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The p - k Mittag-Leffler Function

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Abstract We know that the classical Mittag-Leffler function play an important role as solution of fractional order differential and integral equations. We introduce the p - k Mittag-leffler function and prove some of its properties.

1 Introduction

The two parameter pochhammer symbol is recently introduce by [6], equation 2.1, in the form,

1.1 Definition

Let $x \in C; k, p \in R^{+} - \{0\}$ and $Re(x) > 0, n \in N$, the p - k Pochhammer Symbol (i.e. Two Parameter Pochhammer Symbol), ${}_p(x)_{n,k}$ is given by

$$ {}_p(x)_{n,k} = (\frac{xp}{k})(\frac{xp}{k} + p)(\frac{xp}{k} + 2p)........(\frac{xp}{k} + (n-1)p). \quad (1.1) $$

And the Two Parameter Gamma Function is given by [6], equation 2.6, 2.7 and 2.14,

1.2 Definition

For $x \in C/kZ^{-}; k, p \in R^{+} - \{0\}$ and $Re(x) > 0, n \in N$, the p - k Gamma Function (i.e. Two Parameter Gamma Function), ${}_p\Gamma_k(x)$ as

$$ {}_p\Gamma_k(x) = \frac{1}{k} \lim_{n\to\infty} \frac{n! p^{n+1} (np)^{\frac{x}{k}}}{{}_p(x)_{n+1,k}}. \quad (1.2) $$

or

$$ {}_p\Gamma_k(x) = \frac{1}{k} \lim_{n\to\infty} \frac{n! p^{n+1} (np)^{\frac{x}{k}-1}}{{}_p(x)_{n,k}}. \quad (1.3) $$

The integral representation of p - k Gamma Function is given by

$$ {}_p\Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{p}} t^{x-1} dt. \quad (1.4) $$

The Mittag-Leffler function $E_\alpha(z)$ introduced by Gosta Mittag-Leffler [4] in 1903, defined as

$$ E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}. \quad (1.5) $$

Here $z \in C, \alpha \geqslant 0$.

Wiman [2] generalized $E_\alpha(z)$ in 1905 and gave $E_{\alpha,\beta}(z)$ known as Wiman function, defined as

$$ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}. \quad (1.6) $$

Here $z, \alpha, \beta \in C; Re(\alpha) > 0, Re(\beta) > 0$.

Prabhakar [8] in 1971, gave next generalization of Mittag-Leffler function and denoted as $E^{\gamma}_{\alpha,\beta}(z)$ and defined as

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}. \tag{1.7}$$

Here $z, \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$.

Shukla and Prajapati [1] in 2007, gave second generalization of Mittag-Leffler function and denoted it as $E^{\gamma,q}_{\alpha,\beta}(z)$ and defined as,

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}. \tag{1.8}$$

Here $z, \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0, 1) \cup N$.
The function $E^{\gamma,q}_{\alpha,\beta}(z)$ converges absolutely for all z if $q < Re(\alpha) + 1$ and for $|z| < 1$ if $q = Re(\alpha) + 1$. It is entire function of order $\frac{1}{Re(\alpha)}$.

Gehlot K.S.[5] introduce Generalized k- Mittag-Leffler function in 2012, denoted as $GE^{\gamma,q}_{k,\alpha,\beta}(z)$ and defined for $k \in R; z, \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0, 1) \cup N$, as,

$$GE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}\, z^n}{\Gamma_k(n\alpha + \beta)(n!)}, \tag{1.9}$$

where$(\gamma)_{nq,k}$ is the k- pochhammer symbol and $\Gamma_k(x)$ is the k-gamma function given by [7].
The generalized Pochhammer symbol is given as,

$$(\gamma)_{nq} = \frac{\Gamma(\gamma + nq)}{\Gamma(\gamma)} = q^{qn} \prod_{r=1}^{q} (\frac{\gamma + r - 1}{q})_n, \; if \; q \in N. \tag{1.10}$$

Throughout this paper Let $C, R^+, Re(), Z^- and N$ be the sets of complex numbers, positive real numbers, real part of complex number, negative integer and natural numbers respectively.

2 Main Results

In this section we introduce the p - k Mittag-Leffler function and prove some of its properties and discuss some particular cases.

2.1 Definition

Let $k, p \in R^+ - \{0\}; \alpha, \beta, \gamma \in C/kZ^-; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0, 1) \cup N$.
The p - k Mittag-Leffler function denoted by ${}_pE^{\gamma,q}_{k,\alpha,\beta}(z)$ and defined as

$${}_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha + \beta)} \frac{z^n}{n!}. \tag{2.1}$$

Where ${}_p(\gamma)_{nq,k}$ is two parameter Pochhammer symbol given by equation (1.1) and ${}_p\Gamma_k(x)$ is the two parameter Gamma function given by equation (1.3).

