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Cone associated with frames in Banach Spaces

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Abstract Cone associated with a Banach frame has been defined and studied. A sufficient condition and a necessary condition for a cone associated with a Banach frame to be a generating cone has been given. Also, we prove that a cone associated with an exact Banach frame necessarily has an unbounded base and an extremal subset but it has no weakly compact (compact) base. Finally, we prove that, in a reflexive Banach space, if the cone associated with an exact Banach frame is normal and generating, then the Banach space $\mathcal{X}$ has an unconditional basis.

1 Introduction

Dennis Gabor [19] in 1946 gave a fundamental approach to signal decomposition in terms of elementary signals. Later, in 1952, Duffin and Schaeffer [14] abstracted Gabor's method to define frames for Hilbert spaces. Let $\mathcal{H}$ be a real (or complex) separable Hilbert space with inner product $\langle .,. \rangle$. A countable sequence $\{f_n\} \subset \mathcal{H}$ is called a *frame* (or *Hilbert frame*) for $\mathcal{H}$, if there exist numbers $A,\ B > 0$ such that

$$A\|f\|_{\mathcal{H}}^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \leq B\|f\|_{\mathcal{H}}^2, \text{ for all } f \in \mathcal{H}. \tag{1.1}$$

The scalars A and B are called the *lower* and *upper frame bounds* of the frame, respectively. These bounds are not unique. The inequality in (1.1) is called the *frame inequality* of the frame. For more details related to frames and Riesz bases in Hilbert spaces, one may refer to [6, 10]. These ideas did not generate much interest outside of non-harmonic Fourier series and signal processing for more than three decades until Daubechies, Grossmann and Meyer [12] reintroduced frames. After this land mark paper the theory of frames begin to be studied widely and found new applications to wavelet and Gabor transforms in which frames played an important role. Frames are generalizations of orthonormal bases in Hilbert spaces. The main property of frames which makes them useful is their redundancy. Representation of signals using frames is advantageous over basis expansions in a variety of practical applications in science and engineering. In particular, frames are widely used in sampling theory [2, 15], wavelet theory [13], wireless communication [23, 29], signal processing [8], image processing [27], pseudo-differential operators [22], filter banks [4], geophysics [11], quantum computing [16], wireless sensor network [24], coding theory [30] and many more. The reason for such wide applications is that frames provide both great liberties in design of vector space.

Banach frames were developed for the theory of frames in the context of Gabor and Wavelet analysis. They were introduced by Gröchenig [20] as an extension of frames for Hilbert spaces and were further studied in [6, 7, 9, 10, 17]. Banach frames are used in applied mathematics that provides applications to signal and image processing, sampling theory, etc. The sampling theory in [1] amounts to the construction of Banach frames consisting of reproducing kernels for a large class of shift invariant spaces. Aldroubi et al. [2] used Banach frames in various irregular sampling problems. Gröchenig [21] emphasised that localization of a frame is a necessary condition for its extension to a Banach frame for the associated Banach spaces. He also observed

that localized frames are universal Banach frames for the associated family of Banach spaces. Fornasier [18] studied Banach frames for α-modulation spaces. In fact, he gave a Banach frame characterization for the α-modulation spaces. Carando et al. [5] relate Banach frames to various properties of Banach spaces such as separability and reflexivity. They also observed that a Banach frame for a Banach space with respect to a solid space admits a reconstruction formula whenever the Banach space does not contain a copy of c_0.
The notion of cone in Banach spaces or normed linear spaces had been studied by many authors in various contexts [26, 3]. In the present paper, we relate Banach frames to another geometric notion called cone and associated it with Banach frames and obtain interesting and new results in the context of Banach frames. In fact, we obtain a sufficient condition and a necessary condition for a cone associated with a Banach frame to be a generating cone. Also, we prove that a cone associated with an exact Banach frame necessarily has an unbounded base and an extremal subset but it has no weakly compact (compact) base. Finally, we prove that, in a reflexive Banach space, if the cone associated with an exact Banach frame is normal and generating, then the Banach space $\mathcal{X}$ has an unconditional basis.

