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EXISTENCE AND UNIQUENESS OF SOLUTION OF IMPULSIVE HAMILTON-JACOBI EQUATION

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Abstract. This paper formulates a corresponding Impulsive Hamilton -Jacobi equation and establishes the existence and uniqueness of its solution using the Krasnoselskii theorem.

1 Introduction

Impulsive differential equation are often regarded as Cauchy problems with short term perturbations.Many phenomena in real life which are prone to abrupt changes are modeled using Impulsive differential equation (Lui and Rohlf, 1998 [6]).
These Impulsive Differential Equation or differential equation involving Impulsive effects appear as a natural description of observed evolutional phenomena of several real world problems (Roup (2003 [7])). The occurrence may be as a continuous time differential equation governing the state of a system between Impulses or an Impulsive Jump defined by a Jump function at the instance where it occurs

Hamilton-Jacobi equation is a special example of such modeled equation arising from evolutional phenomena.
Hamiltonian, is in the sense that it indicates the total energy as the sum of the kinetic and potential energies in a system .
Hamilton-Jacobi equation have been introduced in Imbert, Monnaeu and Zidani (2013 [5])for modeling of junction problems to traffic flow, Bessan et al(2014 [1]) , Gotthick etal(2013 [3]) and Han etal(2012 [4]) also in their researches had introduced Hamilton Jacobi formulation for network but these lead to tedious coupling conditions at each junctions.
The application of Hamilton-Jacobi equation can not be over emphasized with regards to time dependent issues. Given that Impulsive differential equation is appearing now in research and are very important concept for solving problems with discontinuities, this paper therefore formulates a corresponding Impulsive Hamilton- Jacobi equation, obtain its solution representation and establishes its existence and uniqueness.

2 Impulsive Hamilton-Jacobi Equation

Let $X = C((0,T)\times\Omega,\mathbb{R})$ be a Banach space of all continuous functions $u(t,x) : (0,\ T)\times\Omega \longrightarrow \mathbb{R}$ with the norm
$\|.\|_\infty = \sup\{|u(t,x)| : (t,x) \in (0,\ T) \times \Omega\}$ such that $u(t,x)$ is continuous at $t \neq t_k$.
Assume that the right limit $u(t_k^+,x)$ and the left limit $u(t_k^-,x)$ exist for $k = 1,2...m$ where $0 < t_1 < ... < t_m = T$. Then the Impulsive Hamilton-Jacobi equation associated with the Hamiltonian H is the equation of the form

$$u_t(t,x) + H(t,x,u,Du(t,x)) = 0 \quad \mathbb{R}^n \setminus C(t_k,\ t_{k+1}) \tag{2.1}$$

$$u(0,x) = g(x) \quad \{t=0\} \times \mathbb{R} \tag{2.2}$$

$$\Delta u(t_k,\ x) = I_k(u(t_k,x)) \quad t=t_k \quad t \in [0,\ T] \setminus \{t_k\} \tag{2.3}$$

Where $u : \mathbb{R}^n \longrightarrow \mathbb{R}, g : \mathbb{R}^n \longrightarrow \mathbb{R}, I_k \in C(\mathbb{R},\ \mathbb{R})$,
$Du = D_x u = (u_{x_1}, u_{x_2}, ...u_{x_n})$ and $\Delta u(t_k\ x) = u(t_k^+,x) - u(t_k^-,x)$ such that

