ON CONVEXITY FOR ENERGY DECAY RATES OF A VISCOELASTIC EQUATION WITH A DYNAMIC BOUNDARY AND NONLINEAR DELAY TERM IN THE NONLINEAR INTERNAL FEEDBACK

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Abstract. In this paper we consider the weak viscoelastic wave equation with a delay term in the nonlinear internal feedback

\[ u_{tt}(x,t) - \Delta u(x,t) + \alpha(t) \int_0^t h(t-s)\Delta u(x,s) ds + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = 0 \]

in a bounded domain, and prove a global existence result which depends on the behavior of both \( \alpha \) and \( h \) using the energy method combined with the Faedo-Galerkin procedure under a condition between the weight of the delay term in the feedback and the weight of the term without delay. Furthermore, we study the asymptotic behavior of solutions using a perturbed energy method.

1 Introduction

In this work, we investigate the existence and decay properties of solutions for the initial boundary value problem of the nonlinear weak viscoelastic wave equation of the type

\[
\begin{align*}
(P) \quad u_{tt}(x,t) - \Delta u(x,t) + \alpha(t) \int_0^t h(t-s)\Delta u(x,s) ds \\
&+ \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = 0 \quad \text{in } \Omega \times [0, +\infty[,
\\
&u(x,t) = 0 \quad \text{on } \Gamma \times [0, +\infty[,
\\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega,
\\
u_t(x,t-\tau) = f_0(x,t-\tau) \quad \text{in } \Omega \times [0, +\infty[,
\end{align*}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), \( n \in \mathbb{N}^* \), with a smooth boundary \( \partial \Omega = \Gamma \), \( \alpha \) and \( h \) are positive non-increasing function defined on \( \mathbb{R}^+ \), \( g_1 \) and \( g_2 \) are two functions, \( \tau > 0 \) is a time delay, \( \mu_1 \) and \( \mu_2 \) are positive real numbers, and the initial data \( (u_0, u_1, f_0) \) belong to a suitable function space. This type of problems arise in viscoelasticity and, for \( \alpha = 1 \), the problem has been discussed by many researchers.

In the absence of the viscoelastic term (that is, if \( h = 0 \)), problem \( (P) \) has been studied by many mathematicians. It is well known that in the further absence of a damping mechanism, the delay term causes instability of system (see, for instance [13]). In the contrast, in the absence of the delay term, the damping term assures global existence for arbitrary initial data and energy decay estimates depending on the rate of growth of \( g_1 \) (see [2],[4],[15],[16] and [19]). In recent years, the PDEs with time delay effects have become an active area of research and arise in many practical real world problems (see for example [1],[32]). To stabilize a hyperbolic system involving delay terms, additional control terms are necessary (see [27],[28] and [33]). In [27] the authors examined problem \( (P) \) in the linear situation (that is, if \( g_1(s) = g_2(s) = s \forall s \in \mathbb{R} \)) and determined suitable relations between \( \mu_1 \) and \( \mu_2 \), for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if
\(\mu_2 < \mu_1\) and they found a sequence of delays for which the corresponding solution of \((P)\) will be instable if \(\mu_2 \geq \mu_1\). The main approach used in [27] is an observability inequality obtained with a Carleman estimate. The same results were obtained if both the damping and the delay acting in the boundary domain. We also recall the result by Xu, Yung and Li [33], where the authors proved the same result as in [27] for the one space dimension by adopting the spectral analysis approach. Very recently, Benaissa and Louhibi [5] extended the result of [27] to the nonlinear case.

In the presence of the viscoelastic term \(h \neq 0\), Cavalcanti et al. [8] studied \((P)\) for \(g_2 \equiv 0\) and in the presence of a linear localized frictional damping \((\alpha(x)u_t)\). They obtained an exponential rate of decay by assuming that the kernel \(h\) is of exponential decay. This work was later improved by Berriimi and Messaoudi [7] by introducing a different functional, which allowed them to weaken the conditions on \(h\). In [23], Messaoudi investigated the decay rate to \((P)\) under a more general condition on \(h\) and improved earlier results in which only the exponential and polynomial rates were considered. Kirane and Said Houari [5] extended the result in [23] to the case when \(g_1, g_2\) are linear and \(\mu_1 \geq \mu_2\).

Motivated by the works, we investigate in this paper system \((P)\) and prove a global solvability and energy decay estimates of the solutions of problem \((P)\) which depends on the behavior of both \(\alpha\) and \(h\) and \(g_1, g_2\) are nonlinear. To obtain global solutions of problem \((P)\), we use the Galerkin approximation scheme (see [20]) together with the energy estimate method. The technic based on the theory of nonlinear semigroups used in [27] does not seem to be applicable in the nonlinear case.

To prove decay estimates, we use a perturbed energy method and some properties of convex functions. These arguments of convexity were introduced and developed by Lasiecka et al. [9], [12], [17], [18] and [19], and used by Liu and Zuazua [21], Eller et al [14] and Alabau-Boussouira [2].

## 2 Preliminaries and main results

For the relaxation function \(g\) and the potential \(\alpha\), we assume that (see [24]):

\[
\text{(H1) (**) } h, \alpha : \mathbb{R}_+ \to \mathbb{R}_+ \text{ are non-increasing differentiable function satisfying} \\
\quad h(0) = h_0 > 0, \quad \int_0^{+\infty} h(s) ds < +\infty \quad \alpha(t) > 0, \quad 1 - \alpha(t) \int_0^t h(s) ds \geq l > 0.
\]

\[
\text{(**) There exists a non-increasing differentiable function } \zeta : \mathbb{R}_+ \to \mathbb{R}_+ \text{ satisfying} \\
\quad \zeta(s) > 0, \quad h'(s) \leq -\zeta(s) h(s), \quad \forall s \geq 0, \quad \lim_{s \to +\infty} \frac{-\alpha'(s)}{\zeta(s) \alpha(s)} = 0
\]

**Remark 2.1.** Note (**) imply \(\lim_{s \to +\infty} \frac{-\alpha'(s)}{\zeta(s) \alpha(s)} = 0\)

**Remark 2.2.** Condition \(1 - \alpha(t) \int_0^t h(s) ds \geq l > 0\) is made so that \((P)\) is hyperbolic and the energy functional (2.11) below is nonnegative.

\[
\text{(H2) } g_1 : \mathbb{R} \to \mathbb{R} \text{ is a non-decreasing function of class } C^1(\mathbb{R}) \text{ such that there exist} \\
\quad c', c_1, c_2, \alpha_1, \alpha_2 > 0 \text{ and a convex and increasing function } H : \mathbb{R}_+ \to \mathbb{R}_+ \text{ of the class} \\
\quad C^1(\mathbb{R}_+) \cap C^2([0, \infty[) \text{ satisfying } H(0) = 0, \text{ and } H \text{ linear on } [0, c'] \text{ or} \\
\quad (H'(0) = 0 \text{ and } H'' > 0 \text{ on } [0, c'] ), \text{ such that} \\
\quad |g_1(s)| \leq c_2 |s|, \quad \text{if } |s| \geq c'. \quad (2.1)
\]

\[
g_1'(s) \leq H^{-1}(s g_1(s)), \quad \text{if } |s| \leq c'. \quad (2.2)
\]

\[
g_2(s) \leq \text{is an odd non-decreasing function of the class } C^1(\mathbb{R}) \text{ such that there exist} \\
\quad c_3, \alpha_1, \alpha_2 > 0 \text{ and} \\
\quad |g_2(s)| \leq c_3. \quad (2.3)
\]
\[ \alpha_1 \, sg_2(s) \leq G_2(s) \leq \alpha_2 \, sg_1(s), \]
where

\[ G_2(s) = \int_0^s g_2(r) \, dr. \]

\[ \alpha_2 \mu_2 < \alpha_1 \mu_1. \]

**Remark 2.3.**

1. By the mean value Theorem for integrals and the monotonicity of \( g_2 \), we find that

\[ G_2(s) = \int_0^s g_2(r) \, dr \leq sg_2(s). \]

Then, \( \alpha_1 \leq 1 \).

2. We need condition (2.3) only to prove global existence. For the energy decay, we can replace the linear growth of the function \( g_2(s) \), for large \( |s| \), by nonlinear polynomial growth.

We also state a Lemma which will be needed later.

**Lemma 2.4** (Sobolev-Poincaré’s inequality). Let \( q \) be a number with \( 2 \leq q < +\infty \) \((n = 1, 2)\) or \( 2 \leq q \leq 2n/(n-2) \) \((n \geq 3)\). Then there is a constant \( c_* = c_*(\Omega, q) \) such that

\[ \|u\|_q \leq c_* \|\nabla u\|_2 \quad \text{for} \quad u \in H_0^1(\Omega). \]

We introduce as in [27] the new variable

\[ z(x, \rho, t) = u_t(x, t - \tau \rho), \quad x \in \Omega, \quad \rho \in [0, 1[, \quad t > 0. \]

Then, we have

\[ \tau z_t(x, \rho, t) + z_x(x, \rho, t) = 0, \quad \text{in} \; \Omega \times]0, 1[ \times ]0, +\infty[, \]

Therefore, problem \((P)\) takes the form:

\[
\begin{cases}
  u_{tt}(x, t) - \Delta_x u(x, t) + \alpha(t) \int_0^t h(t-s) \Delta_x u(x, s) ds \\
  + \mu_1 g_1(u_t(x, t)) + \mu_2 g_2(z(x, 1, t)) = 0, & \text{in} \; \Omega \times]0, +\infty[, \\
  \tau z_t(x, \rho, t) + z_x(x, \rho, t) = 0, & \text{in} \; \Omega \times]0, 1[ \times ]0, +\infty[, \\
  u(x, t) = 0, & \text{on} \; \partial\Omega \times]0, +\infty[, \\
  z(x, 0, t) = u_t(x, t), & \text{on} \; \Omega \times]0, +\infty[, \\
  u(x, 0) = u_0(x), \; u_t(x, 0) = u_1(x), & \text{in} \; \Omega, \\
  z(x, \rho, 0) = f_0(x, -\rho\tau), & \text{in} \; \Omega \times]0, 1[. \\
\end{cases}
\]