Particular cases : For some particular values of the parameters $p, q, k, \alpha, \beta, \gamma$ we can obtain certain Mittag-Leffler functions, defined earlier:

(a) For $q = 1$ equation (2.1), reduces in generalized form of k- Mittag-Leffler functions defined as.

$${}_pE^{\gamma,1}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{n,k}\, z^n}{{}_p\Gamma_k(n\alpha + \beta)(n!)}. \tag{2.2}$$

(b) For $p = k$ equation (2.1), reduces in Generalized k- Mittag-Leffler functions defined by [5].

$$ {}_kE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_k(\gamma)_{nq,k}\, z^n}{{}_k\Gamma_k(n\alpha+\beta)(n!)} = GE^{\gamma,q}_{k,\alpha,\beta}(z). \tag{2.3} $$

(c) For $p = k, q = 1$ equation (2.1), reduces in k - Mittag-Leffler functions defined by [3].

$$ {}_kE^{\gamma,1}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\, z^n}{\Gamma_k(n\alpha+\beta)(n!)} = E^{\gamma}_{k,\alpha,\beta}(z). \tag{2.4} $$

(d) For $p = k$ and $k = 1$ equation (2.1), reduces in Mittag-Leffler functions defined by [1].

$$ {}_1E^{\gamma,q}_{1,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}\, z^n}{\Gamma(n\alpha+\beta)(n!)} = E^{\gamma,q}_{\alpha,\beta}(z). \tag{2.5} $$

(e) For $p = k, q = 1$ and $k = 1$ equation (2.1), reduces in Mittag-Leffler functions defined by [8].

$$ {}_1E^{\gamma,1}_{1,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n}\, z^n}{\Gamma(n\alpha+\beta)(n!)} = E^{\gamma}_{\alpha,\beta}(z), \tag{2.6} $$

(f) For $p = k, q = 1, k = 1$ and $\gamma = 1$ equation (2.1), reduces in Mittag-Leffler functions defined by [3].

$$ {}_1E^{1,1}_{1,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)} = E_{\alpha,\beta}(z), \tag{2.7} $$

(g) For $p = k, q = 1, k = 1, \gamma = 1$ and $\beta = 1$ equation (2.1), reduces in Mittag-Leffler functions defined by [4].

$$ {}_1E^{1,1}_{1,\alpha,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)} = E_{\alpha}(z). \tag{2.8} $$

Theorem 2.1 Let $k, p \in R^+ - \{0\}; \alpha, \beta, \gamma \in C/kZ^-; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0,1) \cup N$, then the functional relation between p - k Mittag-Leffler function and Mittag-Leffler function defined by equation (1.8) is given by

$$ {}_pE^{\gamma,q}_{k,\alpha,\beta}(z) = kp^{-\frac{\beta}{k}} E^{\frac{\gamma}{k},q}_{\frac{\alpha}{k},\frac{\beta}{k}}(zp^{q-\frac{\alpha}{k}}). \tag{2.9} $$

And its counter part is

$$ E^{\frac{\gamma}{k},q}_{\frac{\alpha}{k},\frac{\beta}{k}}(z) = \frac{p^{\frac{\beta}{k}}}{k}\, {}_pE^{\gamma,q}_{k,\alpha,\beta}(zp^{\frac{\alpha}{k}-q}). \tag{2.10} $$

Proof: Using [6], equation 2.19 and 2.20, we get the desired result.

Theorem 2.2 Let $k, p \in R^+ - \{0\}; \alpha, \beta, \gamma \in C/kZ^-; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0,1) \cup N$, then the following result holds

$$ k\, {}_pE^{\gamma,q}_{k,\alpha,\beta}(z) = p\beta\, {}_pE^{\gamma,q}_{k,\alpha,\beta+k}(z) + zp\alpha \frac{d}{dz}\, {}_pE^{\gamma,q}_{k,\alpha,\beta+k}(z). \tag{2.11} $$

Proof: Starting for the right member of (2.11), we have

$$ A \equiv p\beta\, {}_pE^{\gamma,q}_{k,\alpha,\beta+k}(z) + zp\alpha \frac{d}{dz}\, {}_pE^{\gamma,q}_{k,\alpha,\beta+k}(z), $$

$$ A \equiv p\beta \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha+\beta+k)} \frac{z^n}{n!} + zp\alpha \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha+\beta+k)} \frac{nz^{n-1}}{n!}, $$

$$ A \equiv p \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}(n\alpha+\beta)}{{}_p\Gamma_k\,(n\alpha+\beta+k)} \frac{z^n}{n!}, $$

$$A \equiv k \ {}_pE^{\gamma,q}_{k,\alpha,\beta}(z).$$

Here we use the property of the two parameter Gamma Function namely ${}_p\Gamma_k(x+k) = \frac{xp}{k} {}_p\Gamma_k(x)$, cf.([6], Equation 2.22), we get the desire result.

