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Fractional Jensen's Inequality

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Abstract In this paper, we will introduce the fractional analogue of Jensen's inequality using conformable fractional integral operator.

1 Introduction

The idea of fractional derivation is as old as ordinary derivation and integration. In 1695, L'Hospital asked "what would be the one-half derivative of x?" to Leibniz. From that time, many scientists tried to give a definition of fractional derivative to bring about a coherent theory of fractional derivation and integration. By the beginning of 20th century, some definitions of fractional derivation, are introduced, most notedly Riemann-Liouville, Caputo, and Grünwald-Letnikov derivatives. Since fractional derivation and integration has more exponent applications in different disciplines of sciences like engineering, physics, chemistry etc., many mathematicians start to study of aspects of it. For more information about the history and applications, we refer [5, 6, 12, 16] .

The definitions we mentioned above used the integral form of derivative. The idea Riemann-Liouville fractional integral is based on iterating n times and replacing it by one integral, and then using the Cauchy formula with replacing $n!$ with Gamma function. Hence Riemann-Liouville fractional integral is defined as

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x (x-t)^{\alpha-1} f(t)dt.$$

Using this definition, fractional derivative is defined as

$$D_a^\alpha f(x) = D^m J_a^{m-\alpha} f(x),$$

where $m = \lceil \alpha \rceil$ and D represents ordinary derivative.

Riemann-Liouville or any of other definitions for fractional derivative does not satisfies all properties of ordinary derivative. As an example, Riemann-Liouville derivation does not satisfy well-known formula of the product of two functions

$$D(f(t)g(t)) = g(t)Df(t) + f(t)Dg(t).$$

Because of this difficulties, recently some authors tried to give new definitions for fractional derivatives. To handle these difficulties, in 2014 Khalil et al. [11] gave a new definition as

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

This well-behaved definition, called conformable fractional derivative, satisfies many properties of ordinary derivations like product rule, chain rule etc.

Since inequalities are useful tools in mathematics, many mathematicians studied about extensions, generalizations and discretizations of them, see [13, 14, 15] and references cited therein. And mathematicians started to transfer this inequalities into fractional settings both continuous and discrete cases to make a coherent theory of fractional calculus [2, 3, 7, 8, 10].

2 Preliminaries

In this section, we give basic definitions and results of conformable fractional operators derived from [11] .

Definition 2.1. The conformable fractional derivative of a function $f : [0, \infty) \to \mathbb{R}$ of order $0 \leq \alpha \leq 1$ is defined by

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0$.

We note that if the conformable fractional derivative of f of order α exists, we say f is α-differentiable.

Theorem 2.2. *Let $\alpha \in (0, 1]$ and functions f and g be α-differentiable at point $t > 0$. Then following properties are hold:*

(i) $T_\alpha(af + bg)(t) = aT_\alpha(f)(t) + bT_\alpha(g)(t)$, for all $a, b \in \mathbb{R}$.
(ii) $T_\alpha(t^m) = mt^{m-1}$, for all $m \in \mathbb{R}$.
(iii) $T_\alpha(c) = 0$, for all constant functions $f(t) = c$.
(iv) $T_\alpha(fg)(t) = g(t)T_\alpha(f)(t) + f(t)T_\alpha(g)(t)$.
(v) $T_\alpha(\frac{f}{g})(t) = \dfrac{g(t)T_\alpha(f)(t) - f(t)T_\alpha(g)(t)}{(g(t))^2}$
(vi) If, in addition, f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha} \dfrac{df}{dt}$.
Now, we give conformable fractional derivative of some functions:
(1) $T_\alpha(t^m) = mt^{m-1}$, for all $m \in \mathbb{R}$.
(2) $T_\alpha(1) = 0$.
(3) $T_\alpha(e^{at}) = at^{1-\alpha}e^{at},\ a \in \mathbb{R}$.
(4) $T_\alpha(e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha}$.
(5) $T_\alpha(\sin at) = at^{1-\alpha} \cos at,\ a \in \mathbb{R}$.
(6) $T_\alpha(\cos at) = -at^{1-\alpha} \sin at,\ a \in \mathbb{R}$.
(7) $T_\alpha(\sin \frac{1}{\alpha}t^\alpha) = \cos \frac{1}{\alpha}t^\alpha$.
(8) $T_\alpha(\cos \frac{1}{\alpha}t^\alpha) = -\sin \frac{1}{\alpha}t^\alpha$.