Let \( \xi \) be a positive constant such that

\[ \tau \frac{\mu_2 (1 - \alpha_1)}{\alpha_1} < \xi < \tau \frac{\mu_1 - \alpha_2 \mu_2}{\alpha_2}. \]

We define the energy associated to the solution of problem (2.8) by the following formula:

\[
E(t) = \frac{1}{2} \|u'(t)\|_2^2 + \frac{1}{2} \left( 1 - \alpha(t) \int_0^t h(s) \, ds \right) \|\nabla_x u(t)\|_2^2 + \alpha(t) \frac{1}{2} (h \circ \nabla u)(t) \\
+ \xi \int_{\Omega} \int_0^t G_2(z(x, \rho, t)) \, dp \, dx.
\]

Where

\[ (h \circ \nabla u)(t) = \int_0^t h(t-s) \|\nabla u(t) - \nabla u(t-s)\|_2^2 \, ds \]

We have the following theorem.
Theorem 2.5. Let \( (u_0, u_1, f_0) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega; H^1(0, 1)) \) satisfy the compatibility condition
\[ f_0(., 0) = u_1. \]
Assume that the hypotheses (H1) – (H2) hold. Then problem (P) admits a unique weak solution \( u \in L^\infty_{loc}(\Omega) \cap H^2(\Omega) \cap H_0^1(\Omega), u' \in L^\infty_{loc}(\Omega) \cap H_0^1(\Omega), u'' \in L^\infty_{loc}(\Omega) \cap H^2(\Omega) \)
and, for some constants \( \omega, \epsilon_0 \), we obtain the following decay property:
\[ E(t) \leq H_1^{-1}(\omega \int_0^t \alpha(s) \zeta(s) \, ds), \quad \forall t > 0, \]
where
\[ H_1(t) = \int_t^1 \frac{1}{H_2(s)} \, ds \]
and
\[ H_2(t) = \begin{cases} t & \text{if } H \text{ is linear on } [0, \epsilon_1], \\ tH'(\epsilon_0) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on } [0, \epsilon_1]. \end{cases} \]

Example. Let \( g \) be given by \( g(s) = s^p(-\ln s)^q \), where \( 0 \leq p \leq 1 \) and \( q \in \mathbb{R} \) on \( (0, \epsilon_1) \). Then \( g'(s) = s^{p-1}(-\ln s)^q(-(\ln s) - 1) \) which is an increasing function in a right neighborhood of 0 (if \( q = 0 \) we can take \( \epsilon_1 = 1 \)). The function \( H \) is defined in the neighborhood of 0 by
\[ H(s) = \begin{cases} cs^{\frac{n+1}{2p}}(-\ln s)^{-\frac{n-1}{2p}} & \text{if } 0 < p < 1, \quad q \in \mathbb{R} \\ cs(-\ln s)^{-q} & \text{if } p = 1, \quad q > 0 \\ c\sqrt{s} e^{-\frac{1}{2\pi}} & \text{if } p = 0, \quad q < 0 \end{cases} \]
and we have
\[ H'(s) = \begin{cases} cs^{\frac{n+1}{2p}}(-\ln s)^{-\frac{n-1}{2p}} \left( \frac{p+1}{2p} (-\ln s) + \frac{q}{p} \right) & \text{if } 0 < p \leq 1, \quad q \in \mathbb{R} \\ c\frac{1}{\sqrt{s}} \left( 1 - \frac{1}{q} s^{\frac{1}{2\pi}} \right) e^{-\frac{1}{2\pi}} & \text{if } p = 0, \quad q < 0 \end{cases} \]
when \( s \) is near 0. Thus
\[ \varphi(s) = \begin{cases} cs^{\frac{n+1}{2p}}(-\ln s)^{-\frac{n-1}{2p}} \left( \frac{p+1}{2p} (-\ln s) + \frac{q}{p} \right) & \text{if } 0 < p \leq 1, \quad q \in \mathbb{R} \\ c\sqrt{s} \left( 1 - \frac{1}{q} s^{\frac{1}{2\pi}} \right) e^{-\frac{1}{2\pi}} & \text{if } p = 0, \quad q < 0 \end{cases} \]
when \( s \) is near 0. and
\[ \psi(t) = c \int_1^t \frac{1}{s^{\frac{1}{2p}}(-\ln s)^{-\frac{n-1}{2p}} \left( \frac{p+1}{2p} (-\ln s) + \frac{q}{p} \right)} \, ds \]
when \( t \) is near 0.

\[ \psi(t) = c \int_1^\frac{1}{2\pi} \frac{1}{s^{\frac{1}{2p}}(-\ln s)^{-\frac{n-1}{2p}} \left( \frac{p+1}{2p} (-\ln s) + \frac{q}{p} \right)} \, ds, \quad p = 0, q < 0, \text{ when } t \text{ is near 0} \]

We obtain in a neighborhood of 0
\[ \psi(t) = \begin{cases} ct^{\frac{n+1}{2p}}(-\ln t)^{-\frac{n-1}{2p}} & \text{if } 0 < p < 1, \quad q \in \mathbb{R} \\ c(-\ln t)^{1+q} & \text{if } p = 1, \quad q > 0, \\ ct^{\frac{n-1}{2p}} e^{\frac{1}{2\pi}} & \text{if } p = 0, \quad q < 0 \end{cases} \]
and then in a neighborhood of $+\infty$ (see Appendix)

$$
\psi^{-1}(t) = \begin{cases} 
ct^{\frac{2q}{1-p}} (\ln t)^{-\frac{2q}{p-1}} & \text{if } 0 < p \leq 1, \quad q \in \mathbb{R}, \\
ct^{-\frac{2q}{1-p}} & \text{if } p = 1, \quad q > 0, \\
c(tn)^{2q} & \text{if } p = 0, \quad q < 0.
\end{cases}
$$

Using the fact that $h(t) = t$ as $t$ goes to infinity, then

$$
E(t) \leq \begin{cases} 
\sigma(t)^{\frac{1}{2}}(\ln t)^{-\frac{2}{p-1}} & \text{if } 0 < p \leq 1, \quad q \in \mathbb{R}, \\
c^2 - \sigma(t)^{-\frac{2q}{1-p}} & \text{if } p = 1, \quad q < 0, \\
c(tn)^{2q} & \text{if } p > 1 \text{ or } p = 1 \text{ and } q \leq 0.
\end{cases}
$$

where

$$
\sigma(t) = \int_0^t \alpha(s)\zeta(s) \, ds
$$

We finish this section by giving an explicit upper bound for the derivative of the energy.

**Lemma 2.6.** Let $(u, z)$ be a solution to the problem (2.6). Then, the energy functional defined by (2.10) satisfies

$$
E'(t) \leq -\left(\frac{1}{\tau} - \frac{5\alpha^2}{\mu_1} - \mu_2\alpha_2\right) \int_\Omega \alpha^3(u^3) \, dx
- \left(\frac{1}{\tau} - \mu_2(1 - \alpha_1)\right) \int_\Omega \alpha^3(u^3) \, dx
- \frac{1}{\tau} \alpha(t)h(t)\|\nabla u\|_2^2 + \frac{1}{\tau} \alpha(t)(h' \circ \nabla u)(t)
- \frac{1}{\tau} \alpha'(t)\|h's\|_2^2 + \frac{1}{\tau} \alpha'(t)(h' \circ \nabla u)(t)
\leq -\frac{1}{\tau} \alpha'(t)\|h's\|_2^2 + \frac{1}{\tau} \alpha(t)(h' \circ \nabla u)(t).
$$

**Proof.** Multiplying the first equation in (2.8) by $u_t(x, t)$, and integrating the result over $\Omega$, to obtain:

$$
\frac{1}{\tau} \frac{d}{dt} \left(\|u_t(x, t)\|_2^2 + \|\nabla u(x, t)\|_2^2\right) + \mu_2 \int_\Omega g_1(u_t(x, t))u_t(x, t) \, dx
+ \mu_2 \int_\Omega g_1(z(x, 1, t))u_t(x, t) \, dx = \alpha(t) \int_\Omega \alpha^3(u^3) \, dx
$$

The term in the right-hand side of (2.14) can be rewritten as follows

$$
\alpha(t) \int_\Omega \alpha^3(h - s)\nabla u(x, s)\nabla u_t(x, t) \, ds \, dx + \frac{1}{\tau} \alpha(t)(h(t) \circ \nabla u)(t)
+ \frac{1}{\tau} \alpha(t)(h' \circ \nabla u)(t).
$$

Consequently, equality (2.14) becomes

$$
\frac{1}{\tau} \frac{d}{dt} \left[\|u_t(x, t)\|_2^2 + \left(1 - \alpha(t)\right)\int_\Omega \alpha^3(h - s) \, ds \right] + \tau \alpha(t)(h \circ \nabla u)(t)
+ \tau \alpha(t)(h' \circ \nabla u)(t)
\leq -\mu_2 \int_\Omega g_1(u_t(x, t))u_t(x, t) \, dx
- \mu_2 \int_\Omega g_1(z(x, 1, t))u_t(x, t) \, dx
- \frac{1}{\tau} \alpha(t)(h(t) \circ \nabla u)(t)
+ \frac{1}{\tau} \alpha(t)(h' \circ \nabla u)(t).
$$

We multiply the second equation in (2.8) by $\xi g_2(x, 1, t)$ and integrate over $\Omega \times [0, 1]$ to obtain:

$$
\xi \int_\Omega \int_0^1 z_t(x, \rho, \tau) \, G_2(x, \rho, \tau) \, d\rho \, dx = -\xi \int_\Omega \int_0^1 \frac{\partial}{\partial \rho} G_2(x, \rho, \tau) \, d\rho \, dx
= -\xi \int_\Omega (G_2(z(x, 1, t)) - G_2(z(x, 0, t))) \, dx.
$$