2 Preliminaries

Throughout this paper $\mathcal{X}$ will denotes an infinite dimensional real Banach space, $\mathcal{X}^*$ denotes the conjugate space of $\mathcal{X}$. For a sequence $\{x_n\} \in \mathcal{X}$ and $\{f_n\} \in \mathcal{X}^*$, $[x_n]$ denotes the closure of linear span of $\{x_n\}$ in the norm topology of $\mathcal{X}$ and $\widetilde{[f_n]}$ the closure of $\{f_n\}$ in the $weak^*-$topology of $\mathcal{X}^*$. A sequence space S is called a *BK-space* if it is a Banach space and the co-ordinate functionals are continuous on S i.e. the relations $x_n = \{{\alpha_j}^{(n)}\}$, $x = \{\alpha_j\} \in S$ and $\lim\limits_{n\longrightarrow\infty} x_n = x$ imply $\lim\limits_{n\longrightarrow\infty} \alpha_j^{(n)} = \alpha_j \quad (j = 1, 2, 3, ...)$. The notion of Banach frames was introduced and studied by Gröcheing [20]. He gave the following definition:

Definition 2.1. [20] Let $\mathcal{X}$ be a Banach space over $\mathbb{K} \ \ (\mathbb{R}, \mathbb{C})$and $\mathcal{X}_d$ be an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$. Let $\{f_n\} \subset \mathcal{X}^*$ and $S : \mathcal{X}_d \longrightarrow \mathcal{X}$ be given. Then $\Phi = (\{f_n\}, S)$ is called a $Banach \ \ frame$ for $\mathcal{X}$ with respect to $\mathcal{X}_d$ if the following statements holds:

(i) $\{f_n(x)\} \in \mathcal{X}_d$, for each $x \in \mathcal{X}$.

(ii) There exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_{\mathcal{X}} \leq \|\{f_n(x)\}\|_{\mathcal{X}_d} \leq B\|x\|_{\mathcal{X}}, \quad x \in \mathcal{X}. \tag{2.1}$$

(iii) S is a bounded linear operator such that

$$S(\{f_n(x)\}) = x, \quad x \in \mathcal{X}.$$

The positive constants A and B, respectively, are called lower and upper frame bounds of the Banach frame $\Phi = (\{f_n\}, S)$. The operator $S : \mathcal{X}_d \longrightarrow \mathcal{X}$ is called the reconstruction operator (or, the pre-frame operator). The inequality (2.1) is called the frame inequality. The Banach frame $\Phi = (\{f_n\}, S)$ is called tight if $A = B$ and is called normalized tight if $A = B = 1$.

A Banach frame $\Phi = (\{f_n\}, S)$ is called exact if there exists a sequence $\{x_n\} \in \mathcal{X}$ such that $f_i(x_j) = \delta_{i,j}, \quad$ for all $\ i, j \in \mathbb{N}$. The sequence $\{x_n\}$ is called an admissible sequence to the Banach frame Φ. Next, we give the following result in the form of a lemma which will be used throughout the paper.

Lemma 2.2. *[25] . Let $\mathcal{X}$ be a Banach space and $\{f_n\} \subset \mathcal{X}^*$ be a sequence such that $\{x \in \mathcal{X} : f_n(x) = 0, \ \textit{for all} \ \ n \in \mathbb{N}\} = \{0\}$. Then $\mathcal{X}$ is linearly isometric to the Banach space $\mathcal{X}_d = \{\{f_n(x)\} : x \in \mathcal{X}\}$, where the norm is given by $\|\{f_n(x)\}\|_{\mathcal{X}_d} = \|x\|_{\mathcal{X}}, \ \ x \in \mathcal{X}$.*

3 Main Result

Let $\mathcal{X}$ be a real Banach space and let $\Phi = (\{f_n\}, S) \quad (\{f_n\} \subset \mathcal{X}^*, S : \mathcal{X}_d \longrightarrow \mathcal{X})$ be a Banach frame for $\mathcal{X}$. Define $\mathcal{C}_\Phi = \{x \in \mathcal{X} : f_n(x) \geq 0, \quad \text{for all} \quad n \in \mathbb{N}\}$. Then $\mathcal{C}_\Phi$ is a $cone \quad associated \quad with \quad the \quad Banach \quad frame$ Φ and satisfies the following properties:

(i) $\mathcal{C}_\Phi$ is a closed set satisfying

$$\mathcal{C}_\Phi + \mathcal{C}_\Phi \subset \mathcal{C}_\Phi \quad and \quad \lambda\mathcal{C}_\Phi \subset \mathcal{C}_\Phi \quad (\lambda \geq 0)$$

(ii) $\mathcal{C}_\Phi \cap (-\mathcal{C}_\Phi) = \{0\}$.

