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The k-Distance degree index of a Graph

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Abstract In this paper, we introduce a new distance-based topological index of a graph G, called a k-distance degree index. It is defined as $N_k(G) = \sum_{k=1}^{diam(G)} (\sum_{v \in V(G)} d_k(v))k$, where $d_k(v) = |N_k(v)| = |\{u \in V(G) : d(v,u) = k\}|$ is the k-distance degree of a vertex v in G, $d(u,v)$ is the distance between vertices u and v in G and $diam(G)$ is the diameter of G. Exact formulas of the N_k-index for some well-known graphs are presented. Bounds for N_k-index and some other interesting results are established. It is shown that, N_k-index of any graph G is an even integer number. In addition, an explicit formulae of a cartesian product of graphs are presented and we apply this result to compute the N_k-index of some graphs (of chemical and computer science interest) like hypercube Q_d, Hamming graphs $H(d,n)$, nanotube $R = P_n \Box C_m$ and nanotori $S = C_n \Box C_m$, etc.

1 Introduction

Throughout this paper, we consider only simple connected graphs, i.e., finite and connected graph without loops, multiple and directed edges. A graph $G = (V, E)$ is said to be connected if there is a path between every pair of its vertices. As usual, we denote by $n = |V|$ and $m = |E|$ to the number of vertices and edges in a graph G, respectively. The distance $d(u,v)$ between any two vertices u and v of G is equal to the length (number of edges in) a shortest path connecting them. For a vertex $v \in V$ and a positive integer k, the open k-neighborhood of v in a graph G, denoted by $N_k(v/G)$ or simply $N_k(v)$, is defined as, $N_k(v/G) = \{u \in V(G) : d(u,v) = k\}$ and the closed k-neighborhood of v is $N_k[v/G] = N_k(v/G) \cup \{v\}$. The k-degree of a vertex v in G, denoted $d_k(v/G)$ (or simply $d_k(v)$ if no misunderstanding), is defined as $d_k(v/G) = |N_k(v/G)|$. It is clearly that $d_1(v/G) = d(v/G)$ for every $v \in V$. A vertex of degree equals to zero in G is called an isolated vertex and a vertex of degree one is called a pendant vertex. The graph with no vertices (and hence no edges) is the null graph. Any graph with just one vertex is referred to as trivial graph and denoted K_1. The complement $\overline{G}$ of a graph G is a graph with vertex set $V(G)$ and two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. A totally disconnected graph $\overline{K_n}$ is one in which no two vertices are adjacent (that is, one whose edge set is empty). If a graph G consists of $p \geq 2$ disjoint copies of a graph H, then we write $G = pH$. For a vertex v of G, the eccentricity $e(v) = \max\{d(v,u) : u \in V(G)\}$. The radius of G is $rad(G) = \min\{e(v) : v \in V(G)\}$ and the diameter of G is $diam(G) = \max\{e(v) : v \in V(G)\}$.

A topological index of a graph G is a numerical parameter mathematically derived from the graph structure. It is a graph invariant thus it does not depend on the labeling or pictorial representation of the graph and it is the graph invariant number calculated from a graph representing a molecule. The topological indices of molecular graphs are widely used for establishing correlations between the structure of a molecular compound and its physic-chemical properties or biological activity. The topological indices which are definable by a distance function $d(.,.)$ are called a distance-based topological index. All distance-based topological indices can be derived from the distance matrix or some closely related distance-based matrix, for more information on this matter see [2] and a survey paper [18] and the references therein.

There are many examples of such indices, especially those based on distances, which are applicable in chemistry and computer science. The Wiener index (1947), defined as

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v)$$

is the first and most studied of the distance based topological indices [17]. The hyper-Wiener index,

$$WW(G) = \frac{1}{2} \sum_{\{u,v\} \subseteq V} (d(u,v) + d^2(u,v))$$

was introduced in (1993) by M. Randic [13]. The Harrary index

$$H(G) = \sum_{\{u,v\} \subseteq V} \frac{1}{d^2(u,v)}$$

was introduced in (1992) by Mihalic et al. [10]. In spite of this, the Harary index is nowadays defined as [8, 11]

$$H(G) = \sum_{\{u,v\} \subseteq V} \frac{1}{d(u,v)}$$

The Schultz index

$$S(G) = \sum_{\{u,v\} \subseteq V} (d(u) + d(v))d(u,v)$$

was introduced in (1989) by H. P. Schultz [14], A. Dobrynin et al. in (1994) also proposed the Schultz index and called it the degree distance index and denoted $DD(G)$ [1]. S. Klavzar and I Gutman, motivated by Schultz index, introduced in (1997) the second kind of Schultz index

$$S^*(G) = \sum_{\{u,v\} \subseteq V} d(u)d(v)d(u,v)$$

called modified Schultz (or Gutman) index of G [9]. The eccentric connectivity index

$$\xi^c = \sum_{v \in V} d(v)e(v)$$

was proposed by Sharma et al. [15]. For more details and examples of distance-based topological indices, we refer the reader to [2, 18, 12, 6] and the references therein.