$u(t_k^+,\ x) = \lim_{t>t_k \longrightarrow t_k} u(t,\ x)$
and $u(t_k^-, x) = \lim_{t<t_k \longrightarrow t_k} u(t, x)$.
The solution to this Impulsive Hamilton Jacobi Equation would be obtained using the method of characteristics and following the Technique of solution for Hamilton -Jacobi Equation.
Let the equation be written compactly thus:

$$F(t, x, u, u_t, Du(t, x)) = u_t + H(x, p) = 0 \tag{2.4}$$

such that

$$\dot{x} = D_p[u_t + H(x, p)] = D_p H(x, p) \tag{2.5}$$
$$\dot{p} = -D_z[u_t + H(x, P)].P - d_x[u_t + H(x, p)] = D_x H(x, p) \tag{2.6}$$
$$\dot{z} = \frac{\partial u}{\partial x_i}\frac{\partial x_i}{\partial t} = p.D_p[u_t + H(x, P)] = p.D_p H(x, p) \tag{2.7}$$

for $H(x, p) = -u_t$. Let $z(t) = u(t, x), x(t), p(t)$ be the solution of the equations 2.4, 2,5,2.6 respectively. By integrating 2.7 we have

$$\int_0^{t_1} \dot{z}(s)ds + \int_{t_1}^{t_2} \dot{z}(s)ds + ... + \int_{t_{m-1}}^{t_m} \dot{z}(s)ds = \int_0^{t_m} p.D_p H(x, p)ds \tag{2.8}$$

And

$$u(t, x) - u(0, x) - \sum_{k=1}^{m} I_k(u(t_k, x)) = \int_0^{t_m} p.D_p H(x, p)ds \tag{2.9}$$
$$\Longrightarrow u(t, x) = u(0, x) + \int_0^{t_m} p.D_p H(x, p)ds + \sum_{k=1}^{m} I_k(u(t_k, x)) \tag{2.10}$$
$$\Longrightarrow u(t, x) = g(x) + \int_0^{t_m} p.D_p H(x, p)ds + \sum_{k=1}^{m} I_k(u(t_k, x)) \tag{2.11}$$

Equation 2.11 is the solution representation of the Impulsive Hamilton- Jacobi equations 2.1-2.3

3 Existence and Uniqueness of equation 2.11

In order to establish the existence and uniqueness of the solution representation (2.11), we state (w.p) the Kransnoselskii theorem:

Theorem 3.1. *Krasnoselskii theorem Burton 1998 [2]*
Let $(S, \|.\|)$ be a Banach space , with M a closed convex nonempty subset of S. Suppose that $A, B : M \longmapsto S$ are functions such that

(i) $x, y \in M \Longrightarrow Ax + By \in M$ and B is continuous

(ii) BM resides in a compact set

(iii) A is a contraction with constant $0 \leq \alpha < 1$ then there exist a unique $y \in M$ such that $y = Ay_1 + By_2$.

Following the above theorem , the results of this paper is express bellow as a theorem thus:

Theorem 3.2. *Existence and Uniqueness Theorem for equation 2.11*
Let X be a Banach space, W a closed convex, nonempty subset of X. Let $H(t, x, Du)$ and $u_t(t, x)$ be continuous and closed on X such that H is Lipschitzian in the t- variable. Let $|f(I_k)| \leq C_k, C_k$ any positive constant, $k = 1, 2, ...m$ then for any continuous Initial Solution $\phi(0, x) = g(x), \phi \in W \subset X$ there exist a unique solution $u(t, x) = g(x) + \int_0^t pD_p H(x, P)ds + f(I_k)$ for equation 2.1-2.3.

Proof. : The Theorem is proved in steps by establishing the assumptions and the hypothesis of the theorem thus:
Step 1: Since $H(t,x,u,Du)$ and $u_t(t,x)$ are said to be continuous and bounded in X then there exist constants $k_1 > 0, k_2 > 0$ such that $\|H(t,x,u,Du)\| \leq k_1$ and $\|u_t(t,x)\| \leq k_2$ By applying the mean value theorem for $t = t_0$ we have that

$$\|H(t,x,u,Du) - H(t_0,x,u,Du)\| = \|u_t(t,x)\||t - t_0| \leq k_2|t - t_0|$$

holds hence $H(t,x,u,Du(t,x))$ is lipschitzian
Step 2: By Krasnoselskii theorem , Let $A, B \in W$ such that $A, B : W \longrightarrow W$ and A is a contraction with constant $0 \leq \alpha < 1$ also B is continuous with Bw residing in a compact set. Then the solution representation