Hence

$$
\xi \frac{d}{dt} \int_\Omega \int_0^1 G_2(x, \rho, \tau) \, d\rho \, dx + \xi \int_\Omega G_2(x, 1, t) \, dx - \xi \int_\Omega G_2(u_t(x, t)) \, dx = 0.
$$
Remark 2.7. Again, use of \((\nabla u, \nabla v)\) and satisfies the following inequality
\[ E'(t) = \frac{1}{2} \alpha(t)(h' \circ \nabla u)(t) - \frac{1}{2} \alpha(t)h(t)\|\nabla u(x, t)\|_2^2 - \mu_1 \int_{\Omega} g_1(u_t(x, t))u_t(x, t) \, dx \\
- \mu_2 \int_{\Omega} g_2(z(x, 1, t))u_t(x, t) \, dx - \frac{\xi}{p} \int_{\Omega} G_2(z(x, 1, t)) \, dx + \frac{\xi}{p} \int_{\Omega} G_2(u_t(x, t)) \, dx \\
- \frac{1}{a}(t)\left(\int_0^t h(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{a}(t)(h \circ \nabla u(t)). \] 
By recalling (2.4), we arrive at
\[ E'(t) \leq -\left(\mu_1 - \frac{\xi}{p}\right) \int_{\Omega} u_t(x, t)g_1(u_t(x, t)) \, dx - \frac{\xi}{p} \int_{\Omega} z(x, 1, t)g_2(z(x, 1, t)) \, dx \\
+ \mu_2 \int_{\Omega} z(x, 1, t)g_2(z(x, 1, t)) \, dx + \mu_2 \int_{\Omega} G_2(u_t(x, t)) \, dx - \mu_2 \int_{\Omega} G_2(z(x, 1, t)) \, dx \\
+ \frac{1}{2} \alpha(t)h' \circ \nabla u(t) - \frac{1}{2} \alpha(t)h(t)\|\nabla u(x, t)\|_2^2 - \frac{1}{2} \alpha(t)\left(\int_0^t h(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2} \alpha(t)(h \circ \nabla u(t)). \tag{2.18} \]
Let us denote by \(G^*_2\) the conjugate function of the convex function \(G_2\), i.e.,
\[ G^*_2(s) = \sup_{t \in \mathbb{R}^+} (st - G_2(t)). \] 
Then, \(G^*_2\) is the Legendre transform of \(G_2\), which is given by
\[ G^*_2(s) = s(G^*_2)^{-1}(s) - G_2((G^*_2)^{-1}(s)), \quad \forall s \geq 0 \] 
and satisfies the following inequality
\[ st \leq G^*_2(s) + G_2(t), \quad \forall s, t \geq 0. \tag{2.20} \]
(see Arnold [3], p. 61-62, and Lasiecka [9], [12], [17]-[18] for more information).

Then, from the definition of \(G_2\), we get
\[ G^*_2(s) = sg_2^{-1}(s) - G_2(g_2^{-1}(s)). \]
Hence
\[ G^*_2(g_2(z(x, 1, t))) = z(x, 1, t)g_2(z(x, 1, t)) - G_2(z(x, 1, t)). \tag{2.21} \]
Making use of (2.18), (2.20) and (2.21), we arrive at
\[ E'(t) \leq -\left(\mu_1 - \frac{\xi}{p}\right) \int_{\Omega} u_t(x, t)g_1(u_t(x, t)) \, dx - \frac{\xi}{p} \int_{\Omega} z(x, 1, t)g_2(z(x, 1, t)) \, dx \\
+ \mu_2 \int_{\Omega} z(x, 1, t)g_2(z(x, 1, t)) \, dx + \mu_2 \int_{\Omega} G_2(u_t(x, t)) \, dx - \mu_2 \int_{\Omega} G_2(z(x, 1, t)) \, dx \\
+ \frac{1}{2} \alpha(t)h' \circ \nabla u(t) - \frac{1}{2} \alpha(t)h(t)\|\nabla u(x, t)\|_2^2 - \frac{1}{2} \alpha(t)\left(\int_0^t h(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2} \alpha(t)(h \circ \nabla u(t)). \tag{2.22} \]
Again, use of (2.4) yields
\[ E'(t) \leq -\left(\mu_1 - \frac{\xi}{p} - \mu_2\right) \int_{\Omega} u_t(x, t)g_1(u_t(x, t)) \, dx \\
- \left(\frac{\xi}{p} - \mu_2 \right) \int_{\Omega} z(x, 1, t)g_2(z(x, 1, t)) \, dx \\
+ \frac{1}{2} \alpha(t)h' \circ \nabla u(t) - \frac{1}{2} \alpha(t)h(t)\|\nabla u(x, t)\|_2^2 - \frac{1}{2} \alpha(t)\left(\int_0^t h(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2} \alpha(t)(h \circ \nabla u(t)) \tag{2.23} \]
\[ \leq -\frac{1}{2} \alpha(t)\left(\int_0^t h(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2} \alpha(t)(h \circ \nabla u(t)). \]
\[ \square \]

Remark 2.7. Since \(-\alpha(t)(\int_0^t h(s)ds)\|\nabla u\|_2^2 \geq 0\), \(E(t)\) may not be non-increasing.

3 Global Existence

We are now ready to prove Theorem 2.5 in the next two sections.
Throughout this section we assume \(u_0 \in H^2(\Omega) \cap H_0^1(\Omega), u_1 \in H_0^1(\Omega)\) and \(f_0 \in H_0^1(\Omega; H^1(0, 1)).\)

We employ the Galerkin method to construct a global solution. Let \(T > 0\) be fixed and denote by \(V_k\) the space generated by \(\{w_1, w_2, \ldots, w_k\}\) where the set \(\{w_k, k \in \mathbb{N}\}\) is a basis of \(H^2(\Omega) \cap H_0^1(\Omega).\)
Now, we define, for \( 1 \leq j \leq k \), the sequence \( \phi_j(x, \rho) \) as follows:
\[
\phi_j(x, 0) = u_j.
\]
Then, we may extend \( \phi_j(x, 0) \) by \( \phi_j(x, \rho) \) over \( L^2(\Omega \times (0, 1)) \) such that \( \{\phi_j\}_j \) form a basis of \( L^2(\Omega; H^1(0, 1)) \) and denote \( Z_k \) the space generated by \( \{\phi_1, \phi_2, \ldots, \phi_k\} \).

We construct approximate solutions \((u_k, z_k), k = 1, 2, 3, \ldots, \) in the form
\[
\begin{align*}
    u_k(t) &= \sum_{j=1}^{k} g_{jk}(t)w_j, \\
    z_k(t) &= \sum_{j=1}^{k} h_{jk}(t)\phi_j,
\end{align*}
\]
where \( g_{jk} \) and \( h_{jk}, j = 1, 2, \ldots, k, \) are determined by the following ordinary differential equations:
\[
\begin{align*}
    \begin{cases}
        (u''_k(t), w_j) + (\nabla_x u_k(t), \nabla_x w_j) - \alpha(t) \int_0^t h(t-s)(\nabla_x u_k(s), \nabla_x w_j)\, ds + \mu_1(g_1(u'_k), w_j) \\
        + \mu_2(g_2(z_k, 1), w_j) = 0, & 1 \leq j \leq k,
    \end{cases}
    \\
    z_k(x, 0, t) = u'_k(x, t)
\end{align*}
\]
(3.1)
\[
\begin{align*}
    u_k(0) &= u_{0k} = \sum_{j=1}^{k} (u_0, w_j)w_j \rightarrow u_0 \in H^2(\Omega) \cap H^1(\Omega) \text{ as } k \rightarrow +\infty, \\
    u'_k(0) &= u_{1k} = \sum_{j=1}^{k} (u_1, w_j)w_j \rightarrow u_1 \in H^1(\Omega) \text{ as } k \rightarrow +\infty
\end{align*}
\]
(3.2)
(3.3)
and
\[
(\tau z_k + z_k, \phi_j) = 0, \quad 1 \leq j \leq k,
\]
(3.4)
\[
\begin{align*}
    z_k(\rho, 0) &= z_{0k} = \sum_{j=1}^{k} (f_0, \phi_j)\phi_j \rightarrow f_0 \in H^1(\Omega; H^1(0, 1)) \text{ as } k \rightarrow +\infty.
\end{align*}
\]
(3.5)

By virtue of the theory of ordinary differential equations, the system (3.1)-(3.5) has a unique local solution which is extended to a maximal interval \([0, T_k]\) (with \( 0 < T_k \leq +\infty \)) by Zorn lemma since the nonlinear terms in (3.1) are locally Lipschitz continuous. Note that \( u_k(t) \) is \( C^2 \)-class.

In the next step, we obtain a priori estimates for the solution, so that it can be extended outside \([0, T_k]\) to obtain one solution defined for all \( t > 0 \).

In order to use a standard compactness argument with the limiting procedure, it suffices to derive some a priori estimates for \((u_k, z_k)\).

**The first estimate.** Since the sequences \( u_{0k}, u_{1k} \) and \( z_{0k} \) converge, then standard calculations, using (3.1)-(3.5), similar to those used to derive (2.13), yield
\[
\begin{align*}
    E_k(t) &= a_1 \int_0^t \int_0^t u'_k(x, t)g_1(u'_k(x, t))\, dx\, ds + a_2 \int_0^t \int_0^t z_k(x, 1, t)g_2(z_k(x, 1, t))\, dx\, ds \\
    &\quad + \frac{1}{2} \alpha(t)h(t)\|\nabla u_k\|^2_2 - \frac{1}{2} \alpha(t)(h' \circ \nabla u_k)(t) + \frac{1}{2} \alpha'(t)(\int_0^t h(s)\, ds)\|\nabla u_k\|^2_2 - \frac{1}{2} \alpha'(t)(h' \circ \nabla u_k)(t) \\
    &\leq E_k(0) \leq C,
\end{align*}
\]
(3.6)
where
\[
E(t) = \frac{1}{2}\|u'_k(t)\|^2_2 + \frac{1}{2} \left(1 - \alpha(t) \int_0^t h(s)\, ds\right)\|\nabla u_k(t)\|^2_2 + \alpha(t)\|h \circ \nabla u_k(t)\|_2 + \int_0^t G_2(z(x, \rho, t))\, d\rho\, dx.
\]
Multiplying by $h$ obtain:

First, we estimate $\frac{\xi_1}{\tau} - \mu_2(1 - \alpha_1)$.

for some $C$ independent of $k$. These estimates imply that the solution $(u_k, z_k)$ exists globally in $[0, +\infty]$.