Definition 3.1. The cone $\mathcal{C}_\Phi$ associated with a Banach frame Φ is called

(a) $generating$ if

$$\mathcal{X} = \{y - z : y, z \in \mathcal{C}_\Phi\}$$

(b) $normal$ if there exists a constant $L > 0$ such that

$$0 \leq x \leq y \Rightarrow \|x\| \leq L\|y\|; \quad x, y \quad \in \mathcal{X}.$$

Recall that the cone $\mathcal{C}_\Phi$ induces a natural partial order relation on $\mathcal{X}$ namely $x \geq y$ if and only if $x - y \in \mathcal{C}_\Phi$.
A subset $\mathcal{B}$ of $\mathcal{C}_\Phi$ is called a $base$ of $\mathcal{C}_\Phi$ if it is closed and convex and if for every $x \in \mathcal{C}_\Phi \setminus \{0\}$ has a unique representation of the form $x = \lambda y, \;\; \lambda > 0, \;\; y \in \mathcal{B}$.
A set $\mathcal{E}$ contained in $\mathcal{C}_\Phi$ is called an $extremal \;\; subset$ of $\mathcal{C}_\Phi$ if $x, \; y \in \mathcal{C}_\Phi$ with $\lambda x + (1-\lambda)y \in \mathcal{E}$ and $(0 \leq \lambda \leq 1)$ imply $x, \; y \in \mathcal{E}$.

In the following examples, we show the existence of normal and generating cones.

Example 3.2. Let $\mathcal{X} = c_0$. Define $\{x_n\} \subset \mathcal{X}$ and $\{f_n\} \subset \mathcal{X}^*$ by

$$x_n = (\underbrace{1, 1, ...1}_{n-terms}, 0, 0, ...), \quad n \in \mathbb{N},$$

$$f_n(x) = \eta_n - \eta_{n+1}, \quad \text{for all} \quad x = \{\eta_n\} \in \mathcal{X}.$$

Then, by Lemma 2.2, there exists an associated Banach space $\mathcal{X}_d = \{f_n(x) : x \in \mathcal{X}\}$ with norm given by $\|\{f_n(x)\}\|_{\mathcal{X}_d} = \|x\|_{\mathcal{X}}, \quad x \in \mathcal{X}$ and a bounded linear operator $S : \mathcal{X}_d \longrightarrow \mathcal{X}$ such that $\Phi = (\{f_n\}, S)$ is a Banach frame for $\mathcal{X}$. Define

$$\mathcal{C}_\Phi = \{x = \{\eta_n\} \in \mathcal{X} : \eta_1 \geq \eta_2 \geq \eta_3 ...\}.$$

Then $\mathcal{C}_\Phi$ is a cone associated with the Banach frame Φ. Also, note that if $x = \{\eta_n\}$ and $y = \{\xi_n\}$ are any two elements in $\mathcal{X}$ such that $0 \leq x \leq y$. Then

$$\|x\| = \sup_{1 \leq n < \infty} |\eta_n| \leq \sup_{1 \leq n < \infty} |\xi_n| = \|y\|.$$

Thus $\mathcal{C}_\Phi$ is a normal cone.

Example 3.3. Let $\mathcal{X} = c_0$ and let $\{e_n\}$ be the sequence of standard unit vectors in $\mathcal{X}$. Define $\{x_n\} \subset \mathcal{X}$ and $\{f_n\} \subset \mathcal{X}^*$ by

$$x_n = \sum_{i=1}^{n} (-1)^{n+i} e_i, n \in \mathbb{N}$$

and

$$f_n(x) = \eta_n + \eta_{n+1}, \quad \text{for all} \;\; n \in \mathbb{N} \quad \text{and} \quad x = \{\eta_n\} \in \mathcal{X}.$$

Then, by Lemma 2.2, there exists an associated Banach space $\mathcal{X}_d = \{f_n(x) : x \in \mathcal{X}\}$ and a bounded linear operator $S : \mathcal{X}_d \longrightarrow \mathcal{X}$ such that $\Phi = (\{f_n\}, S)$ is a Banach frame for $\mathcal{X}$. Define

$$\mathcal{C}_\Phi = \{x = \{\eta_n\} \in c_0 : \eta_1 + \eta_2 \geq 0, \quad \eta_2 + \eta_3 \geq 0, ...\}.$$