For any terminology or notation not mention here, we refer to books [3, 5].

In this paper, we introduce a new distance-based topological index of a graph $G = (V, E)$, called a k-distance degree index (shortly N_k-index). It is defined as $N_k(G) = \sum_{k=1}^{diam(G)} (\sum_{v \in V(G)} d_k(v))k$. We present the exact formulas of the N_k-index for some well-known graphs as the complete graph K_n, the path P_n, the cycle C_n, the star $K_{1,n-1}$, the complete bipartite $K_{r,s}$ and the wheel $W_n = K_1 + C_{n-1}$. Upper and lower bounds on N_k-index of G and other some interesting results are established. In addition, an explicit formula for the cartesian product of graphs are computed. Finally, the N_k-index formula of the cartesian product applied to some graphs like hypercube Q_n, Hamming graphs $H(r, s)$, nanotube $R = P_r \square C_s$ and nanotori $S = C_r \square C_s$, etc.

2 The N_k-index of graphs

Definition 2.1. For a connected graph G with n vertices, the N_k-index of G, is defined as

$$N_k(G) = \sum_{k=1}^{diam(G)} \Big(\sum_{v \in V(G)} d_k(v) \Big)\, k.$$

To illustrate the N_k-index of a graph, firstly, we consider the following remarks.

Remark 2.2. Let G be a connected graph. Then for a vertex $v \in V(G)$

(i) Since, $d(v, u) = 0$, for $u \in V(G)$, if and only if $v = u$, it follows that $d_0(v) = |N_0(v)| = 1$.

(ii) If $k > e(v)$, then $d_k(v) = 0$.

Then, we discuss the following example.

Example 2.3. Let G be a graph with four vertices v_1, v_2, v_3, v_4 as in Figure 1.

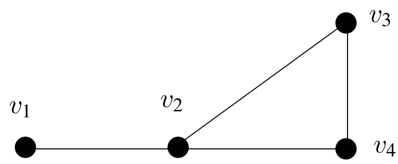

Figure 1

It is clear that $diam(G) = 2$.
Hence,

$$\begin{aligned}
N_k(G) &= \sum_{k=1}^{e(v)} \left(\sum_{v \in V} d_k(v) \right) k \\
&= (\sum_{v \in V} d_1(v)).1 + (\sum_{v \in V} d_2(v)).2 \\
&= (d_1(v_1) + d_1(v_2) + d_1(v_3) + d_1(v_4)).1 + (d_2(v_1) + d_2(v_2) + d_2(v_3) + d_2(v_4)).2 \\
&= (1 + 3 + 2 + 2) + 2(2 + 0 + 1 + 1) = 16.
\end{aligned}$$

Since, for any two vertices u and v in a graph G, either u and v are adjacent and then $u \in N_1(v/G)$ (also $v \in N_1(u/G)$) or u and v are not adjacent in G, then $u \notin N_1(v/G)$ and $v \notin N_1(u/G)$. If, without loss of the generality, $u \notin N_1(v/G)$, then $u \in N_k(v/G)$, for some $2 \leq k \leq diam(G)$. Using the definition of the complement $\overline{G}$ of G, if $u \notin N_1(v/G)$, then $u \in N_1(v/\overline{G})$. Thus, $\bigcup_{k=2}^{diam(G)} N_k(v/G) = N_1(v/\overline{G})$. That means $\sum_{k=2}^{diam(G)} \sum_{v \in V(G)} d_k(v/G) = \sum_{v \in V(G)} d_1(v/\overline{G})$. Then, by using the well-known result $d_1(v/\overline{G}) = n - 1 - d_1(v/G)$, the following result follows.

Lemma 2.4. *Let G be a connected graph with $n \geq 2$ vertices. Then*

(i) $\displaystyle\sum_{k=1}^{diam(G)} \sum_{v \in V(G)} d_k(v) = n(n-1)$.

(ii) $\displaystyle\sum_{k=0}^{diam(G)} \sum_{v \in V(G)} d_k(v) = n^2$.

Note that we can rewrite N_k-index of a graph G as $N_k(G) = \displaystyle\sum_{v \in V(G)} \left(\sum_{k=1}^{e(v)} d_k(v).k \right)$.