Theorem 2.3 Let $k, p \in R^+ - \{0\}; \alpha, \beta, \gamma \in C/kZ^-; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0,1) \cup N$, then the following result holds

$$zpq \ {}_p(\gamma)_{q-1,k} \ {}_pE^{\gamma+kq-k,q}_{k,\alpha,\alpha+\beta}(z) = {}_pE^{\gamma,q}_{k,\alpha,\beta}(z) - {}_pE^{\gamma-k,q}_{k,\alpha,\beta}(z). \tag{2.12}$$

Proof: Consider the right hand side of (2.12),

$$A \equiv {}_pE^{\gamma,q}_{k,\alpha,\beta}(z) - {}_pE^{\gamma-k,q}_{k,\alpha,\beta}(z),$$

using the definition (2.1), we have

$$A \equiv \sum_{n=0}^{\infty} \frac{z^n}{{}_p\Gamma_k(n\alpha+\beta)n!}\left({}_p(\gamma)_{nq,k} - {}_p(\gamma-k)_{nq,k}\right).$$

Using [6] equations (2.20), (2.21) and relations $n(x)_{n-1} = (x)_n - (x-1)_n$ and $(\delta)_{n+j} = (\delta)_j(\delta+j)_n$, we obtain the desire result,

$$A \equiv zpq \ {}_p(\gamma)_{q-1,k} \ {}_pE^{\gamma+kq-k,q}_{k,\alpha,\alpha+\beta}(z).$$

Theorem 2.4 Let $k, p \in R^+ - \{0\}; \alpha, \beta, \gamma \in C/kZ^-; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0,1) \cup N$, then the derivative of the p - k Mittag-Leffler Function is given by,

$$\left(\frac{d}{dz}\right)^j \ {}_pE^{\gamma,q}_{k,\alpha,\beta}(z) = {}_p(\gamma)_{jq,k} \times {}_pE^{\gamma+jqk,q}_{k,\alpha,j\alpha+\beta}(z). \tag{2.13}$$

Proof: Consider the Left hand side,

$$A \equiv \left(\frac{d}{dz}\right)^j \ {}_pE^{\gamma,q}_{k,\alpha,\beta}(z),$$

using the definition (2.1) and differentiate j times we have,

$$A \equiv \sum_{n=j}^{\infty} \frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{z^{n-j}}{(n-j)!},$$

using [6], equation (2.33), we obtain desire result.

Theorem 2.5 Let $k, p \in R^+ - \{0\}; \alpha, \beta, \gamma \in C/kZ^-; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0,1) \cup N$. Then

$$\sum_{n=0}^{\infty}(x+y)^n \ {}_pE^{nqk+k,q}_{k,0,(n+1)k}(xy) = \frac{k}{p}\sum_{r=0}^{\infty} \frac{{}_p\Gamma_k(rqk+k)(xy)^r}{r!} \times {}_pE^{rqk+k,q}_{k,qk,k}\left(\frac{x+y}{p}\right). \tag{2.14}$$

Proof: Consider the Left hand side,

$$A \equiv \sum_{n=0}^{\infty}(x+y)^n \ {}_pE^{nqk+k,q}_{k,0,(n+1)k}(xy),$$

using equation (2.1), we have

$$A \equiv \sum_{n=0}^{\infty}(x+y)^n \sum_{r=0}^{\infty} \frac{{}_p(nqk+k)_{rq,k}}{{}_p\Gamma_k(nk+k)} \frac{(xy)^r}{r!} \tag{2.15}$$

now simplifying, by using [6], equation (2.19) and (2.20)

$${}_p(nqk+k)_{rq,k} = p^{rq}(nq+1)_{rq},$$

$$= p^{rq}\frac{\Gamma(nq+rq+1)}{\Gamma(nq+1))},$$

$$= p^{rq}\frac{\Gamma(rq+1+nq)}{\Gamma(rq+1)}\frac{\Gamma(rq+1)}{\Gamma(nq+1)},$$

$$= {}_p\Gamma_k(rqk+k)\frac{{}_p(rqk+k)_{nq,k}}{{}_p\Gamma_k(nqk+k)},$$

then equation (2.15) becomes,

$$A \equiv \sum_{n=0}^{\infty}(x+y)^n\sum_{r=0}^{\infty}\frac{{}_p(rqk+k)_{nq,k}\,{}_p\Gamma_k(rqk+k)}{{}_p\mathit{\Gamma}_k(nk+k){}_p\Gamma_k(nqk+k)}\frac{(xy)^r}{r!},$$

rearranging the terms, we have

$$A \equiv \frac{k}{p}\sum_{r=0}^{\infty}\frac{{}_p\Gamma_k(rqk+k)(xy)^r}{r!}\sum_{n=0}^{\infty}\frac{{}_p(rqk+k)_{nq,k}}{{}_p\Gamma_k(nqk+k)n!}(\frac{x+y}{p})^n.$$

This completes the proof.

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