Then $\mathcal{C}_\Phi$ is a cone associated with the Banach frame Φ. Also, let $x = \{\eta_n\} \in c_0$ be any arbitrary element. Write $\eta_n = \phi_n - \psi_n, \quad n \in \mathbb{N}$, where $\phi_n \geq 0$ and $\psi_n \geq 0$. Then $y = \{\phi_n\} \in \mathcal{C}_\Phi$ and $z = \{\psi_n\} \in \mathcal{C}_\Phi$ are such that every $x \in \mathcal{X}$ can be expressed as $x = y - z$. Hence $\mathcal{C}_\Phi$ is a generating cone.

The following result gives a sufficient condition for a cone associated with a Banach frame to be a generating cone.

Theorem 3.4. *Let $\Phi = (\{f_n\}, S)$ be a Banach frame for $\mathcal{X}$ with associated cone $\mathcal{C}_\Phi$. If for every $x \in \mathcal{X}$, there exists an element $z \in \mathcal{X}$ such that*

$$f_n(z) = |f_n(x)|, \quad \textit{for all} \quad n \in \mathbb{N}, \tag{3.1}$$

then the cone $\mathcal{C}_\Phi$ is generating.

Proof. Let $x \in \mathcal{X}$ be any element and let $z \in \mathcal{X}$ satisfies (3.1). Write $c_1 = \frac{x+z}{2}$ and $c_2 = \frac{-x+z}{2}$. Then $c_1 - c_2 = x$. Thus, in order to show that $\mathcal{C}_\Phi$ is generating , we need to show that $f_n(c_i) \geq 0, \quad$ for all $n \in \mathbb{N}$ and $i = 1, 2$. Note that, for all $n \in \mathbb{N}$, $f_n(c_1) = \frac{1}{2}(f_n(x) + |f_n(x)|) \geq 0$ and $f_n(c_2) = \frac{1}{2}(-f_n(x) + |f_n(x)|) \geq 0$. Hence $\mathcal{C}_\Phi$ is generating. □

Remark 3.5. The converse of Theorem 3.4 is not true.(see the following example)

Example 3.6. Let $\mathcal{C}_\Phi$ be the cone associated with the Banach frame Φ as given in Example 3.3. Note that Φ is a generating cone. We claim that the condition (3.1) in Theorem 3.4 is not satisfied. Suppose on the contrary that for every $x = \{\eta_n\} \in \mathcal{X}$ there exists an element $z = \{\xi_n\} \in \mathcal{X}$ such that $f_n(z) = |f_n(x)|, \quad$ for all $\quad n \in \mathbb{N}$. Then

$$\begin{aligned} \sum_{i=1}^{\infty}(-1)^i|\eta_i + \eta_{i+1}| &= \sum_{i=1}^{\infty}(-1)^i|f_i(x)| \\ &= \sum_{i=1}^{\infty}(-1)^i(\xi_i + \xi_{i+1}) \\ &= -\xi_1. \end{aligned}$$

Thus, for every $x = \{\eta_n\} \in \mathcal{X}$, the infinite series $\sum_{i=1}^{\infty}(-1)^i|\eta_i + \eta_{i+1}|$ is convergent. But if we choose

$$\eta_1 = 0 \quad \text{and} \quad \eta_{2n} = -\eta_{2n+1} = \sum_{i=n+1}^{\infty} \frac{(-1)^i}{i}, \quad n \in \mathbb{N},$$

then $x = \{\eta_n\} \in \mathcal{X}$ is such that

$$|\eta_{2n} + \eta_{2n+1}| = 0 \quad \text{and} \quad |\eta_{2n+1} + \eta_{2n+2}| = \frac{1}{n+1}.$$

Hence $\sum_{i=1}^{\infty}(-1)^i|\eta_i + \eta_{i+1}| \to -\infty$, a contradiction.

Next, we give a necessary condition for a generating cone associated with an exact Banach frame Φ satisfying certain conditions.