Theorem 2.5. *For any a connected graph G of order n, size m and $diam(G) = 2$*

$$N_k(G) = 2n(n-1) - 2m.$$

Proof. Let G be a connected graph of order n, size m and diameter $diam(G) = 2$ and let $\overline{m}$ be the size of $\overline{G}$. Since for any two distinct vertices v and u in G, either $uv \in E(G)$ or $uv \in E(\overline{G})$, it follows that $d_2(v/G) = d_1(v/\overline{G})$, for every $v \in V(G)$. Hence,

$$\begin{aligned} N_k(G) &= \sum_{k=1}^{2}\Big(\sum_{v\in V(G)} d_k(v/G)\Big).k \\ &= \Big(\sum_{v\in V(G)} d_1(v/G)\Big).1 + \Big(\sum_{v\in V(G)} d_2(v/G)\Big).2 \\ &= \Big(\sum_{v\in V(G)} d_1(v/G)\Big).1 + \Big(\sum_{v\in V(G)} d_1(v/\overline{G})\Big).2 \\ &= 2m + (2\overline{m}).2 = 2m + 4\overline{m} \\ &= 2m + 4\Big(\frac{n(n-1)}{2} - m\Big) = 2n(n-1) - 2m. \end{aligned}$$

□

We need the following definition to prove the next result.

Definition 2.6. [3] **Power of a Graph:** For a positive integer number k, k^{th} power of a simple graph $G = (V, E)$ is the graph G^k whose vertex set is $V(G)$, two distinct vertices being adjacent in G^k if and only if their distance in G is at most k.

Theorem 2.7. *For a positive integer number k and a connected nontrivial graph G, N_k-index is an even integer number.*

Proof. Let G be a connected nontrivial graph Of order $n \geq 2$, size m and diameter $diam(G)$. Since $V(G) = V(G^k)$ for every $1 \leq k \leq diam(G)$ and $G = G^1$, it follows that $d_k(v/G) = d_1(v/G^k)$, for every $v \in V(G)$. By the well-known results, for any graph G, $\sum_{v\in V(G)} d_1(v/G) = 2|E(G)|$, we obtain $\sum_{v\in V(G)} d_k(v/G) = \sum_{v\in V(G)} d_1(v/G^k) = 2|E(G^k)|$. Hence,

$$N_k(G) = \sum_{k=1}^{diam(G)} \Big(\sum_{v\in V(G)} d_k(v/G)\Big).k = \sum_{k=1}^{diam(G)} (2|E(G^k)|).k = 2\Big(\sum_{k=1}^{diam(G)} (|E(G^k)|).k\Big).$$

Since $|E(G^k)|$ and k are integer numbers for every $1 \leq k \leq diam(G)$, it follows that $\sum_{v\in V(G)} |E(G^k)|.k$ is an integer number. Therefore, N_k-index is an even integer number. □

3 The N_k-index of some standard graphs

In this section, we compute the N_k-index of some well-known graphs such as complete graphs K_n, paths P_n, cycles C_n, wheel $W_{1,n}$, complete bipartite $K_{r,s}$ and multipartite graphs $K_{n_1,n_2,\ldots,n_t}$, $t \geq 3$.

Proposition 3.1. *For $n \geq 2$,*

$$N_k(K_n) = n(n-1).$$

Proof. Consider a complete graph K_n of order $n \geq 2$. Since $diam(K_n) = 1$, it follows that $N_k(K_n) = \sum_{k=1}^{diam(K_n)} \left(\sum_{v\in V(K_n)} d_k(v)\right).k = \sum_{k=1}^{1}\sum_{v\in V(K_n)} d(v) = n(n-1)$. □

Proposition 3.2. *For $n \geq 2$,*

$$N_k(P_n) = \frac{n^3 - n}{3}.$$

Proof. Consider a path graph P_n of order $n \geq 2$. We prove the result of N_k-index of P_n only for n is even. The proof for n is odd is analogous. Since $diam(P_n) = n-1$, it follows that

$$N_k(P_n) = \sum_{k=1}^{n-1}\left(\sum_{v\in V(P_n)} d_k(v)\right).k$$

$$= (\sum_{v\in V(P_n)} d_1(v)).1 + (\sum_{v\in V(P_n)} d_2(v)).2 + ... + (\sum_{v\in V(P_n)} d_i(v)).i + ... + (\sum_{v\in V(P_n)} d_{n-1}(v)).(n-1)$$

$$= (1 + \overbrace{2+2+...+2}^{\text{n-2 times}}+1).1 + (1+1+\overbrace{2+2+...+2}^{\text{n-4 times}}+1+1).2 + ...$$

$$+ (\overbrace{1+1+...+1}^{i\ times}+\overbrace{2+2+...+2}^{\text{n-2i times}}+\overbrace{1+1+...+1}^{i\ times}).i + ... + (\overbrace{1+1+...+1}^{n\ times}).\frac{n}{2}+$$