Estimate (3.6) yields

\[ u_k \text{ is bounded in } L^\infty_{loc}(0, \infty; H^1_0(\Omega)), \]

(3.7)

\[ u'_k \text{ is bounded in } L^\infty_{loc}(0, \infty; L^2(\Omega)), \]

(3.8)

\[ u'_k(t)g_1(u'_k(t)) \text{ is bounded in } L^1(\Omega \times (0, T)), \]

(3.9)

\[ G_2(z_k(x, \rho, t)) \text{ is bounded in } L^\infty_{loc}(0, \infty; L^1(\Omega \times (0, 1))), \]

(3.10)

\[ z_k(x, 1, t)g_2(z_k(x, 1, t)) \text{ is bounded in } L^1(\Omega \times (0, T)). \]

(3.11)

The second estimate. first, we estimate $u''_k(0)$. Testing (3.1) by $g''_k(t)$ and choosing $t = 0$, we obtain:

\[ \|u''_k(0)\|_2 \leq \|\Delta x u_0k\|_2 + \mu_1\|g'_1(u_{1k})\|_2 + \mu_2\|g_2(z_{20k})\|_2. \]

Since $g_1(u_{1k}), g_2(z_{20k})$ are bounded in $L^2(\Omega)$ hence, from (3.2), (3.3) and (3.5),

\[ \|u''_k(0)\|_2 \leq C. \]

Differentiating (3.1) with respect to $t$, we get

\[ \left( u''_k(t) + \Delta x u'_k(t) + \frac{d}{dt}\left( \alpha(t) \int_0^t h(t-s) \Delta x u_k(s) ds \right) + \mu_1u'_k(t)g'_1(u_k) + \mu_2z'_k g'_2(z_k, w_j) \right) = 0. \]

Multiplying by $g''_k(t)$ and summing over $j$ from 1 to $k$, it follows that

\[ \frac{1}{2} \frac{d}{dt} \left( \|u''_k(t)\|^2 + \|\nabla_x u'_k(t)\|^2 \right) - \alpha(t)h(0) \frac{d}{dt} \left( \Delta_x u_k(t), \nabla_x u'_k(t) + \alpha(t)h(0) \|\nabla_x u'_k(t)\|^2 \right) \]

\[ -\alpha(t) \frac{d}{dt} \int_0^t h(t-s) (\Delta x u_k(s), \nabla_x u'_k(t)) ds - \alpha(t)h'(0) (\nabla_x u_k(t), \nabla_x u'_k(t)) \]

\[ + \alpha(t) \int_0^t h''(t-s) (\nabla_x u_k(s), \nabla_x u'_k(t)) ds + \mu_1 \int_\Omega u''_k(t) g'_1(u_k(t)) dx \]

\[ + \mu_2 \int_\Omega u''_k(t) z'_k(x, 1, t) g'_2(z_k(x, 1, t)) dx = 0. \]

(3.12)

Differentiating (3.4) with respect to $t$, we get

\[ (\tau z'_k(t) + \frac{d}{d\rho} z'_k, \phi_j) = 0. \]

Multiplying by $h'_k(t)$ and summing over $j$ from 1 to $k$, it follows that

\[ \frac{1}{2} \tau \frac{d}{dt} \|z'_k(t)\|^2 + \frac{1}{2} \frac{d}{d\rho} \|z'_k(t)\|^2 = 0. \]

(3.13)

Taking the sum of (3.12) and (3.13), we obtain

\[ \frac{1}{2} \frac{d}{dt} \left( \|u''_k(t)\|^2 + \|\nabla_x u'_k(t)\|^2 \right) + \alpha(t)h(0) \|\nabla_x u'_k(t)\|^2 \]

\[ + \mu_1 \int_\Omega u''_k(t) g'_1(u_k(t)) dx + \frac{1}{2} \int_\Omega (\tau z'_k(x, 1, t))^2 dx = \alpha(t)h(0) \frac{d}{dt} (\Delta_x u_k(t), \nabla_x u'_k(t)) \]

\[ + \alpha(t) \frac{d}{dt} \int_0^t h(t-s) (\Delta x u_k(s), \nabla_x u'_k(t)) ds - \alpha(t)h'(0) (\nabla_x u_k(t), \nabla_x u'_k(t)) \]

\[ - \alpha(t) \int_0^t h''(t-s) (\nabla_x u_k(s), \nabla_x u'_k(t)) ds - \mu_2 \int_\Omega u''_k(t) z'_k(x, 1, t) g'_2(z_k(x, 1, t)) dx \]

\[ + \frac{1}{2} \|u''_k(t)\|^2. \]

Using (2.3), Cauchy-Schwarz and Young’s inequalities, we obtain

\[ |\alpha(t)h'(0)(\nabla_x u_k(t), \nabla_x u'_k(t))| \leq \varepsilon \alpha(t) \|\nabla_x u_k(t)\|^2 + \frac{\alpha(t)h'(0)^2}{4\varepsilon} \|\nabla_x u'_k(t)\|^2, \]
Consequently, equality (3.19) becomes

\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla u_k(t)\|^2 + \|\Delta u_k(t)\|^2 \right) - \alpha(t) \int_0^t h(s) \|\nabla u_k(s)\|^2 ds + \frac{\alpha(t)}{4\varepsilon} \|\nabla u_k(t)\|^2 + \frac{\alpha(t)}{4\varepsilon} \|\Delta u_k(t)\|^2 = 0.
\]

Then from (3.14), choosing ε small enough and using Gronwall’s lemma, we obtain

\[
\|u_k(t)\|^2 + \|\Delta u_k(t)\|^2 + \|\Delta u_k(t)\|^2 \leq \alpha(t)\varepsilon + \frac{\alpha(t)}{4\varepsilon} \|\nabla u_k(t)\|^2 + \frac{\alpha(t)}{4\varepsilon} \|\Delta u_k(t)\|^2 + M\alpha(t),
\]

for all \( t \in [0, T] \) and \( M \) is a positive constant independent of \( k \in \mathbb{N} \). Therefore, we conclude that

\[
\begin{align*}
\|u_k\|^2 & \text{ is bounded in } L^\infty_{loc}(0, +\infty; L^2(\Omega)), \\
\|u_k\|^2 & \text{ is bounded in } L^\infty_{loc}(0, +\infty; H^1_0(\Omega)), \\
\|z_k\| & \text{ is bounded in } L^\infty_{loc}(0, +\infty; L^2(\Omega \times (0, 1))).
\end{align*}
\]
Replacing \( \phi_j \) by \(-\Delta_x \phi_j \) in (3.4), multiplying by \( h_{j,k}(t) \), summing over \( j \) from 1 to \( k \), it follows that

\[
\frac{1}{2} \frac{d}{dt} \| \nabla_x z_k(t) \|^2 + \frac{1}{2} \frac{d}{d\rho} \| \nabla_x z_k(t) \|^2 = 0. \tag{3.21}
\]

From (3.19) and (3.21), we have

\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla_x u_k'(t) \|^2 + \left( 1 - \alpha(t) \int_0^t h(s) \, ds \right) \| \Delta_x u_k(t) \|^2 + \tau \| \nabla_x z_k(x, \rho, t) \|^2 \right) + \alpha(t) h(t) \| \nabla_x u_k(t) \|^2 \right)
\]

\[
= -\mu_2 \int_{\Omega} \nabla_x u_k'(t) \nabla_x z_k(x, 1, t) \, dx + \frac{\tau}{2} \| \nabla_x z_k(x, 1, t) \|^2 \]

Using (2.3), Cauchy-Schwarz and Young’s inequalities, we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla_x u_k'(t) \|^2 + \left( 1 - \alpha(t) \int_0^t h(s) \, ds \right) \| \Delta_x u_k(t) \|^2 + \tau \| \nabla_x z_k(x, \rho, t) \|^2 \right)
\]

\[
+ \mu_1 \int_{\Omega} \| \nabla_x u_k(t) \|^2 g'(1(u_k(t)) \, dx + \epsilon \int_{\Omega} \| \nabla_x z_k(x, 1, t) \|^2 \leq \epsilon \| \nabla_x u_k(t) \|^2.
\]

Integrating the last inequality over \((0, t)\) and using Gronwall’s lemma, we obtain

\[
\| \nabla_x u_k'(t) \|^2 + \left( 1 - \alpha(t) \int_0^t h(s) \, ds \right) \| \Delta_x u_k(t) \|^2 + \tau \| \nabla_x z_k(x, \rho, t) \|^2 \right)
\]

\[
\leq e^{\epsilon t} \left( \| \nabla_x u_k'(0) \|^2 + \| \Delta_x u_k(0) \|^2 + \tau \| \nabla_x z_k(x, 0, 0) \|^2 \right)
\]

for all \( t \in \mathbb{R}_+ \), therefore, we conclude that

\[
\begin{align*}
\| \Delta_x u_k(t) \| &\leq C, \\
\| \nabla_x z_k(t) \| &\leq C.
\end{align*}
\]

Applying Dunford-Pettis’s theorem, we conclude from (3.7), (3.8), (3.9), (3.10), (3.16), (3.17), (3.18), (3.22) and (3.23), replacing the sequences \( u_k \) and \( z_k \) with subsequence if necessary, that

\[
\begin{align*}
\| \Delta_x u_k(t) \| &\leq C, \\
\| \nabla_x z_k(t) \| &\leq C.
\end{align*}
\]

for suitable functions \( u \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \), \( z \in L^\infty(0, T; L^2(\Omega \times 0, 1)) \), \( \chi \in L^2(\Omega \times 0, T) \), \( \psi \in L^2(\Omega \times 0, T) \) for all \( t \geq 0 \). We have to show that \((u, z)\) is a solution of (2.8).