Theorem 3.7. *Let $\mathcal{C}_\Phi$ be the cone associated with an exact Banach frame $\Phi = (\{f_n\}, S)$ with admissible sequence $\{x_n\} \subset \mathcal{X}$. If $\mathcal{C}_\Phi$ is a generating cone and if for every $x \in \mathcal{C}_\Phi$, the set $\mathcal{C}_\Phi \cap (x - \mathcal{C}_\Phi)$ is bounded (norm), then for every $x \in \mathcal{X}$,*

$$\sup_{1 \leq n < \infty} \| \sum_{i=1}^{n} f_i(x)x_i \| < \infty.$$

Proof. Note that for $n \in \mathbb{N}$ and $x \in \mathcal{C}_\Phi$, we have

$$f_j(\sum_{k=1}^{n} f_k(x)x_k) = \begin{cases} f_j(x), & j = 1,2,3,...n \\ 0, & j > n. \end{cases}$$

This gives

$$0 \leq \sum_{i=1}^{n} f_i(x)x_i \leq x, \quad \text{for all} \quad n \in \mathbb{N}, \ x \in \mathcal{C}_\Phi.$$

Since $\mathcal{C}_\Phi \cap (x - \mathcal{C}_\Phi)$ is norm bounded, we have

$$0 \leq \|\sum_{i=1}^{n} f_i(x)x_i\| < \infty, \quad \text{for every} \ x \ \in \mathcal{C}_\Phi.$$

Let $x \in \mathcal{X}$ be any arbitrary element. Since $\mathcal{C}_\Phi$ is a generating cone, $x = y - z$, where $y, z \in \mathcal{C}_\Phi$. Hence, for every $x \in \mathcal{X}$,

$$\begin{aligned} \sup_{1\leq n<\infty} \|\sum_{i=1}^{n} f_i(x)x_i\| &\leq \sup_{1\leq n<\infty} \|\sum_{i=1}^{n} f_i(y)x_i\| + \sup_{1\leq n<\infty} \|\sum_{i=1}^{n} f_i(z)x_i\| \\ &< \infty. \end{aligned}$$

□

Next, we prove that a cone associated with an exact Banach frame necessarily has an unbounded base and an extremal subset but it has no weakly compact base.

Theorem 3.8. *Let $\mathcal{X}$ be an infinite dimensional real Banach space, $\Phi = (\{f_n\}, S)$ be an exact Banach frame for $\mathcal{X}$ with admissible sequence $\{x_n\} \subset \mathcal{X}$ and let $\mathcal{C}_\Phi$ be the cone associated with Φ. Then*

(a) $\mathcal{C}_\Phi$ has an unbounded base.

(b) $\mathcal{C}_\Phi$ has no weakly compact (compact) base.

(c) For each $j \in \mathbb{N}$, the set $S_j = \{\lambda x_j : 0 \leq \lambda < \infty\}$ is an extremal subset of $\mathcal{C}_\Phi$.

Proof. (a) Define $f \in \mathcal{X}^*$ by

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{2^k \|f_k\|} f_k(x), \quad x \in \mathcal{X}$$

and $\mathcal{A} = \{a \in \mathcal{C}_\Phi : f(a) = 1\}$. Clearly $\mathcal{A}$ is a closed and convex set. We claim that $\mathcal{A}$ is a base for $\mathcal{C}_\Phi$. Let $0 \neq x \in \mathcal{C}_\Phi$. Take $y = \frac{x}{f(x)}$. Then $f(y) = 1$. So $y \in \mathcal{A}$. Now $x = \lambda y$, where $\lambda = f(x) > 0$. If $x = \lambda_1 y_1 = \lambda_2 y_2$, where $\lambda_1, \ \lambda_2 > 0$ and $y_1, \ y_2 \in \mathcal{A}$, then $f(x) = \lambda_1 = \lambda_2$. Therefore the representation $x = \lambda y$ is unique. Hence $\mathcal{A}$ is a base for $\mathcal{C}_\Phi$. Write $a_n = 2^n \|f_n\| x_n, \quad n \in \mathbb{N}$. Since $f_i(x_j) = \delta_{i,j}$, for all $i, j \in \mathbb{N}$, $\|a_n\| > 2^n$, for all n $\in \mathbb{N}$. Hence the base $\mathcal{A}$ is unbounded.