$$(\overbrace{1+1+...+1}^{\frac{n}{2}-1\ times}+0+0+\overbrace{1+1+...+1}^{\frac{n}{2}-1\ times}).(\frac{n}{2}+1) + ... + (1+1+\overbrace{0+0+...+0}^{n-4\ times}+1+1).(n-2)+$$

$$(1+\overbrace{0+0+...+0}^{n-2\ times}+1).(n-1)$$

$$= 2(n-1).1 + 2(n-2).2 + ... + 2(n-i).i + ... + 2(\frac{n}{2}).\frac{n}{2} + ... + 2(n-(n-2)).(n-2)+$$

$$2(n-(n-1)).(n-1)$$

$$= 2(n-1).1 + 2(n-2).2 + ... + 2(n-i).i + ... + 2(2).(n-2) + 2(1).(n-1)$$

$$= \sum_{k=1}^{n-1} 2(n-k).k = 2n\sum_{k=1}^{n-1} k - 2\sum_{k=1}^{n-1} k^2$$

$$= \frac{n^3-n}{3}.$$

□

Proposition 3.3. *For* $n \geq 3$*,*

$$N_k(C_n) = \begin{cases} \frac{n^3}{4}, & \textit{if } n \textit{ even;} \\ \frac{n(n^2-1)}{4}, & \textit{if } n \textit{ odd.} \end{cases}$$

Proof. Consider a cycle graph C_n of order $n \geq 3$. Since $diam(C_n) = \lfloor \frac{n}{2} \rfloor$ then we consider the following cases

Case 1: If n is even, then $diam(C_n) = \frac{n}{2}$ and $d_k(v) = 2$, $v \in V(C_n)$ and for every $2 \leq k \leq \frac{n}{2} - 1$ and $d_{\frac{n}{2}}(v) = 1$, for every $v \in V(C_n)$. sequentially,

$$N_k(C_n) = \sum_{k=1}^{\frac{n}{2}}\left(\sum_{v\in V(C_n)} d_k(v)\right).k$$

$$= \sum_{k=1}^{\frac{n}{2}-1}\left(\sum_{v\in V(C_n)} 2\right).k + \left(\sum_{v\in V(C_n)} 1\right).\frac{n}{2}$$

$$= \sum_{k=1}^{\frac{n}{2}-1}(2n).k + \frac{n^2}{2}$$

$$= 2n\sum_{k=1}^{\frac{n}{2}} k + \frac{n^2}{2} = \frac{n^3}{4}.$$

Case 2: If n is odd, then $diam(C_n) = \frac{n-1}{2}$ and $d_k(v) = 2$, $v \in V(C_n)$. sequentially,

$$\begin{aligned}
N_k(C_n) &= \sum_{k=1}^{\frac{n-1}{2}} \left(\sum_{v\in V(C_n)} d_k(v) \right) .k \\
&= \sum_{k=1}^{\frac{n-1}{2}} \left(\sum_{v\in V(C_n)} 2 \right) .k \\
&= \sum_{k=1}^{\frac{n-1}{2}} (2n)\, .k \\
&= 2n \sum_{k=1}^{\frac{n-1}{2}} .k = \frac{n(n^2-1)}{4}.
\end{aligned}$$

Thus, $N_k(C_n) = \begin{cases} \frac{n^3}{4}, & \text{if } n \text{ even;} \\ \frac{n(n^2-1)}{4}, & \text{if } n \text{ odd.} \end{cases}$ □

A graph G is said to be a complete t-partite graph if there is a partition $V_1 \cup V_2 \cup ... \cup V_t = V(G)$ of the vertex set, such that $uv \in E(G)$, if and only if u and v are in different parts of the partition. If $|V_i| = n_i$, for every $1 \le i \le t$, then G is denoted by $K_{n_1,n_2,...,n_t}$.

Corollary 3.4. *[16] For any complete k-partite graph $K_{n_i,n_2,...,n_k}$, the number of its edge is*

$$m = \frac{1}{2}\Big[\Big(\sum_{i=1}^{k} n_i\Big)^2 - \sum_{i=1}^{k} n_i^2\Big].$$

From Theorem 2.5 and Corollary 3.4, the following results are immediately follows .

