1}\mu(a) = 0$.*

Proof. Let $(h\,(I)\,, a)_{\lambda,\mu} = 0$ and $k = d\tau^{-1}\mu(a)$. Then we get

$$\begin{aligned} 0 &= \left(h\left(\tau^{-1}\mu(a)y\right), a\right)_{\lambda,\mu} = (d\tau^{-1}\mu(a)\sigma(y) + \mu(a)h(y), a)_{\lambda,\mu} \\ &= k\left[\sigma(y),\lambda(a)\right] + (k,a)_{\lambda,\mu}\sigma(y) + \mu(a)(h\,(y)\,, a)_{\lambda,\mu} - \left[\mu(a),\mu(a)\right]h(y),\ \forall y \in I. \end{aligned}$$

This gives that

$$k\left[\sigma(y),\lambda(a)\right] + (k,a)_{\lambda,\mu}\sigma(y) = 0,\ \forall y \in I. \tag{2.3}$$

Replacing y by $yr, r \in R$ in (2.3) and using (2.3) we get,

$$\begin{aligned} 0 &= k\sigma(y)\left[\sigma(r),\lambda(a)\right] + k\left[\sigma(y),\lambda(a)\right]\sigma(r) + (k,a)_{\lambda,\mu}\sigma(y)\sigma(r) \\ &= k\sigma(y)\left[\sigma(r),\lambda(a)\right], \forall y \in I, r \in R. \end{aligned}$$

That is

$$k\sigma(I)\left[R,\lambda(a)\right] = 0. \tag{2.4}$$

Since $\sigma(I)$ is a nonzero ideal of R then we have $a \in Z$ or $d\tau^{-1}\mu(a) = 0$ by (2.4) in prime rings. □

Theorem 2.17. *Let U be a nonzero right $(\sigma,\tau)-$Lie ideal of R and $b\in R$. Let V be a nonzero left $(\sigma,\tau)-$Lie ideal of R.*

(i) If $(V,b)_{\lambda,\mu}=0$ then $b\in Z$ or $\sigma(v)+\tau(v)\in Z$ for all $v\in V$.
(ii) If $(b,V)_{\lambda,\mu}=0$ then $b\in C_{\lambda,\mu}$ or $\sigma(v)+\tau(v)\in Z$ for all $v\in V$.
(iii) If $(U,b)_{\lambda,\mu}=0$ then $b\in Z$ or $U\subset C_{\sigma,\tau}$.

Proof. (i) Let $g(r)=(r,b)_{\lambda,\mu},\forall r\in R$ and $d(r)=-\left[r,\mu(b)\right],\forall r\in R$. Since,

$$g(rs)=(rs,b)_{\lambda,\mu}=r(s,b)_{\lambda,\mu}-[r,\mu(b)]s=d(r)s+rg(s),\forall r,s\in R$$

then g is a left-generalized derivation with derivation $d(r)=-\left[r,\mu(b)\right],\forall r\in R$. If $g=0$ then we have $d=0$ by the above relation. This gives that $b\in Z$.

Let us consider that $g\neq 0$. If $(V,b)_{\lambda,\mu}=0$ then $g(V)=0$. This implies that $d=0$ (and so $b\in Z$) or $\sigma(v)+\tau(v)\in Z,\forall v\in V$ by Lemma (2.15).

(ii) The mapping defined by $h(r)=(b,r)_{\lambda,\mu},\forall r\in R$ is a left-generalized $(\lambda,\mu)-$ derivation with $(\lambda,\mu)-$derivation $d_1(r)=\left[b,r\right]_{\lambda,\mu},\forall r\in R$. Because,

$$d_1(rs)=\left[b,rs\right]_{\lambda,\mu}=\mu(r)\left[b,s\right]_{\lambda,\mu}+\left[b,r\right]_{\lambda,\mu}\lambda(s)=d_1(r)\lambda(s)+\mu(r)d_1(s),\forall r,s\in R$$

and

$$h(rs)=(b,rs)_{\lambda,\mu}=\mu(r)(b,s)_{\lambda,\mu}+\left[b,r\right]_{\lambda,\mu}\lambda(s)=d_1(r)\lambda(s)+\mu(r)h(s),\forall r,s\in R.$$

If $h=0$ then, considering as in the proof of (i), we have $d_1=0$ and so $b\in C_{\lambda,\mu}$.

Assume that $h\neq 0$. If $(b,V)_{\lambda,\mu}=0$ then we have $h(V)=0$. Using Lemma (2.15) we obtain that $d_1=0$ (and so $b\in C_{\lambda,\mu}$) or $\sigma(v)+\tau(v)\in Z,\forall v\in V$.

(iii) Let $d(r)=[u,r]_{\sigma,\tau},\forall r\in R$, for any $u\in U$. Then d is a $(\sigma,\tau)-$derivation and so left (and right)-generalized $(\sigma,\tau)-$ derivation associated with d. Let $d\neq 0$.