(b) Suppose that $\mathcal{C}_\Phi$ has a weakly compact base say $\mathcal{B}$. Let $\{x_n\}$ be a sequence in $\mathcal{C}_\Phi$ such that $x_n \neq 0, \quad n \in \mathbb{N}$. Then for each n, there exist $\lambda_n > 0$ and $y_n \in \mathcal{B}$ such that $x_n = \lambda_n y_n, \ n \in \mathbb{N}$. Also the representation of each x_n is unique. By assumption, $\mathcal{B}$ is weakly sequentially compact and so the sequence $\{y_n\}$ has a subsequence $\{y_{n_k}\}$ that converges weakly to an element say $y_0 \in \mathcal{B}$. Note that $f_l(y_{n_k}) = 0$, for $n_k > l$; $l, k \ \in \mathbb{N}$. So $\lim_{k\to\infty} f_l(y_{n_k}) = 0$. This gives $f_l(y_0) = 0$, for all $l \in \mathbb{N}$. Therefore, by the lower frame inequality for the Banach frame $(\{f_n\}, S)$, $y_0 = 0$. This is a contradiction as $\mathcal{B}$ is a base to the cone $\mathcal{C}_\Phi$. Hence, we conclude that the cone $\mathcal{C}_\Phi$ has no weakly compact (or compact) base.

(c) Suppose that $a,\ b \in \mathcal{C}_\Phi$ be any two elements such that $\alpha_0 a + (1-\alpha_0)b \in S_j \quad (0 < \alpha_0 < 1)$. Then $\alpha_0 a + (1-\alpha_0)b = \lambda_0 x_j \ \ (\lambda_0 \geq 0)$. Therefore

$$\alpha_0 f_i(a) + (1-\alpha_0) f_i(b) = \begin{cases} \lambda_0, & \text{if} \quad i = j \\ 0, & \text{if} \quad i \neq j. \end{cases}$$

Since $a, b \in \mathcal{C}_\Phi$, for each $i \in \mathbb{N}$, $f_i(a) \geq 0$ and $f_i(b) \geq 0$. Therefore $f_i(a) = 0 = f_i(b)$, for all $i \neq j$. This gives $f_i(f_j(a)x_j - a) = 0$, for all $i \in \mathbb{N}$. Hence by the Banach frame inequality for the Banach frame $(\{f_n\}, S)$, $a = f_j(a)x_j \in S_j$. Similarly, $b \in S_j$. Hence S_j is an extremal subset of $\mathcal{C}_\Phi$.

□

In the following result, we show that if the cone associated with an exact Banach frame is both normal and generating then the admissible sequence to the exact Banach frame become an unconditional basis of $\mathcal{X}$ provided $\mathcal{X}$ is reflexive.

Theorem 3.9. *Let $\mathcal{X}$ be a reflexive Banach space and let $\Phi = (\{f_n\}, S)$ be an exact Banach frame with admissible sequence $\{x_n\} \in \mathcal{X}$. If the cone $\mathcal{C}_\Phi$ associated with Φ is normal and generating, then $\{x_n\}$ is an unconditional basis of $\mathcal{X}$.*

Proof. Let $x \in \mathcal{C}_\Phi$ be any element. Define

$$S_n(x) = \sum_{i=1}^{n} f_i(x) x_i, \quad n \in \mathbb{N}.$$

Then, for each $n \in \mathbb{N}$

$$f_i(S_n(x)) = \begin{cases} f_i(x), & i = 1, 2, ..., n \\ 0, & \text{for} \quad i = n+1, n+2, ... \end{cases}$$

Thus, for each $i \in \mathbb{N}$, $\{f_i(S_n(x))\}$ is a bounded above and monotonically increasing sequence in $\mathbb{R}$. Therefore for each $i \in \mathbb{N}$, $\lim_{n\to\infty} f_i(S_n(x))$ exists. Since $\mathcal{X}$ is reflexive, $\lim_{n\to\infty} f(S_n(x))$ exists for all $f \in \mathcal{X}^*$. Also, we have $0 \leq S_n(x) \leq x, \quad n \in \mathbb{N}$. Since $\mathcal{C}_\Phi$ is normal, there exists a constant L such that

$$\|S_n(x)\| \leq L\|x\|, \quad \text{for all n} \in \mathbb{N}.$$

Thus $\{S_n(x)\}$ is a weak Cauchy sequence in $\mathcal{X}$. Now $\mathcal{X}$ being reflexive is weakly complete. Therefore $S_n(x)$ converges weakly to some elements say $s \in \mathcal{X}$. Thus, for each $i \in \mathbb{N}$, we have