Proposition 3.5. *For $t \ge 2$, $n = n_1 + ... + n_t$ and $n_1 \ge n_2 \ge ... \ge n_t$ the N_k-index of a complete t-partite $K_{n_1,...,n_t}$ graph is*

$$N_k(K_{n_1,n_2,...,n_t}) = n(n-2) + \sum_{i=1}^{t} n_i^2.$$

Proposition 3.6. *For $2 \le r \le s$, the N_k-index of a complete bipartite graph $K_{r,s}$ is*

$$N_k(K_{r,s}) = 2(r+s)(r+s-1) - 2rs.$$

Proposition 3.7. *For $n \ge 2$, the N_k-index of a star graph is*

$$N_k(K_{1,n-1}) = 2(n-1)^2.$$

Proposition 3.8. *For $n \ge 4$ the n_k-index of a wheel $W_{1,n} = K_1 + C_n$ with $n+1$ vertices is*

$$N_k(W_{1,n}) = 2n(n-1).$$

4 Bounds for N_k-index of graphs

In this section, upper and lower bounds for N_k-index of a graph G and some interesting result are established.

Theorem 4.1. *Let G be a connected graph with $n \ge 2$ vertices. Then*

$$n(n-1) \le N_k(G) \le n(n-1)^2.$$

The lower bound attains on complete graphs K_n, for $n \ge 2$, whereas the upper bound attains on K_2.

Proof. Let G be a connected graph with $n \geq 2$ vertices. Then for $1 \leq k \leq diam(G)$,

$$\sum_{k=1}^{diam(G)} \left(\sum_{v \in V(G)} d_k(v) \right) .1 \leq \sum_{k=1}^{diam(G)} \left(\sum_{v \in V(G)} d_k(v) \right) .k \leq \sum_{k=1}^{diam(G)} \left(\sum_{v \in V(G)} d_k(v) \right) .diam(G).$$

Then by Theorem 2.4, $n(n-1) \leq N_k(G) \leq n(n-1)diam(G)$. Since for any connected graph G, $diam(G) \leq n-1$, it follows that $n(n-1) \leq N_k(G) \leq n(n-1)^2$. □

Theorem 4.2. *Let G be a connected graph with $n \geq 2$ vertices. Then $N_k(G) = n(n-1)$, if and only if $G = K_n$.*

Proof. If $G = K_n$, for $n \geq 2$, then $N_k(G) = n(n-1)$. Conversely, Suppose, to the contrary, that $G \neq K_n$. Then $diam(G) \geq 2$ and $m = |E(G)| < \frac{n(n-1)}{2}$. Thus by Theorem 2.5,

$$N_k(G) \geq \sum_{k=1}^{2} (\sum_{v \in V(G)} d_k(v)_G).k = 2n(n-1) - 2m > n(n-1).$$

□

Corollary 4.3. *Let G be a graph with n vertices and diameter $diam(G)$. Then*

$$(diam(G)+1)\ diam(G) \leq N_k(G) \leq n(n-1)\ diam(G).$$

In a connected graph G, a cut edge is an edge $e \in E(G)$ that when removed (the vertices stay in place) from a graph creates more components than previously in G or an if $G - e$ results in a disconnected graph.

Theorem 4.4. *Let G be a connected graph and let e be not a cut edge of G. Then*

$$N_k(G) \leq N_k(G-e).$$

Proof. The proof is immediately consequences of the result $diam(G-e) \geq diam(G)$ and Corollary 4.3. □

Corollary 4.5. *Let G be a connected graph with n vertices such that $G \neq K_n$. Then*

$$N_k(K_n) < N_k(G).$$

Corollary 4.6. *Let G be a connected graph and let H be a connected spanning subgraph of G. Then*

$$N_k(G) \leq N_k(H).$$

5 Cartesian product

Definition 5.1. [4] For given graphs G and H their Cartesian product, denoted by $G \Box H$, is defined as the graph on the vertex set $V(G) \times V(H)$, and vertices $u = (u_1, v_1)$ and $v = (u_2, v_2)$ of $V(G) \times V(H)$ are connected by an edge if and only if either $(u_1 = u_2 \text{ and } v_1v_2 \in E(H))$ or $(v_1 = v_2 \text{ and } u_1u_2 \in E(G))$.

It is a well known fact that the Cartesian product of graphs is commutative and associative up to isomorphism, $|V(G \Box H)| = |V(G)||V(H)|$, the distance between any two vertices $u = (u_1, v_1)$ and $v = (u_2, v_2)$ in $G \Box H$ is given by $d_{G \Box H}(u, v) = d_G(u_1, u_2) + d_H(v_1, v_2)$. The eccentricity $e(u, v)$ is obtained in the same way. Also, $diam(G \Box H) = diam(G) + diam(H)$. Let $diam(G) \leq diam(H)$. If $1 \leq i \leq diam(H) - diam(G) - 1$ and $1 \leq j \leq diam(G) - 1$, then $d_{diam(G)+i}(u/G) = 0$ and $d_{diam(H)+i}(v/H) = 0$.
For more details on cartesian product properties, see [4].

The following result is required to prove the next our main result.