From (3.7) and (3.8) we have \((u_k')\) is bounded in \(L^2(0, T; H_0^1(\Omega))\). Then \((u_k')\) is bounded in \(L^2(0, T; H_0^1(\Omega))\). Since \((u_k'')\) is bounded in \(L^\infty(0, T; L^2(\Omega))\), \( (u_k'') \) is bounded in \(L^2(0, T; L^2(\Omega))\). Consequently, \((u_k')\) is bounded in \(H^1(\Omega)\).

Since the embedding \(H^1(\Omega) \hookrightarrow L^2(\Omega)\) is compact, using Aubin-Lions’ theorem [20], we can extract a subsequence \((u'_\nu)\) of \((u_k')\) such that

\[
\begin{align*}
\| u'_\nu - u' \| &\to 0 \quad \text{strongly in } L^2(\Omega).
\end{align*}
\]

Therefore

\[
\begin{align*}
u' &\to u' \quad \text{a.e in } Q. \tag{3.27}
\end{align*}
\]

Similarly we obtain

\[
\begin{align*}
\nu &\to z \quad \text{a.e in } Q. \tag{3.28}
\end{align*}
\]
Lemma 3.1. For each $T > 0$, $g(u'), g(z(x, 1, t)) \in L^1(Q)$ and 
$\|g(u')\|_{L^1(Q)}, \|g(z(x, 1, t))\|_{L^1(Q)} \leq K_1$, where $K_1$ is a constant independent of $t$.

Proof. By (H2) and (3.27) we have

$$g_1(u'_k(x, t)) \to g_1(u'(x, t)) \text{ a.e. in } Q;$$

$$0 \leq g_1(u'_k(x, t))u'_k(x, t) \to g_1(u'(x, t))u'(x, t) \text{ a.e. in } Q$$

Hence, by (3.9) and Fatou’s lemma we have

$$\int_0^T \int_{\Omega} u'(x, t)g_1(u'(x, t)) \, dx \, dt \leq K \text{ for } T > 0. \quad (3.29)$$

By Cauchy-Schwarz inequality, using (3.29), we have

$$\int_0^T \int_{\Omega} |g_1(u'(x, t))| \, dx \, dt \leq c|Q|^{\frac{1}{2}} \left( \int_0^T \int_{\Omega} u'(w) \, dw \, dt \right)^{\frac{1}{2}} \leq c|Q|^{\frac{1}{2}} K_1 \equiv K_1$$

$\square$

Lemma 3.2. $g(u'_k) \to g(u')$ in $L^1(\Omega \times (0, T))$ and $g(z_k) \to g(z)$ in $L^1(\Omega \times (0, T))$.

Proof. Let $E \subset \Omega \times [0, T]$ and set

$$E_1 = \left\{ (x, t) \in E; |g_1(u'_k(x, t))| \leq \frac{1}{\sqrt{|E|}} \right\}, \quad E_2 = E \setminus E_1,$$

where $|E|$ is the measure of $E$. If $M(r) := \inf\{|s|; s \in \mathbb{R} \text{ and } |g_1(s)| \geq r\}$,

$$\int_E |g_1(u'_k)| \, dx \, dt \leq c\sqrt{|E|} + \left( M \left( \frac{1}{\sqrt{|E|}} \right) \right)^{-1} \int_{E_2} |u'_k| \, dx \, dt.$$

Applying (3.9) we deduce that $\sup_k \int_E |g_1(u'_k)| \, dx \, dt \to 0$ as $|E| \to 0$. From Vitali’s convergence theorem we deduce that $g_1(u'_k) \to g_1(u')$ in $L^1(\Omega \times (0, T))$, hence

$$g_1(u'_k) \to g_1(u') \text{ weak in } L^2(Q).$$

Similarly, we have

$$g_2(z'_k) \to g_2(z') \text{ weak in } L^2(Q),$$

and this implies that

$$\int_0^T \int_{\Omega} g_1(u'_k) v \, dx \, dt \to \int_0^T \int_{\Omega} g_1(u') v \, dx \, dt, \text{ for all } v \in L^2(0, T; H^1_0) \quad (3.30)$$

as $k \to +\infty$. $\square$

It follows at once from (3.24), (3.25), (3.30), (3.31) and (3.26) that for each fixed $v \in L^2(0, T; H^1_0)$ and $w \in L^2(0, T; H^1_0(\Omega \times (0, 1)))$

$$\int_0^T \int_{\Omega} (u''_k - \Delta u_k + \alpha(t) \int_0^t h(t-s)\Delta u_k(s) \, ds + \mu_1 g_1(u'_k) + \mu_2 g_2(z_k)) \, dx \, dt$$

$$\to \int_0^T \int_{\Omega} (u'' - \Delta u + \alpha(t) \int_0^t h(t-s)\Delta u(s) \, ds + \mu_1 g_1(u') + \mu_2 g_2(z)) \, dx \, dt$$

as $k \to +\infty$. Hence

$$\int_0^T \int_{\Omega} (u'' + \Delta u + \alpha(t) \int_0^t h(t-s)\Delta u(s) \, ds + \mu_1 g_1(u') + \mu_2 g_2(z)) \, dx \, dt = 0, \quad \forall v \in L^2(0, T; H^1_0).$$

Thus the problem (P) admits a global weak solution $u$. 

4 Asymptotic behavior

For \( M > 0 \) and \( \varepsilon_1, \varepsilon_2 > 0 \), we define the perturbed modified energy by

\[
L(t) = ME(t) + \varepsilon_1 \alpha(t) \Psi(t) + \varepsilon_2 \alpha(t) I(t) + \alpha(t) \chi(t),
\]

where

\[
\Psi(t) = \int_{\Omega} u_t(x, t) u(x, t) \, dx,
\]

\[
I(t) = \int_{\Omega} \int_0^1 e^{-2r\rho} G_2(z(x, \rho, t)) \, d\rho \, dx,
\]

\[
\chi(t) = -\int u_t(x, t) \int_0^t h(t-s)(u(t) - u(s)) \, ds \, dx.
\]

Proof. We consider the functional

\[
K(t) = L(t) - ME(t) = \varepsilon_1 \alpha(t) \Psi(t) + \varepsilon_2 \alpha(t) I(t) + \alpha(t) \chi(t)
\]

and show that

\[
|K(t)| \leq C\alpha(t) E(t), \quad C > 0.
\]

Using Young’s inequality and Poincaré’s inequality, we obtain

\[
|\alpha(t) \chi(t)| = \left| \alpha(t) \int_{\Omega} u_t(x, t) \int_0^t h(t-s)(u(t) - u(s)) \, ds \, dx \right|
\leq \frac{\alpha(t)}{2} \int_{\Omega} u_t^2 \, dx + \frac{\alpha(t)}{2} \left( \int_0^t h(t-s)(u(t) - u(s)) \, ds \right)^2 \, dx
\leq \frac{\alpha(t)}{2} \int_{\Omega} u_t^2 \, dx + \frac{\alpha(t)}{2} \left( \int_0^t h(s) \, ds \right)^2 c_s^2 (h \circ \nabla u)(t).
\]

Where \( c_s \) is the Poincaré constant.

Similarly, we have

\[
|\varepsilon_1 \alpha(t) \Psi(t) + \varepsilon_2 \alpha(t) I(t)| \leq \varepsilon_1 \alpha(t) \int_{\Omega} |u_t| |u| \, dx + \varepsilon_2 \alpha(t) \int_{\Omega} \int_0^1 e^{-2r\rho} G_2(z(x, \rho, t)) \, d\rho \, dx
\leq \varepsilon_1 \alpha(t) \int_{\Omega} u_t^2 \, dx + \varepsilon_2 \alpha(t) \int_{\Omega} |\nabla u|^2 \, dx
+ \varepsilon_2 \alpha(t) \int_{\Omega} \int_0^1 G_2(z(x, \rho, t)) \, d\rho \, dx.
\]

Using \( 1 - \alpha(t) \int_0^t h(s) \, ds \geq l > 0, (2.10), (4.6) \) and (4.7), we get (4.5) for some positive constant \( C \). By choosing \( M \) large enough, our result follows from (4.1), (4.5).

\[
\Box
\]

Proposition 4.3. For each \( t_0 > 0 \) and sufficiently large \( M > 0 \) and appropriately small \( \varepsilon_1, \varepsilon_2 > 0 \), there exist positive constants \( C_3, C_4, \) and \( C_5 \) such that

\[
\frac{d}{dt} L(t) \leq -C_3 \alpha(t) E(t) + C_4 \alpha(t)(h \circ \nabla u)(t) + C_5 \alpha(t) \|g_1(u_t)\|_2^2 \quad \forall t \geq t_0.
\]
The proof of this proposition will be carried out through three lemmas.