If $(U,b)_{\lambda,\mu}=0$ then we can write $([u,R]_{\sigma,\tau},b)_{\lambda,\mu}=0,\forall u\in U$ and so $(d(R),b)_{\lambda,\mu}=0$. If we use Lemma (2.16) we obtain that $b\in Z$ or $d\tau^{-1}\mu(b)=0$. That is

$$b\in Z \text{ or } [u,\tau^{-1}\mu(b)]_{\sigma,\tau}=0.$$

If $d=0$ then we have $u\in C_{\sigma,\tau}$ and so $[u,\tau^{-1}\mu(b)]_{\sigma,\tau}=0$. Considering same thing for all $u\in U$ we get

$$b\in Z \text{ or } [U,\tau^{-1}\mu(b)]_{\sigma,\tau}=0.$$

On the other hand, $[U,\tau^{-1}\mu(b)]_{\sigma,\tau}=0$ gives that $b\in Z$ or $U\subset C_{\sigma,\tau}$ by [8, Lemma3]. □

Lemma 2.18. *Let M be a nonzero ideal of R. If $c\in R$ such that $[[c,M],c]c=0$ then $c\in Z$ or $c^2=0$.*

Proof. If $[[c,M],c]c=0$ then we can write

$$[c,x]cc=c[c,x]c,\forall x\in M. \tag{2.5}$$

Using the hypothesis and (2.5) we get

$$\begin{aligned}
0 &= [[c,yx],c]c=[y[c,x]+[c,y]x,c]c=[y[c,x],c]c+[[c,y]x,c]c\\
&= y[[c,x],c]c+[y,c][c,x]c+[c,y][x,c]c+[[c,y],c]xc\\
&= [y,c][c,x]c+[c,y][x,c]c+[[c,y],c]xc\\
&= [y,c][c,x]c+[y,c][c,x]c+[[c,y],c]xc,\forall x,y\in M.
\end{aligned}$$

That is

$$2[y,c][c,x]c+[[c,y],c]xc=0,\forall x,y\in M. \tag{2.6}$$

Replacing x by cx in (2.6) and using hypothesis,$charR\neq 2$, we get

$$\begin{aligned}
0 &= 2[y,c][c,cx]c+[[c,y],c]cxc=2[y,c][c,cx]c\\
&= 2[y,c]c[c,x]c+2[y,c][c,c]xc,\forall x,y\in M
\end{aligned}$$

and so

$$[y,c]c[c,x]c=0,\forall x,y\in M \tag{2.7}$$

Taking $ry, r\in R$ instead of y in (2.7) we have

$0=[ry,c]c[c,x]c=r[y,c]c[c,x]c+[r,c]yc[c,x]c=[r,c]yc[c,x]c$ for all $x,y\in M, r\in R$ and so

$$[R,c]Mc[c,M]c=0. \tag{2.8}$$

Since R is prime then we obtain that $c\in Z$ or $c[c,M]c=0$ by (2.8) and so $c[c,x]c=0,\forall x\in M$ for two case. Using (2.5) we get $[c,x]cc=0,\forall x\in M$. Replacing x by $sx, s\in R$ in the last relation we have $[c,R]Mcc=0$ and so $c\in Z$ or $c^2=0$. □

Lemma 2.19. *Let $h:R\longrightarrow R$ be a nonzero **right-generalized** $(\sigma,\tau)-$ derivation associated with a nonzero $(\sigma,\tau)-$derivation d and $a,b\in R$.*

(i) If $ah(R)b=0$ then $a=0$ or $d\sigma^{-1}(b)b=0$.
(ii) If $ah(R)b\subset C_{\lambda,\mu}$ then $a=0$ or $[d\sigma^{-1}(b),b]b=0$.

Proof. (i) If $ah(R)b=0$ then we have

$$0=ah(x\sigma^{-1}(b))b=ah(x)bb+a\tau(x)d\sigma^{-1}(b)b=a\tau(x)d\sigma^{-1}(b)b,\forall x\in R.$$

That is, $aRd\sigma^{-1}(b)b=0$. This means that $a=0$ or $d\sigma^{-1}(b)b=0$ in prime rings.
(ii) Let $ah(R)b\subset C_{\lambda,\mu}$ and $k=d\sigma^{-1}(b)$. Then we get

$$\begin{aligned}
0 &= [ah(x\sigma^{-1}(b))b,\lambda^{-1}(b)]_{\lambda,\mu}=[ah(x)bb+a\tau(x)d\sigma^{-1}(b)b,\lambda^{-1}(b)]_{\lambda,\mu}\\
&= ah(x)b[b,b]+[ah(x)b,\lambda^{-1}(b)]_{\lambda,\mu}b+a\tau(x)[kb,b]+[a\tau(x),\lambda^{-1}(b)]_{\lambda,\mu}kb\\
&= a\tau(x)[kb,b]+[a\tau(x),\lambda^{-1}(b)]_{\lambda,\mu}kb\\
&= a\tau(x)k[b,b]+a\tau(x)[k,b]b+[a\tau(x),\lambda^{-1}(b)]_{\lambda,\mu}kb,\forall x\in R
\end{aligned}$$

which gives that

$$a\tau(x)[k,b]b+[a\tau(x),\lambda^{-1}(b)]_{\lambda,\mu}kb=0,\forall x\in R. \tag{2.9}$$