Theorem 5.2. *[16] Let G and H be connected graphs of orders n_G and n_H, respectively. Then for any vertex $w = (u, v) \in G\Box H$,*

$$d_k(w/G\Box H) = \sum_{i=1}^{k} d_i(u/G)\, d_{k-i}(v/H).$$

Theorem 5.3. *Let G and H be nontrivial connected graphs. Then*

$$N_k(G\Box H) = |V(H)|^2\, N_k(G) + |V(G)|^2\, N_k(H).$$

Proof. Let G and H be connected graphs of orders $|V(G)| \geq 2$ and $|V(H)| \geq 2$, respectively and let $D_1 = diam(G)$ and $D_2 = diam(H)$. Then $G\Box H$ is connected graph with $|V(G)||V(H)|$ vertices. Let $w = (u, v) \in V(G\Box H)$ and suppose, without loss of generality, that $D_1 \leq D_2$. Then by Theorem 5.2, and properties of summation notion, we get

$$\begin{aligned}
N_k(G\Box H) &= \sum_{k=1}^{diam(G\Box H)} \Big(\sum_{w\in V(G\Box H)} d_k(w/G\Box H)\Big).\, k \\
&= \sum_{k=1}^{D_1+D_2} \Big(\sum_{(u,v)\in V(G\Box H)} d_k((u,v)/G\Box H)\Big).\, k \\
&= \sum_{k=1}^{D_1+D_2} \Big(\sum_{(u,v)\in V(G\Box H)} \sum_{i=0}^{k} d_i(u/G)\, d_{k-i}(v/H)\Big).\, k \\
&= \sum_{(u,v)\in G\Box H} \Big(\sum_{k=1}^{D_1+D_2} \sum_{i=0}^{k} d_i(u/G)\, d_{k-i}(v/H)\Big).\, k \\
&= \sum_{(u,v)\in G\Box H} \Big[(d_0(u/G)d_1(v/H) + d_1(u/G)d_0(v/H)).\, 1 \\
&+ (d_0(u/G)d_2(v/H) + d_1(u/G)d_1(v/H) + d_2(u/G)d_0(v/H)).\, 2 + ... \\
&+ (d_0(u/G)d_{D_1}(v/H) + d_1(u/G)d_{D_1-1}(v/H) + ... + d_{D_1}(u/G)d_0(v/H)).\, D_1 + ... \\
&+ (d_0(u/G)d_{D_1+i}(v/H) + d_1(u/G)d_{D_1+i-1}(v/H) + ... \\
&+ d_{D_1+i}(u/G)d_0(v/H)).(D_1 + i) + ... \\
&+ (d_0(u/G)d_{D_2}(v/H) + d_1(u/G)d_{D_2-1}(v/H) + ... + d_{D_2}(u/G)d_0(v/H)).\, D_2 \\
&+ (d_0(u/G)d_{D_2+1}(v/H) + d_1(u/G)d_{D_2}(v/H) + ... \\
&+ d_{D_2+1}(u/G)d_0(v/H)).\, (D_2 + 1) + ... \\
&+ (d_0(u/G)d_{D_2+j}(v/H) + d_1(u/G)d_{D_2+j-1}(v/H) + ... \\
&+ d_{D_2+j}(u/G)d_0(v/H)).\, (D_2 + j) + ... \\
&+ (d_0(u/G)d_{D_1+D_2}(v/H) + d_1(u/G)d_{D_1+D_2-1}(v/H) + ... \\
&+ d_{D_2}(u/G)d_0(v/H)).\, (D_1 + D_2)\Big].
\end{aligned}$$