$$u(t,x) = g(x) + \int_0^{t_m} p.D_pH(x,p)ds + \sum_{k=1}^{m} I_k(u(t_k,x))$$

can be written in the form

$$u(t,x) = Au_1(t,x) + Bu_2(t,x) \tag{3.1}$$

where

$$Au_1(t,x) = g(x) + \int_{t_0}^{t} p.D_pH(x,p)ds$$

and

$$Bu_2(t,x) = \sum_{k=1}^{m} I_k(u(t_k,x))$$

Step 3: Given that

$$Au_1(t,x) = g(x) + \int_{t_0}^{t} p.D_pH(x,p)ds$$

Assume $\phi \in W$ is a continuous initial function such that $\phi(0,x) = g(x)$ then there exist another continuous function $u(0,x,\phi) \in W$ satisfying the Impulsive Problem 2.1-2.3 since X is a Banach space then $A : W \longrightarrow W$ is such that

$$Aw = \phi(t,x) = g(x) + \int_{t_0}^{t} pD_pH(\phi(x,p))ds \tag{3.2}$$

$\Longrightarrow$

$$\begin{aligned}
\|\phi(t,x) - g(x)\| &= \|\int_{t_0}^{t} pD_pH(\phi(x,P))ds\| && (3.3)\\
&\leq \int_{t_0}^{t} \|pD_pH(\phi(x,P))\|ds && (3.4)\\
&\leq k_1\|\phi(t,x) - \phi(t_0,x)\| && (3.5)\\
&\leq k_1\alpha && (3.6)
\end{aligned}$$

again for $w_1, w_2 \in W \subset X$ let $Aw_1 = \phi_1$ and $Aw_2 = \phi_2$ we have that

$$\begin{aligned}
\delta(Aw_1 \;\; , Aw_2) &= \delta(\phi_1 \;\; \phi_2)\\
&\leq \sup \int_{t_0}^{t} \|pD_pH(\phi_2(x,P)) - pH_pH(\phi_1(x,P))\|ds\\
&\leq k_1 \int_{t_o}^{t} \sup \|\phi_2(x,p) - \phi_1(x,p)\|ds\\
&\leq k_1\delta(\phi_2,\phi_1) \sup \|\phi(t,x) - \phi(t_0,x)\|\\
&\leq k_1\alpha\delta(w_1,w_2)
\end{aligned}$$

since $0 \leq k_1, \alpha < 1$, we have that the map A is a contraction map.

Step 4: We show that $Bu_2(t,x)$ is compact in W
Let $R^*\{I_k(u(t_k,x)) : \|I_k(u(t_k,x))\| \leq C_k, u(t_0,x) - \alpha \leq u(t,x) \leq u(t_0,x) + \alpha\}$
such that $0 \leq k_2\alpha < 1$ for $u_2 \in R^*$, and $Bu_2(t,x) \in R^*$, hence R^* is a bounded subset of W.
Now let δ be defined on W such that for $u_2^1(t,x), u_2^2(t,x) \in W$ we have that $Bu_2^1(x,t) = \varphi_1$, and $Bu_2^2(x,t) = \varphi_2$, then

$$\delta(Bu_2^1(t,x), Bu_2^2(t,x) = \delta(\varphi_1\varphi_2) = sup\|Bu_2^2(t,x) - BU_2^1(t,x)\| \leq k_2\alpha \tag{3.7}$$

holds and for BW being compact we conclude that $Bu_2(t,x)$ is compact. □

Conclusion: Given any Impulsive Hamilton- Jacobi equations 2.1-2.3, a continuous function $u(t,x)$ satisfying the assumptions and hypothesis of this theorem will have its representation as in equation 2.11 which does exist and is unique.

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