Definition 2.3. The conformable fractional integral of a function $f : [0, \infty) \to \mathbb{R}$ of order α is defined by

$$I_\alpha^a(f)(t) = I_1^a(t^{\alpha-1}f)(t) = \int_a^t \frac{f(s)}{t^{1-\alpha}} ds,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1)$.

Theorem 2.4. $T_\alpha I_\alpha^a(f)(t) = f(t),$ *for* $t \geq a$, *where f is any continuous function in the domain of I_α.*

$$I_{1/2}^0(\sqrt{t} \cos t) = \int_0^t \cos s ds = \sin t.$$

For more information and applications on conformable fractional derivative and integral, we refer [1, 4, 11, 17].

3 Main results

In this section, we will give fractional analogue of Jensen inequality given below:

Theorem 3.1. *If $g \in C([a,b],(c,d))$ and $F \in C([a,b],\mathbb{R})$ is convex, then*

$$F\left(\frac{\int_a^b g(s)ds}{b-a}\right) \leq \frac{1}{b-a}\int_a^b F(g(s))ds. \tag{3.1}$$

Before stating our results, we begin with a lemma that will be used in the proofs from [9].

Lemma 3.2. *Let $f \in C([a,b],\mathbb{R})$ be convex. Then for each $t \in (c,d)$, there exists $\lambda \in \mathbb{R}$ such that*

$$f(x) - f(t) \geq \lambda(x-t), \text{ for all } x \in (c,d). \tag{3.2}$$

If f is strictly convex, then inequality sign "$\geq$" in (3.2) should be replaced with sign "$>$".

We start with our first result.

Theorem 3.3. *Let $a,t,c,d \in \mathbb{R}$, with $a < t$. Let $g \in C([a,t],(c,d))$ and $F \in C([c,d],\mathbb{R})$ is convex. Then*

$$F\left(\frac{\alpha I_\alpha^a g(t)}{t^\alpha - a^\alpha}\right) \leq \frac{\alpha}{t^\alpha - a^\alpha} I_\alpha^a F(g(t)) \tag{3.3}$$

holds.
If F is strictly convex, then inequality sign "$\leq$" in (3.3) should be replaced with sign "$<$".

Proof. Take

$$\tau = \frac{\alpha}{t^\alpha - a^\alpha} I_\alpha^a g(t).$$

Now,

$$\begin{aligned} & I_\alpha^a F(g(t)) - \frac{t^\alpha - a^\alpha}{\alpha} F\left(\frac{\alpha I_\alpha^a g(t)}{t^\alpha - a^\alpha}\right) \\ = \; & I_\alpha^a F(g(t)) - \frac{t^\alpha - a^\alpha}{\alpha} F(\tau) \\ = \; & I_\alpha^a \{F(g(t)) - F(\tau)\}. \end{aligned}$$

Since F is convex, there is a $\lambda \in \mathbb{R}$ such that (3.2) holds. Hence, we have

$$\begin{aligned} & I_\alpha^a F(g(t)) - \frac{t^\alpha - a^\alpha}{\alpha} F\left(\frac{\alpha I_\alpha^a g(t)}{t^\alpha - a^\alpha}\right) \\ \geq \; & \lambda I_\alpha^a \{g(t) - \tau\} \\ = \; & \lambda I_\alpha^a \{g(t) - \tau \frac{t^\alpha - a^\alpha}{\alpha}\} \\ = \; & \lambda \left[I_\alpha^a g(t) - I_\alpha^a g(t)\}\right] \\ = \; & 0. \end{aligned}$$

This completes the proof. □

Remark 3.4. When we take $\alpha = 1$ in (3.3), we have inequality

$$F\left(\frac{\int_a^t g(s)ds}{t-a}\right) \leq \frac{1}{t-a}\int_a^t F(g(s))ds, \tag{3.4}$$

where upper limit of integral is a variable. And taking $t = b$ in (3.4) gives

$$F\left(\frac{\int_a^b g(s)ds}{b-a}\right) \leq \frac{1}{b-a}\int_a^b F(g(s))ds,$$

and this inequality is Jensen's inequality given in (3.1).