$$
\begin{aligned}
N_k(G\Box H) &= \sum_{(u,v)\in V(G\Box H)} \Big[\Big(d_0(u/G)d_1(v/H)+d_1(u/G)d_0(v/H)\Big)\\
&+2\Big(d_0(u/G)d_2(v/H)+d_1(u/G)d_1(v/H)+d_2(u/G)d_0(v/H)\Big)+...\\
&+D_1\Big(d_0(u/G)d_{D_1}(v/H)+d_1(u/G)d_{D_1-1}(v/H)+...+d_{D_1}(u/G)d_0(v/H)\Big)+...\\
&+(D_1+i)\Big(d_0(u/G)d_{D_1+i}(v/H)+d_1(u/G)d_{D_1+i-1}(v/H)+...\\
&+d_{D_1}(u/G)d_i(v/H)\Big)+...\\
&+D_2\Big(d_0(u/G)d_{D_2}(v/H)+d_1(u/G)d_{D_2-1}(v/H)+...\\
&+d_{D_1}(u/G)d_{D_2-D_1}(v/H)\Big)\\
&+(D_2+j)\Big(d_j(u/G)d_{D_2}(v/H)+d_{j+1}(u/G)d_{D_2-1}(v/H)+...\\
&+d_{D_1}(u/G)d_{D_2+i-D_1}(v/H)\Big)+...\\
&+(D_1+D_2-1)\Big(d_{D_1-1}(u/G)d_{D_2}(v/H)+d_{D_1}(u/G)d_{D_2-1}(v/H))\\
&+(D_1+D_2)(d_{D_1}(u/G)d_{D_2}(v/H))\Big]\\
&=\sum_{(u,v)}\Big[(d_0(u/G)d_1(v/H)+2d_0(u/G)d_2(v/H)+...+D_2d_0(u/G)d_{D_2}(v/H))\\
&+(d_1(u/G)d_0(v/H)+2d_1(u/G)d_1(v/H)+...+(D_2+1)d_1(u/G)d_{D_2}(v/H))+\\
&(2d_2(u/G)d_0(v/H)+3d_2(u/G)d_1(v/H)+...+(D_2+2)d_2(u/G)d_{D_2}(v/H))+...\\
&+(jd_j(u/G)d_0(v/H)+(j+1)d_2(u/G)d_1(v/H)+...\\
&+(D_2+j)d_j(u/G)d_{D_2}(v/H))+...\\
&+(D_1d_{D_1}(u/G)d_0(v/H)+(D_1+1)d_{D_1}(u/G)d_1(v/H)+...\\
&+(D_2+D_1)d_{D_1}(u/G)d_{D_2}(v/H))\Big]\\
&=\sum_{(u,v)}\Big[d_0(u/G)(\sum_{k=0}^{D_2}d_k(v/H).\,k)+d_1(u/G)(\sum_{k=0}^{D_2}d_k(v/H).\,(k+1))+...\\
&+d_j(u/G)(\sum_{k=0}^{D_2}d_k(v/H).\,(k+j))+...+d_{D_1}(u/G)(\sum_{k=0}^{D_2}d_k(v/H).\,(k+D_1))\Big]\\
&=\sum_{(u,v)}\Big[\sum_{i=0}^{D_1}d_i(u/G)(\sum_{k=0}^{D_2}d_k(v/H).\,(k+i))\Big]\\
&=\sum_{(u,v)}\Big[\sum_{i=0}^{D_1}d_i(u/G)(\sum_{k=0}^{D_2}d_k(v/H).\,(k))+\sum_{i=0}^{D_1}d_i(u/G)(\sum_{k=0}^{D_2}d_k(v/H).\,(i))\Big]\\
&=(\sum_{i=0}^{D_1}\sum_{u\in V(G)}d_i(u/G))(\sum_{k=1}^{D_2}\sum_{v\in V(H)}d_k(v/H).\,k)\\
&+(\sum_{k=0}^{D_2}\sum_{v\in V(H)}d_k(v/H))(\sum_{i=1}^{D_1}\sum_{u\in V(G)}d_i(u/G).\,i)\\
&=|V(G)|^2N_k(H)+|V(H)|^2N_k(G).
\end{aligned}
$$

□

The Cartesian product of more than two graphs is defined inductively,

$$G_1 \Box G_2 \Box ... \Box G_k = G_1 \Box (G_2 \Box ... \Box G_k).$$

We denote by $\prod_{i=1}^{k} G_i$ to $G_1 \Box G_2 \Box ... \Box G_k$. It is clear that $|V(\prod_{i=1}^{k} G_i)| = \prod_{i=1}^{k} |V(G_i)|$.

Theorem 5.4. *Let $G_1, G_2, ..., G_t$, for $t \geq 2$ be nontrivial connected graphs with $n_1, n_2, ..., n_t$ vertices, respectively. Then*

$$N_k(\prod_{i=1}^{t} G_i) = \sum_{i=1}^{t} \left(\prod_{\substack{j=1 \\ j \neq i}}^{t} n_j^2 \right) N_k(G_i).$$

Proof. Let $G_1, G_2, ..., G_t$, for $t \geq 2$, be connected graphs with $n_1, n_2, ..., n_t$ vertices, respectively. Then we set $\prod_{i=1}^{t} n_i = n_1 n_2 .... n_t$ is a usual product of integer numbers. We prove this result by mathematical induction.

(i) The result is true for $t = 2$, by Theorem 5.3.

(ii) Assume there is a $t \geq 2$ such that $N_k(\prod_{i=1}^{t} G_i) = \sum_{i=1}^{t} \left(\prod_{\substack{j=1 \\ j \neq i}}^{t} n_j^2 \right) N_k(G_i)$.