**Lemma 4.4.** Let \((u, z)\) be the solution of \((2.8)\), then for any \(\gamma > 0\), we have

\[
\Psi'(t) \leq \frac{\mu_1}{4\gamma} \int_{\Omega} |g_1(u(t, x))|^2 \, dx + \frac{\mu_2}{4\gamma} \int_{\Omega} |g_2(z(x, 1, t))|^2 \, dx + \frac{(1-l)(\alpha(t)(h \circ \nabla u(t))}{4\gamma}, \tag{4.8}
\]

**Proof.** Using the first equation in \((2.8)\), a direct computation leads to

\[
\Psi'(t) = \int_{\Omega} u_t^2(x, t) \, dx + \int_{\Omega} u_t(x, t)u(x, t) \, dx
\]

\[
= \|u_t\|_2^2 + \int_{\Omega} \Delta u(x, t) - \alpha(t) f_0^t \, h(t - s) \Delta u(x, s) \, ds
\]

\[
- \mu_1 g_1(u_t(x, t)) - \mu_2 g_2(u_t(x, t - \tau)) \, u(x, t) \, dx
\]

\[
= ||u_t||_2^2 - ||\nabla u||_2^2 + \alpha(t) f_0^t \, h(t - s) \, \nabla u(x, s) \, ds
\]

\[
- \mu_1 \int_{\Omega} g_1(u_t(x, t)) \, u(x, t) \, dx - \mu_2 \int_{\Omega} g_2(z(x, 1, t)) \, u(x, t) \, dx.
\]

Since \(f_0^t h(s) \, ds \leq f_0^\infty h(s) \, ds \leq \frac{1}{2\gamma t} \),

Now, the third term in the right-hand side of \((4.9)\) can be estimated as follows:

\[
\alpha(t) \int_{\Omega} \nabla u(x, t) f_0^t h(t - s) \, \nabla u(x, s) \, ds \, dx
\]

\[
= \alpha(t) f_0^l f_0^t h(t - s) \left[ \nabla u(x, s) - \nabla u(x, t) \right] \nabla u(x, t) \, ds \, dx + \alpha(t) f_0^t f_0^h h(t - s) \, ||\nabla u(x, t)||^2 \, ds \, dx
\]

\[
\leq (1 - l) ||\nabla u(x, t)||^2 + \alpha(t) f_0^l h(t - s) \, ||\nabla u(x, s) - \nabla u(x, t)|| \, ds \, dx
\]

\[
\leq (1 - l) ||\nabla u(x, t)||^2 + \alpha(t) f_0^l h(t - s) \, ||\nabla u(x, s) - \nabla u(x, t)||^2 \, ds \, dx
\]

\[
\leq (1 - l) ||\nabla u(x, t)||^2 + (1 - l)^2 ||\nabla u(x, t)||^2 \left[ f_0^l \alpha(t) \left( f_0^h h(t - s) \, ||\nabla u(x, s) - \nabla u(x, t)||^2 \, ds \, dx \right)^{1/2}
\]

\[
\leq (1 - l) ||\nabla u(x, t)||^2 + (1 - l) \frac{1}{2} ||\nabla u(x, t)||^2 + \gamma ||\nabla u(x, t)||^2 + \left( \frac{1}{4\gamma} (h \circ \nabla u)(t) \right)
\]

\[
\leq (1 - l + \gamma) ||\nabla u(x, t)||^2 + \frac{(1-l)(\alpha(t)(h \circ \nabla u(t))}{4\gamma},
\]

then we conclude

\[
\Psi'(t) \leq ||u_t||_2^2 - ||\nabla u||_2^2 + (1 - l + \gamma) ||\nabla u||_2^2 + \frac{(1-l)(\alpha(t)(h \circ \nabla u(t))}{4\gamma}
\]

\[
+ \mu_1 \int_{\Omega} g_1(u_t(x, t)) \, u(x, t) \, dx + \mu_2 \int_{\Omega} g_2(z(x, 1, t)) \, u(x, t) \, dx.
\]

Since

\[
\int_{\Omega} |g_1(u_t(x, t))| \, u(x, t) \, dx \leq \gamma c_2^2 ||\nabla u||_2^2 + \frac{1}{4\gamma} \int_{\Omega} |g_1(u_t(x, t))|^2 \, dx
\]

\[
\int_{\Omega} |g_2(z(x, 1, t))| \, u(x, t) \, dx \leq \gamma c_2^2 ||\nabla u||_2^2 + \frac{1}{4\gamma} \int_{\Omega} |g_2(z(x, 1, t))|^2 \, dx
\]

we obtain

\[
\Psi'(t) \leq ||u_t||_2^2 - (l - \gamma - \gamma c_2^2 (\mu_1 + \mu_2)) ||\nabla u||_2^2
\]

\[
+ \frac{\mu_1}{4\gamma} \int_{\Omega} |g_1(u_t(x, t))|^2 \, dx + \frac{\mu_2}{4\gamma} \int_{\Omega} |g_2(z(x, 1, t))|^2 \, dx + \frac{(1-l)(\alpha(t)(h \circ \nabla u(t))}{4\gamma}.
\]

\(\square\)

**Lemma 4.5.** Let \((u, z)\) be the solution of \((2.8)\), then we have

\[
\frac{d}{dt} I(t) \leq -2I(t) - \frac{e^{-2\tau}}{\tau} \int_{\Omega} G_2(z(x, 1, t)) \, dx + \frac{1}{\tau} \int_{\Omega} G_2(u(x, t)) \, dx. \tag{4.10}
\]
Proof. Differentiating (4.3) and using the second equation in (2.8), we have
\[
\frac{d}{dt} I(t) = \int_{\Omega} \int_{0}^{1} e^{-2\tau \rho} z_{t}(x, \rho, t) g_{2}(z(x, \rho, t)) \, d\rho \, dx = -\frac{1}{t} \int_{\Omega} \int_{0}^{1} e^{-2\tau \rho} z_{\rho}(x, \rho, t) g_{2}(z(x, \rho, t)) \, d\rho \, dx = -\frac{1}{t} \int_{\Omega} \int_{0}^{1} e^{-2\tau \rho} \frac{d}{dt} G_{2}(z(x, \rho, t)) \, d\rho \, dx
\]
\[
= -\frac{1}{t} \int_{\Omega} \int_{0}^{1} \left[ \frac{d}{dt} \left( e^{-2\tau \rho} G_{2}(z(x, \rho, t)) \right) + 2\tau e^{-2\tau \rho} G_{2}(z(x, \rho, t)) \right] \, d\rho \, dx
\]
\[
= -\frac{1}{t} \int_{\Omega} \int_{0}^{1} e^{-2\tau \rho} G_{2}(z(x, 1, t)) - G_{2}(u_{t}(x, t)) \, dx
\]
\[
-2 \int_{\Omega} \int_{0}^{1} e^{-2\tau \rho} G_{2}(z(x, \rho, t)) \, d\rho \, dx \leq -2 \int_{\Omega} \int_{0}^{1} e^{-2\tau \rho} G_{2}(z(x, 1, t)) \, dx + \frac{1}{t} \int_{\Omega} G_{2}(u_{t}(x, t)) \, dx
\]
\[
\leq -2 I(t) - e^{-2\tau} \int_{\Omega} G_{2}(z(x, 1, t)) \, dx + \frac{1}{t} \int_{\Omega} G_{2}(u_{t}(x, t)) \, dx.
\]
\[
\square
\]

Lemma 4.6. Let \((u, z)\) be the solution of (2.8), then we have the estimate
\[
\frac{d}{dt} \chi(t) \leq \eta(1 + 2(1 - l)^{2}) \|\nabla u\|_{2}^{2} - \left( \int_{0}^{t} h(s) \, ds \right) - \eta \|u_{t}\|_{2}^{2}
\]
\[
+ 
\left( 2\eta \alpha^{2}(t) + \frac{1}{4\eta} \mu_{1} + \mu_{2} \right) \left( \int_{0}^{t} h(s) \, ds \right)(h \circ \nabla u)(t) - \frac{h_{0} \alpha^{2}}{4\eta}(h' \circ \nabla u)(t)
\]
\[
+ \eta \mu_{1} \|g_{1}(u_{t})\|_{2}^{2} + \eta \mu_{2} \|g_{2}(z(x, 1, t))\|_{2}^{2},
\]
(4.11)
for any \(\eta\) a positive constant.

Proof. A differentiation of (4.4) leads to
\[
\chi(t) = -\int_{\Omega} u_{t}(x, t) \int_{0}^{t} h(t - s) \left( u(t) - u(s) \right) \, ds \, dx,
\]
we have
\[
\chi'(t) = -\int_{\Omega} u_{tt}(x, t) \int_{0}^{t} h(t - s) \left( u(t) - u(s) \right) \, ds \, dx
\]
\[
- \int_{\Omega} u_{t}(x, t) \left[ u_{t}(x, t) \int_{0}^{t} h(t - s) \, ds + \int_{0}^{t} h'(t - s) \left( u(t) - u(s) \right) \, ds \right] \, dx
\]
\[
= -\int_{\Omega} \left[ \Delta u_{t}(x, t) - \alpha \int_{0}^{t} h(t - s) \Delta u(x, s) \, ds - \mu \int_{0}^{t} g_{1}(u_{t}(x, t)) \right]
\]
\[
- \mu_{2} g_{2}(z(x, 1, t)) \right] \int_{0}^{t} h(t - s) \left( u(t) - u(s) \right) \, ds \, dx
\]
\[
- \int_{\Omega} u_{t}(x, t) \int_{0}^{t} h(t - s) \left( u(t) - u(s) \right) \, ds \, dx
\]
\[
\int_{\Omega} \nabla u_{t}(x, t) \int_{0}^{t} h(t - s) \left( \nabla u(x, s) - \nabla u(x, s) \right) \, ds \, dx
\]
\[
- \alpha \int_{\Omega} \int_{0}^{t} h(t - s) \nabla u(x, s) \, ds \, dx
\]
\[
+ \int_{\Omega} g_{1}(u_{t}(x, t)) \int_{0}^{t} h(t - s) \left( u(t) - u(s) \right) \, ds \, dx
\]
\[
+ \int_{\Omega} g_{1}(u_{t}(x, 1, t)) \int_{0}^{t} h(t - s) \left( u(t) - u(s) \right) \, ds \, dx
\]
\[
- \left( \int_{0}^{t} h(s) \, ds \right) \|u_{t}\|_{2}^{2}
\]
\[
- \int_{\Omega} u_{t}(x, t) \int_{0}^{t} h'(t - s) \left( u(t) - u(s) \right) \, ds \, dx.
\]
(4.12)
Using Young’s inequality and the embedding \(H_{1}^{1}(\Omega) \to L^{2}(\Omega)\), we infer
\[
\int_{\Omega} u_{t}(x, t) \int_{0}^{t} h'(t - s) \left( u(t) - u(s) \right) \, ds \, dx
\]
\[
\leq \eta \|u_{t}\|_{2}^{2} + \frac{1}{4\eta} \left( \int_{0}^{t} h'(t - s) \|u(t) - u(s)\|_{2} \, ds \right)^{2}
\]
\[
\leq \eta \|u_{t}\|_{2}^{2} + \frac{1}{4\eta} \left( \int_{0}^{t} h'(t - s) \|u(t) - u(s)\|_{2} \, ds \right)^{2}
\]
\[
\leq \eta \|u_{t}\|_{2}^{2} - \frac{h_{0} \alpha^{2}}{4\eta} (h' \circ \nabla u)(t),
\]
\[ \left| \int_{\Omega} \nabla u(x,t) \int_{0}^{t} h(t-s) \left( \nabla u(x,t) - \nabla u(x,s) \right) ds \right| \leq \eta \| \nabla u \|_{2}^{2} + \frac{1}{4\eta} \left( \int_{0}^{t} h(s) ds \right) (h \circ \nabla u)(t), \]