Secondly, we present a more general result given below.

Theorem 3.5. *Let $a, t, c, d \in \mathbb{R}$, with $a < t$. Let $g \in C([a,t],(c,d))$ and $h \in C([a,t],\mathbb{R})$ with*

$$I_\alpha^a h(t) > 0.$$

If $F \in C([c,d],\mathbb{R})$ is convex, then

$$F\left(\frac{I_\alpha^a\{|h(t)|\, g(t)\}}{I_\alpha^a\{|h(t)|\}}\right) \leq \frac{I_\alpha^a\{|h(t)|\, F(g(t))\}}{I_\alpha^a\{|h(t)|\}} \tag{3.5}$$

holds.
If F is strictly convex, then inequality sign "$\leq$" in (3.5) should be replaced with sign "$<$".

Proof. Take

$$\tau = \frac{I_\alpha^a\{|h(t)|\, g(t)\}}{I_\alpha^a\{|h(t)|\}}.$$

Now,

$$\begin{aligned}
& I_\alpha^a\{|h(t)|\, F(g(t))\} - I_\alpha^a\{|h(t)|\}\, F\left(\frac{I_\alpha^a\{|h(t)|\, g(t)\}}{I_\alpha^a\{|h(t)|\}}\right) \\
= \; & I_\alpha^a\{|h(t)|\, F(g(t))\} - I_\alpha^a\{|h(t)|\}\, F(\tau) \\
= \; & I_\alpha^a\{|h(t)|\, [F(g(t)) - F(\tau)]\}.
\end{aligned}$$

Since F is convex, there is a $\lambda \in \mathbb{R}$ such that (3.2) holds. Therefore, we have

$$\begin{aligned}
& I_\alpha^a\{|h(t)|\, F(g(t))\} - I_\alpha^a\{|h(t)|\}\, F\left(\frac{I_\alpha^a\{|h(t)|\, g(t)\}}{I_\alpha^a\{|h(t)|\}}\right) \\
\geq \; & \lambda I_\alpha^a\{|h(t)|\, [g(t) - \tau]\} \\
= \; & \lambda \left[I_\alpha^a\{|h(t)|\, g(t)\} - \tau I_\alpha^a\{|h(t)|\}\right] \\
= \; & \lambda \left[I_\alpha^a\{|h(t)|\, g(t)\} - \frac{I_\alpha^a\{|h(t)|\, g(t)\}}{I_\alpha^a\{|h(t)|\}} I_\alpha^a\{|h(t)|\}\right] \\
= \; & 0.
\end{aligned}$$

This completes the proof. ◻

Remark 3.6. If the convexity of F changed by concavity, then the sign in (3.5) should be reversed.

Remark 3.7. One can show easily that the function defined as $F(t) = t^r$ is concave for $r \in (0,1)$ and convex for $r < 0$ or $r > 1$.

As a consequence of these two remarks, we state following result.

Corollary 3.8. *Let $g \in C([a,t],(c,d))$ with $g(t) \geq 0$ on $[a,t]$ and $h \in C([a,t],\mathbb{R})$ with $I_\alpha^a h(t) > 0$. Then, for $r < 0$ or $r > 1$*

$$\left(\frac{I_\alpha^a\{|h(t)|\, g(t)\}}{I_\alpha^a\{|h(t)|\}}\right)^r \leq \frac{I_\alpha^a\{|h(t)|\, g^r(t)\}}{I_\alpha^a\{|h(t)|\}},$$

and for $r \in (0,1)$

$$\left(\frac{I_\alpha^a\{|h(t)|\, g(t)\}}{I_\alpha^a\{|h(t)|\}}\right)^r \geq \frac{I_\alpha^a\{|h(t)|\, g^r(t)\}}{I_\alpha^a\{|h(t)|\}}$$

are hold.

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