(iii) Now we have to prove that the result is true for $t+1$. So let $\prod_{i=1}^{t+1} G_i = (\prod_{i=1}^{t} G_i) \Box G_{t+1}$, where G_{t+1} is a connected graph of order n_{t+1}. Then

$$\begin{aligned} N_k(\prod_{i=1}^{t+1} G_i) &= N_k \left((\prod_{i=1}^{t} G_i) \Box G_{t+1} \right) \\ &= (n_{t+1}^2) N_k(\prod_{i=1}^{t} G_i) + \left(\prod_{i=1}^{t} n_i \right)^2 N_k(G_{t+1}) \\ &= (\prod_{j=2}^{t+1} n_j^2) N_k(G_1) + (\prod_{\substack{j=1 \\ j \neq 2}}^{t+1} n_j^2) N_k(G_2) + ... \\ &+ (\prod_{\substack{j=1 \\ j \neq t}}^{t+1} n_j^2) N_k(G_t) + (\prod_{\substack{j=1 \\ j \neq t+1}}^{t+1} n_j^2) N_k(G_{t+1}) \\ &= \sum_{i=1}^{t+1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{t+1} n_j^2 \right) N_k(G_i). \end{aligned}$$

Therefore, the result is true for every positive integer $t \geq 2$. □

Corollary 5.5. *Let G be a connected graph with $n \geq 2$ vertices. Then for $t \geq 1$*

$$N_k(\prod_{i=1}^{t} G) = t n^{2t-2} N_k(G).$$

By Theorem 5.4 and Corollary 5.5, we can compute the N_k-index for several classes of graphs which defined as a cartesian product of graphs. For examples, hypercube graph, Hamming graphs, $(n \times m)$-grid graphs, n-prism graph and nanotube graphs. etc. see[3, 7]. Such graphs appear in many applications, for instance in the theory of communication networks and in chemistry.

Definition 5.6. [7]

(i) A Hypercube graph Q_d is the Cartesian product of d copies of K_2.

(ii) The Hamming graph $H(d, n)$ is, equivalently, the Cartesian product of d complete graphs K_n.

Example 5.7. For $d \geq 1$,

(i) $N_k(Q_d) = d2^{2d-1}$.

(ii) $N_k(H(d, n)) = dn^{2d}(1 - \frac{1}{n})$.

Definition 5.8. [3]

(i) The $(n \times m)$-grid graphs $G(n, m)$ is the cartesian product of the path P_n by the path P_m.

(ii) A prism graph Y_n is the Cartesian product of a cycle C_n by K_2.

(iii) The C_4 nanotube graph R is the Cartesian product of a cycle C_n by a path P_m.

(iv) The nanotori graph S is the Cartesian product of a cycle C_n by a cycle C_m.

Example 5.9. For $n \geq 3$ and $m \geq 2$,

(i) $N_k(G(n, m)) = \frac{nm(n+m)(nm-1)}{3}$.

(ii) $N_k(Y_n) = \begin{cases} n^3 + 2n^2, & \text{if } n \text{ is even;} \\ n^3 + 2n^2 - n, & \text{if } n \text{ is odd.} \end{cases}$

(iii) $N_k(R) = N_k(C_n \square P_m) = \begin{cases} \frac{n^2m(3nm+4m^2-4)}{12}, & \text{if } n \text{ is even;} \\ \frac{nm(3n^2m-3m+4nm^2-4n)}{12}, & \text{if } n \text{ is odd.} \end{cases}$

(iv) $N_k(S) = N_k(C_n \square C_m) = \begin{cases} \frac{n^2m^2(n+m)}{4}, & \text{if } n \text{ and } m \text{ are even;} \\ \frac{n^2m(m^2+nm-1)}{4}, & \text{if } n \text{ is even and } m \text{ is odd;} \\ \frac{nm^2(n^2+nm-1)}{4}, & \text{if } n \text{ is odd and } m \text{ is even;} \\ \frac{nm(n+m)(nm-1)}{4}, & \text{if } n \text{ and } m \text{ are odd.} \end{cases}$

6 Conclusions

In this paper, the new distance-based topological index, called a k-distance degree index (Shortly, N_k-index), of graphs is introduced. It is shown that the N_k-index of a graph is even integer number. Bounds and interesting result for N_k-index are obtained. Exact formulaes of the N_k-index for some well-known graphs are presented. Finally, the exact formulaes of the N_k-index for Cartesian product of graphs are computed.

Open Problems

- Compute the values of N_k-index of some others families of graphs.
- Compute the values of N_k-index of some others operations on graphs, as line graph, complement of graph, corona product of graphs, etc.
- Find the relationships between N_k-index with other indices of a graph.
- Find the relationships between N_k-index of a graph with other parameters of a graph, such as maximum degree $\Delta(G)$, minimum degree $\delta(G)$, clique number $\omega(G)$, chromatic number $\chi(G)$ and etc.
- Find the relationships between N_k-index of a graph with other distance-based topological indices of a graph.

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