\[ \mu_1 \int_{\Omega} g_1(u_t(x,t)) \int_{0}^{t} h(t-s)(u(x,t) - u(x,s)) ds \ dx \]

\[ \leq \eta \mu_1 \| g_1(u_t(x,t)) \|_{2}^{2} + \frac{\mu_1^{2}}{4\eta} \left( \int_{0}^{t} h(s) ds \right) (h \circ \nabla u)(t), \]

\[ \mu_2 \int_{\Omega} g_2(z(x,1,t)) \int_{0}^{t} h(t-s)(u(x,t) - u(x,s)) ds \ dx \]

\[ \leq \eta \mu_2 \| g_2(z(x,1,t)) \|_{2}^{2} + \frac{\mu_2^{2}}{4\eta} \left( \int_{0}^{t} h(s) ds \right) (h \circ \nabla u)(t) \]

and

\[ \left| \alpha(t) \int_{\Omega} \left( \int_{0}^{t} h(t-s) \nabla u(x,s) ds \right) \left( \int_{0}^{t} h(t-s) \nabla u(x,t) - \nabla u(x,s) ds \right) dx \right| \]

\[ = \left| \alpha(t) \int_{\Omega} \left[ \int_{0}^{t} h(t-s) \left( \nabla u(x,t) - \nabla u(x,s) \right) ds - \int_{0}^{t} h(t-s) \nabla u(x,t) ds \right] dx \right| \]

\[ \leq \eta \alpha^2(t) \int_{\Omega} \left( \int_{0}^{t} h(t-s) \nabla u(x,s) ds \right)^2 dx + \frac{1}{4\eta} \int_{\Omega} \left( \int_{0}^{t} h(t-s) \nabla u(x,t) ds \right)^2 dx \]

\[ \leq (2\eta \alpha^2(t) + \frac{1}{4\eta}) \int_{\Omega} \left( \int_{0}^{t} h(s) ds \right) \nabla u(x,t) - \nabla u(x,s) ds \right)^2 dx \]

\[ + 2\eta \alpha^2(t) \left( \int_{0}^{t} h(s) ds \right) \int_{\Omega} \nabla u(x,t) dx \]

\[ \leq 2\eta (1 - l)^2 \| \nabla u(x,t) \|_{2}^{2} + \left( 2\eta \alpha^2(t) + \frac{1}{4\eta} \right) \left( \int_{0}^{t} h(s) ds \right) (h \circ \nabla u)(t). \]

Combining all estimates above, we get

\[ \chi'(t) \leq \eta (1 + 2(1 - l)^2) \| \nabla u \|_{2}^{2} - \left( \left( \int_{0}^{t} h(s) ds \right) - \eta \right) \| u_t \|_{2}^{2} \]

\[ + \left( 2\eta \alpha^2(t) + \frac{1}{4\eta} + \frac{\alpha^2}{4\eta} (\mu_1 + \mu_2) + \frac{1}{4\eta} \right) \left( \int_{0}^{t} h(s) ds \right) (h \circ \nabla u)(t) - \frac{\alpha^2}{4\eta} (h' \circ \nabla u)(t) \]

\[ + \eta \mu_1 \| g_1(u_t) \|_{2}^{2} + \eta \mu_2 \| g_2(z(x,1,t)) \|_{2}^{2}. \] (4.13)

Proof of proposition 4.3. Since \( h \) is positive, then for any \( t_0 > 0 \) we have

\[ \int_{0}^{t} h(s) ds \geq \int_{0}^{t_0} h(s) ds = \bar{h}_0 \text{ for all } t \geq t_0. \]
Thus, making use of this and combining (2.23), (4.8), (4.10) and (4.13) we have
\[
L'(t) = \frac{d}{dt} L(t) = ME'(t) + \varepsilon_1 \alpha(t) \Psi(t) + \varepsilon_1 \alpha'(t) \Psi(t) + \varepsilon_2 \alpha(t) I'(t) + \varepsilon_2 \alpha'(t) I(t) + \alpha(t) \chi'(t)
\]
\[
\leq M \left[ -\frac{1}{2} \alpha'(t) (\int_0^t h(s) ds) \|\nabla u\|^2 + \frac{1}{2} \alpha(t) (h' \circ \nabla u)(t) \right]
\]
\[
+ \varepsilon_1 \alpha(t) \left[ \|u\|_2^2 - (l - \gamma - \gamma_c^2 (\mu_1 + \mu_2)) \|\nabla u\|^2_2 \right]
\]
\[
+ \frac{M}{4 \gamma} \int_\Omega |g_1(u_t(x, t))|^2 dx + \frac{M}{4 \gamma} \int_\Omega |g_2(z(x, 1, t))^2 dx + \frac{(1 - l_0)(\alpha(t))}{4 \gamma} (h \circ \nabla u)(t)
\]
\[
+ \varepsilon_2 \alpha(t) \left[ -2 I(t) - \frac{2 \alpha - \varepsilon_2}{\gamma} \int_\Omega G_2(z(x, 1, t)) dx + \frac{1}{\gamma} \int_\Omega G_2(u_t(x, t)) dx \right]
\]
\[
+ \alpha(t) \left[ \eta (1 + 2 + 2 - l^2) \|\nabla u\|^2 - (\int_0^t h(s) ds) - \eta \|u_t\|^2_2 \right]
\]
\[
+ (2 \eta \alpha_c^2 t + \frac{4}{4 \gamma} (\mu_1 + \mu_2) + \frac{4}{4 \gamma}) \left( \int_0^t h(s) ds \right) (h \circ \nabla u)(t)
\]
\[
- \frac{\varepsilon_1}{4} (h' \circ \nabla u)(t) + \eta_1 \|g_1(u_t)\|^2_2 + \eta_2 \|g_2(z(x, 1, t))\|^2_2 \right]
\]
\[
+ \varepsilon_1 \alpha'(t) \Psi(t) + \varepsilon_2 \alpha'(t) I(t) + \alpha' \chi(t)
\]
\[
\leq -\alpha(t) \left[ l_0 - \varepsilon_1 \|u_t\|^2_2 + \alpha(t) \left( \frac{M}{2} - \frac{\varepsilon_1}{4} \|\nabla u\|^2 \right) (h' \circ \nabla u)(t)
\]
\[
- \alpha(t) \left[ \eta (1 + 2 + 2 - l^2) \|\nabla u\|^2 \right] \left[ \int_\Omega \left( \int_0^t h(s) ds \right) \right] (h \circ \nabla u)(t)
\]
\[
+ \alpha(t) \left( \frac{\varepsilon_1}{4} (h' \circ \nabla u)(t) + \eta_1 |g_1(u_t)|^2_2 + \eta_2 |g_2(z(x, 1, t))|^2_2 \right]
\]
\[
+ \varepsilon_1 \alpha'(t) \Psi(t) + \varepsilon_2 \alpha'(t) I(t) + \alpha' \chi(t).
\]
\[(4.14)\]

By using (4.2), (4.4), Young’s and Poincaré’s inequalities, we have
\[
\int_\Omega u_t^2 dx + \int_\Omega |\nabla u|^2 dx \leq (1 + \varepsilon_2) \alpha(t) \int_\Omega u_t^2 dx + \varepsilon_2 \alpha(t) \int_\Omega \int_\Omega h(s) ds \|u_t\|^2_2 dx
\]
\[
+ \alpha(t) \left( \frac{\varepsilon_1}{4} (h' \circ \nabla u)(t) + \varepsilon_2 \alpha'(t) \int_\Omega \left( \int_0^t h(s) ds \right) G_2(z(x, 1, t)) \right) \right) \|u_t\|^2_2
\]
\[
+ \varepsilon_1 \alpha'(t) \Psi(t) + \varepsilon_2 \alpha'(t) I(t) + \alpha' \chi(t).
\]
\[(4.15)\]
At this point, we choose, first, \( \varepsilon_1 > 0 \) so small that
\[
\tilde{h}_0 - \varepsilon_1 > 0.
\]
Next, we choose \( \gamma > 0 \) so small such that
\[
l - \gamma - \gamma c_s^2 (\mu_1 + \mu_2) > 0.
\]
and \( \eta > 0 \) sufficiently small such that
\[
\varepsilon_1 (l - \gamma - \gamma c_s^2 (\mu_1 + \mu_2)) - \eta (1 + 2(1 - l)^2) > 0.
\]
and
\[
\tilde{h}_0 - \eta - \varepsilon_1 > 0.
\]
Then, we pick \( M > 0 \) sufficiently large so that
\[
\frac{M}{2} - \frac{\varepsilon_2 \gamma c_s^2}{4 \eta} > 0
\]
\[
MC + \varepsilon_2 \frac{2 \gamma c_s^2}{\eta} - \left( \eta \mu_2 + \frac{2 \eta \mu_2}{\gamma} \right) c_3 > 0
\]
\[
MC - \frac{\varepsilon_2}{2} \alpha_2 > 0
\]
We then use \( \lim_{t \to +\infty} \frac{\alpha'(t)}{\alpha(t)} = 0 \) (which can be deduced from (H1)) to choose \( t_0 \geq t_0 \) so That, (4.14) takes the form
\[
\frac{d}{dt} L(t) \leq -C_3 \alpha(t) E(t) + C_4 \alpha(t) (h \circ \nabla u)(t) + C_5 \alpha(t) ||g_1(u_t)||^2
\]
where \( C_3, C_4 \) and \( C_5 \) are three positive constants. This completes the proof of Proposition 4.3.