Replacing x by $\tau^{-1}h(x)\tau^{-1}(b)$ in (2.9) and using hypothesis we get $ah(R)b[k,b]b=0$.
Since $ah(R)b\subset C_{\lambda,\mu}$ and R is prime ring then we have $ah(R)b=0$ or $[k,b]b=0$. That is

$$ah(R)b=0 \text{ or } [d\sigma^{-1}(b),b]b=0. \tag{2.10}$$

If $ah(R)b=0$ in (2.10) then we get $a=0$ or $d\sigma^{-1}(b)b=0$ by (i). On the other hand, if $d\sigma^{-1}(b)b=0$ then we obtain that

$$[d\sigma^{-1}(b),b]b=d\sigma^{-1}(b)bb-bd\sigma^{-1}(b)b=0.$$

□

Lemma 2.20. *If I is a nonzero ideal of R and $a,b\in R$ such that $a[R,I]_{\sigma,\tau}b=0$ then $a=0$ or $b\in Z$.*

Proof. For any $x\in I$, the mapping defined by $h(r)=[r,x]_{\sigma,\tau},\forall\, r\in R$ is a right-generalized derivation with derivation $d=[r,\sigma(x)],\forall\, r\in R$, (see Theorem (2.13)).
If $a[R,I]_{\sigma,\tau}b=0$ then we have $ah(R)b=0$. This gives that $a=0$ or $d(b)b=0$ by Lemma (2.19). That is, $a=0$ or $[b,\sigma(x)]b=0$. If we consider same argument for all $x\in I$ then we get

$$a=0 \text{ or } [b,\sigma(I)]b=0.$$

The mapping defined by $d_1(r)=[b,r],\forall r\in R$ is a derivation. If $[b,\sigma(I)]b=0$ then we get $d_1\sigma(I)b=0$. Since $\sigma(I)\neq 0$ an ideal of R then we have $b=0$ or $d_1=0$ by Lemma (2.2). If $d_1=0$ then $b\in Z$. Finally we obtain that $a=0$ or $b\in Z$ for all case. □

Theorem 2.21. *Let V be a nonzero left $(\sigma,\tau)-$Lie ideal of R and $a,b\in R$. If $aVb\subset C_{\lambda,\mu}$ then $a=0$ or $b\in Z$ or $\sigma(v)+\tau(v)\in Z, \forall v\in V$.*

Proof. If $V\subset Z$ then $\sigma(v)+\tau(v)\in Z, \forall v\in V$. If $V\not\subseteq Z$ then there exist a nonzero ideal M of R such that

$$([R,M]_{\sigma,\tau}\subset V \text{ and } [R,M]_{\sigma,\tau}\not\subseteq C_{\sigma,\tau}) \text{ or } \sigma(v)+\tau(v)\in Z,\ \forall v\in V$$

by Lemma (2.1). Let us take any element $m\in M$. Then the mapping defined by $h(r)=[r,m]_{\sigma,\tau}, \forall r\in R$ is a right-generalized derivation associated with derivation $d(s)=[s,\sigma(m)], \forall s\in R$.

Let $[R,M]_{\sigma,\tau}\subset V$ and $[R,M]_{\sigma,\tau}\not\subseteq C_{\sigma,\tau}$. If $aVb\subset C_{\lambda,\mu}$ then we have $a[R,M]_{\sigma,\tau}b\subset aVb\subset C_{\lambda,\mu}$ and so $ah(R)b\subset C_{\lambda,\mu}$. This means that $a=0$ or $[d(b),b]b=0$ or $h=0$ by Lemma (2.19). That is,

$$a=0 \text{ or } h=0 \text{ or } [[b,\sigma(m)],b]b=0.$$

If $h=0$ then we have $d=0$ by the relation $h(rs)=h(r)s+rd(s), \forall r,s\in R$ and so $m\in Z$. That is, again we have $[[b,\sigma(m)],b]b=0$. If we consider the same argument for all $m\in M$ then we obtain that

$$a=0 \text{ or } [[b,\sigma(M)],b]b=0.$$

Since $\sigma(M)$ is a nonzero ideal of R then $[[b,\sigma(M)],b]b=0$ means that $b\in Z$ or $b^2=0$ by Lemma (2.18). If $b^2=0$ then using that $a[R,M]_{\sigma,\tau}b\subset C_{\lambda,\mu}$ we get

$$a[r,m]_{\sigma,\tau}b\lambda(s)b=\mu(s)a[r,m]_{\sigma,\tau}b^2=0 \text{ for all } r,s\in R, m\in M$$

and so $a[R,m]_{\sigma,\tau}bRb=0$. Using primeness of R we obtain that $a[R,M]_{\sigma,\tau}b=0$ or $b=0$. That is $a[R,M]_{\sigma,\tau}b=0$ for two case. This gives that $a=0$ or $b\in Z$ by Lemma (2.20). Finally we obtain that $a=0$ or $b\in Z$ or $\sigma(v)+\tau(v)\in Z, \forall v\in V$ □

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