$$f_i(s) = f_i(x), \quad \text{for all} \quad i \in \mathbb{N}.$$

Then, by the lower frame inequality for the Banach frame $\Phi = (\{f_n\}, S)$, $\quad s = x$. Thus, for every $x \in \mathcal{C}_\Phi$, $\quad \{S_n(x)\}$ converges weakly to x. Now, let $x \in \mathcal{X}$ be any arbitrary element. Since $\mathcal{C}_\Phi$ is generating, we may write $x = x_1 - x_2$, where $x_1,\ x_2 \in \mathcal{C}_\Phi$. Then

$$S_n(x) = S_n(x_1) - S_n(x_2), \quad \text{for all } n \in \mathbb{N}.$$

As $x_1,\ x_2 \in \mathcal{C}_\Phi$, $S_n(x_1)$ and $S_n(x_2)$ converges weakly to x_1 and x_2 respectively. So $S_n(x)$ converges weakly to x. Consequently $x \in [x_n]$. Therefore $\{x_n\}$ is a basis of $\mathcal{X}$. Let Π denote the set of all permutation on $\mathbb{N}$ and $\sigma \in \Pi$ be any arbitrary elements of Π. Then, by Lemma 2.2, there exist a Banach space $\mathcal{X}_{\sigma(d)} = \{\{f_{\sigma(n)}(x)\} : x \in \mathcal{X}\}$ with norm $\|\{f_{\sigma(n)}(x)\}\|_{\mathcal{X}_{\sigma(d)}} = \|x\|_{\mathcal{X}}, \quad x \in \mathcal{X}$ and a bounded linear operator $S_\sigma : \mathcal{X}_{\sigma(d)} \longrightarrow \mathcal{X}$ such that $(\{f_{\sigma(n)}\}, S_\sigma)$ is an exact Banach frame for $\mathcal{X}$ with admissible sequence $\{x_{\sigma(n)}\}$. Thus $\{x_{\sigma(n)}\}$ is a basis of $\mathcal{X}$. Hence $\{x_n\}$ is an unconditional basis of $\mathcal{X}$. □

Finally, towards the converse of the Theorem 3.9, we prove the following result.

Theorem 3.10. *Let $\mathcal{X}$ be a real Banach space and let $\Phi = (\{f_n\}, S)$ be an exact Banach frame with admissible sequence $\{x_n\} \in \mathcal{X}$. Then $\{x_n\}$ is an unconditional basis of $\mathcal{X}$ with associated sequence of coefficient functional $\{f_n\}$ if and only if the cone $\mathcal{C}_\Phi$ associated with Φ is normal, generating and*

$$0 \leq \eta_1 \leq \eta_2 \leq ..., \leq \eta \qquad \Rightarrow \{\eta_n\} \quad \text{is norm convergent.} \tag{3.2}$$

Proof. Suppose first that $\mathcal{C}_\Phi$ is generating and the relation (3.2) holds. Let $x \in \mathcal{C}_\Phi$ be any arbitrary element. Define

$$S_n(x) = \sum_{i=1}^{n} f_i(x)x_i, \quad n \in \mathbb{N}.$$

Then, for each $i \in \mathbb{N}$, we have

$$0 \leq S_n(x) \leq S_{n+1}(x) \leq ... \leq x.$$

Therefore, by relation (3.2), $\lim_{n\to\infty} S_n(x)$ exists. Let $\lim_{n\to\infty} S_n(x) = x_0 \in \mathcal{X}$. Then, for each $i \in \mathbb{N}$, we have

$$f_i(x) = f_i(x_0), \quad \text{for all} \quad i \in \mathbb{N}.$$

Then, by the lower frame inequality for the Banach frame Φ, $x = x_0$. So $\lim_{n\to\infty} S_n(x) = x$. Now let $x \in \mathcal{X}$ be arbitrary, since $\mathcal{C}_\Phi$ is generating $x = x_1 - x_2$, where $x_1, \ x_2 \in \mathcal{C}_\Phi$. Then, for each $x \in \mathcal{X}$, we have

$$\begin{aligned} \lim_{n\to\infty} S_n(x) &= \lim_{n\to\infty} S_n(x_1) - \lim_{n\to\infty} S_n(x_2) \\ &= x_1 - x_2 \\ &= x. \end{aligned}$$