Now, we estimate the last term in the right hand side of (4.16). We denote by
\[
\Omega^+ = \{ x \in \Omega : |u'| \geq \varepsilon' \}, \quad \Omega^- = \{ x \in \Omega : |u'| \leq \varepsilon' \}.
\]
From (2.1) and (2.2), it follows that
\[
\int_{\Omega^+} |g_1(u')|^2 \, dx \leq \mu_1 \int_{\Omega^-} u' g_1(u') \, dx \leq -\mu_1 E'(t).
\]
Case 1: \( H \) is linear on \( [0, \varepsilon'] \). In this case one can easily check that there exists \( \mu'_1 > 0 \), such that
\[
|g_1(s)| \leq \mu'_1 |s| \text{ for all } |s| \leq \varepsilon',
\]
and thus
\[
\int_{\Omega^-} |g_1(u')|^2 \, dx \leq \mu'_1 \int_{\Omega^-} u' g_1(u') \, dx \leq -\mu'_1 E'(t).
\]
Substitution of (4.17) and (4.18) into (4.16) gives
\[
(L(t) + \mu E(t))' \leq -c_1 \alpha(t) H_2(E(t)) + C_4 \alpha(t) h \circ \nabla u
\]
where \( \mu = C_3 (\mu_1 + \mu'_1) \) and here and in the sequel we take \( C_i \) to be a generic positive constant.
Case 2: \( H'(0) = 0 \) and \( H'' > 0 \) on \( [0, \varepsilon'] \).

Since \( H \) is convex and increasing, \( H^{-1} \) is concave and increasing. By (2.1), the reversed Jensen’s inequality for concave function, and (2.23), it follows that
\[
\int_{\Omega^-} |g_1(u')|^2 \, dx \leq \int_{\Omega^-} H^{-1}(u' g_1(u')) \, dx \leq |\Omega| H^{-1} \left( \frac{1}{|\Omega|} \int_{\Omega^-} u' g_1(u') \, dx \right) \leq C H^{-1}(-C' E'(t)).
\]
A combination of (4.16), (4.17) and (4.20) yields
\[
(L(t) + C_3 \mu_1 E(t))' \leq -C_3 \alpha(t) E(t) + C_4 \alpha(t) (h \circ \nabla u)(t) + \tilde{C}_5 \alpha(t) H^{-1}(-C' E'(t)), \quad t \geq t_0.
\]
Let us denote by $H^*$ the conjugate function of the convex function $H$, i.e.,

$$H^*(s) = \sup_{t \in \mathbb{R}} (st - H(t)).$$

Then $H^*$ is the Legendre transform of $H$, which is given by

$$H^*(s) = s(H')^{-1}(s) - H[(H')^{-1}(s)], \quad \forall s \geq 0$$  \hspace{1cm} \text{(4.22)}

and satisfies the following inequality

$$st \leq H^*(s) + H(t), \quad \forall s, t \geq 0.$$  \hspace{1cm} \text{(4.23)}

The relation (4.22), the fact that $H'(0) = 0$ and $(H')^{-1}$, $H$ are increasing functions yield

$$H^*(s) \leq s(H')^{-1}(s), \quad \forall s \geq 0.$$  \hspace{1cm} \text{(4.24)}

Making use of $E'(t) \leq 0$, $H''(t) \geq 0$, (4.21) and (4.24) we derive for $\varepsilon_0 > 0$ small enough

$$[H'(\varepsilon_0 E(t))\{L(t) + C_5\mu_1 E(t)\} + \tilde{C}_5 C'E(t)]' = \varepsilon_0 E'(t)H''(\varepsilon_0 E(t))\{L(t) + C_5\mu_1 E(t)\} + H'(\varepsilon_0 E(t))\{L(t) + C_5\mu_1 E'(t)\} + \tilde{C}_5 C'E(t) \leq -C_3 \alpha(t)H'(\varepsilon_0 E(t))E(t) + C_4 \alpha(t)H'(\varepsilon_0 E(t))(h \circ \nabla u)(t) + \tilde{C}_3 \alpha(t)H'(\varepsilon_0 E(t))H^{-1}(\tilde{C}^{-1} E'(t)) + \tilde{C}_5 C'E(t) \leq -C_3 \alpha(t)H'(\varepsilon_0 E(t))E(t) + \tilde{C}_5 \alpha(t)H'(\varepsilon_0 E(t))(h \circ \nabla u)(t) \leq -C_3 \alpha(t)H'(\varepsilon_0 E(t))E(t) + C_4 \alpha(t)H'(\varepsilon_0 E(0))(h \circ \nabla u)(t) = -\tilde{C}_3 \alpha(t)H_2(E(t)) + C_4 \alpha(t)H'(\varepsilon_0 E(0))(h \circ \nabla u)(t).$$  \hspace{1cm} \text{(4.25)}

We note that, in the second inequality, we have used (4.23) and $0 \leq H'(\varepsilon_0 E(t)) < H'(\varepsilon_0 E(0))$.

Let

$$\tilde{L}(t) = \begin{cases} L(t) + \mu E(t) & \text{if } H \text{ is linear on } [0, \varepsilon'] \\ H'(\varepsilon_0 E(t))\{L(t) + C_5\mu_1 E(t)\} + \tilde{C}_5 C'E(t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on } [0, \varepsilon'], \end{cases}$$

then from (4.19) and (4.25), it holds that

$$\tilde{L}'(t) \leq -c_4 \alpha(t)H_2(E(t)) + c_5 \alpha(t)\alpha(t)(h \circ \nabla u)(t), \quad \forall t \geq t_0.$$  \hspace{1cm} \text{(4.26)}

On the other hand, by choosing $M > 0$ larger if needed, we can observe from Lemma 4.2 that $L(t)$ is equivalent to $E(t)$. So, $\tilde{L}(t)$ is also equivalent to $E(t)$. Moreover, because the fact that $\zeta'(t) \leq 0$, there exists $\bar{\varepsilon} > 0$, such that

$$\zeta(t)\tilde{L}(t) + 2c_5 E(t) \leq \bar{\varepsilon} E(t), \quad \forall t \geq t_0.$$  \hspace{1cm} \text{(4.28)}

Finally, let

$$L(t) = \varepsilon(\zeta(t)\tilde{L}(t) + 2c_5 E(t)), \quad \text{for } 0 < \varepsilon < \frac{1}{\bar{\varepsilon}},$$

then we observe, from (4.27), (H1), (2.23) and (4.28), that

$$L'(t) = \begin{cases} \varepsilon(\zeta'(t)\tilde{L}(t) + \zeta(t)\tilde{L}'(t) + 2c_5 E'(t)) & \text{if } \zeta'(t) \geq 0 \\ -c_4 \varepsilon\alpha(t)\zeta(t)H_2(E(t)) + c_5 \varepsilon\alpha(t)\zeta(t)(h \circ \nabla u)(t) + 2c_5 \varepsilon E'(t) & \text{if } \zeta'(t) < 0 \end{cases} \leq -c_4 \varepsilon\alpha(t)\zeta(t)H_2(E(t)) - c_5 \varepsilon\alpha(t)(h' \circ \nabla u)(t) + 2c_5 \varepsilon E'(t) \leq -c_4 \varepsilon\alpha(t)\zeta(t)H_2(E(t)) \leq -c_4 \varepsilon\alpha(t)\zeta(t)H_2\left(\frac{1}{\varepsilon} (\zeta(t)\tilde{L}(t) + 2c_5 E(t))\right) \leq -c_4 \varepsilon\alpha(t)\zeta(t)H_2(\varepsilon(\zeta(t)\tilde{L}(t) + 2c_5 E(t))) = -c_4 \varepsilon\alpha(t)\zeta(t)H_2(L(t)).$$  \hspace{1cm} \text{(4.29)}

We have used the fact $H_2$ is increasing in the last two inequalities. Noting that $H_1' = -1/H_2$ (see (2.12)), we infer from (4.29)

$$L'(t)H_1'(L(t)) \geq c_4 \varepsilon\alpha(t)\zeta(t), \quad \forall t \geq t_0.$$
A simple Integration over \((t_0, t)\) then yields

\[
H_1(\mathcal{L}(t)) \geq H_1(\mathcal{L}(t_0)) + c_2 \varepsilon \int_0^t \alpha(t) \zeta(s) \, ds - c_4 \varepsilon \int_0^{t_0} \alpha(t) \zeta(s) \, ds.
\]

Choose \(\varepsilon > 0\) sufficiently small so that \(H_1(\mathcal{L}(t_0)) - c_4 \varepsilon \int_0^{t_0} \alpha(t) \zeta(s) \, ds > 0\), then, thanks to the fact \(H_1^{-1}\) is decreasing, we infer

\[
\mathcal{L}(t) \leq H_1^{-1} \left( H_1(\mathcal{L}(t_0)) - c_4 \varepsilon \int_0^{t_0} \alpha(t) \zeta(s) \, ds + c_4 \varepsilon \int_0^t \alpha(t) \zeta(s) \, ds \right)
\]

\[
\leq H_1^{-1} \left( c_4 \varepsilon \int_0^t \alpha(t) \zeta(s) \, ds \right).
\]

Consequently, the equivalence of \(\mathcal{L}, \bar{L}, L\) and \(E\), yield

\[
E(t) \leq C_0 H_1^{-1} \left( \omega \int_0^t \alpha(t) \zeta(s) \, ds \right).
\]

References


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