Thus $\{x_n\}$ is a basis of $\mathcal{X}$. Further, it is easy to verify that $\{x_n\}$ is an unconditional basis of $\mathcal{X}$. Conversely, let $x, \ y \in \mathcal{X}$ be such that $0 \leq x \leq y$. Then for each $i \in \mathbb{N}$, $\ 0 \leq f_i(x) \leq f_i(y)$. Therefore, for each $i \in \mathbb{N}$, there exists a real no $\lambda_i \ (0 \leq \lambda_i \leq 1)$ such that

$$f_i(x) = \lambda_i f_i(y), \quad i \in \mathbb{N}. \tag{3.3}$$

Since $\{x_n\}$ is an unconditional basis of $\mathcal{X}$ with associated sequence of coefficient functional $\{f_n\}$, we have

$$x = \lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)x_i$$

Therefore using (3.3) one can find a constant $L > 0$ such that $\|x\| \leq L\|y\|$. Thus $\mathcal{C}_\Phi$ is normal. Also, for every $x \in \mathcal{X}$, there exists a $z \in \mathcal{X}$ such that $f_n(z) = |f_n(x)|, \quad n \in \mathbb{N}$. Then, by Theorem 3.4, $\mathcal{C}_\Phi$ is generating.
Now, let $0 \leq \eta_1 \leq \eta_2 \leq ... \leq \eta$. Then for each $i \in \mathbb{N}$, we have $0 \leq f_i(\eta_n) \leq f_i(\eta_{n+1}) \leq ... \leq f_i(\eta), n \in \mathbb{N}$. Clearly, $\lim_{n\to\infty} f_i(\eta_n)$ exist for each $i \in \mathbb{N}$. Let $\lim_{n\to\infty} f_i(\eta_n) = \alpha_i \ \ (i \in \mathbb{N})$. Then $0 \leq \alpha_i \leq f_i(\eta), \quad i \in \mathbb{N}$. So

$$0 \leq \sum_{i=n+1}^{n+m} \alpha_i x_i \leq \sum_{i=n+1}^{n+m} f_i(\eta)x_i, \quad \text{for all} \quad n, \ \ m \in \mathbb{N}.$$

Since the cone $\mathcal{C}_\Phi$ is normal, there exists a constant $M > 0$ such that

$$\|\sum_{i=n+1}^{n+m} \alpha_i x_i\| \leq M\|\sum_{i=n+1}^{n+m} f_i(\eta)x_i\|, \quad \text{for all} \quad n, \ \ m \in \mathbb{N}.$$

Thus $\lim_{n\to\infty}\sum_{i=1}^{n}\alpha_i x_i$ exists $(= a \in \mathcal{X})$.
Finally, we prove that $\lim_{n\to\infty}\|\eta_n - a\| = 0$. Let $\epsilon > 0$ be given. Choose N such that

$$\|\sum_{i=N+1}^{\infty} f_i(\eta)x_i\| \leq \frac{\epsilon}{3M}.$$

Since $\mathcal{C}_\Phi$ is normal, we have

$$\begin{aligned}\max\{\|\sum_{i=N+1}^{\infty} f_i(a)x_i\|, \|\sum_{i=N+1}^{\infty} f_i(\eta_n)x_i\|\} &\leq M\|\sum_{i=N+1}^{\infty} f_i(\eta)x_i\| \\ &< \frac{\epsilon}{3}, \quad \text{for all } \ n \in \mathbb{N}.\end{aligned}$$

Also, for each $i \in \mathbb{N}$

$$\begin{aligned} f_i(a) &= f_i(\lim_{n\to\infty}\sum_{i=1}^{n}\alpha_i x_i) \\ &= \alpha_i \\ &= \lim_{n\to\infty} f_i(\eta_n). \end{aligned}$$

Thus, for each $\epsilon > 0$, there exists a positive integer K such that

$$\|\sum_{i=1}^{N}(f_i(a) - f_i(\eta_n))x_i\| < \frac{\epsilon}{3}, \ \text{for all } \ n \geq K.$$

Therefore

$$\begin{aligned} \|\eta_n - a\| &= \|\sum_{i=1}^{\infty}(f_i(a) - f_i(\eta_n))x_i\| \\ &\leq \|\sum_{i=1}^{N}[f_i(a) - f_i(\eta_n)]x_i\| + \|\sum_{i=N+1}^{\infty} f_i(a)x_i\| + \|\sum_{i=N+1}^{\infty} f_i(\eta_n)x_i\| \\ &< \epsilon, \quad \text{for all } \ n \geq K. \end{aligned}$$

Hence $\lim_{n\to\infty}\|\eta_n - a\| = 0$. □

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