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NONEXISTENCE RESULTS OF GLOBAL WEAK SOLUTION IN FUJITA-TYPE SYSTEM ON THE HEISENBERG GROUP

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Abstract. Our aim here is to prove nonexitence results for systems of the following type:

$$\begin{cases} u_t + |x|^{\alpha}(-\Delta_{\mathbb{H}})^{\frac{\alpha_1}{2}}(a_{11}u) = f(\eta,t)|v|^p \\ v_t + |x|^{\beta}\left\{(-\Delta_{\mathbb{H}})^{\frac{\alpha_2}{2}}(a_{21}u) + (-\Delta_{\mathbb{H}})^{\frac{\alpha_3}{2}}(a_{22}v)\right\} = h(\eta,t)|w|^q \\ w_t + |x|^{\gamma}\left\{(-\Delta_{\mathbb{H}})^{\frac{\alpha_4}{2}}(a_{31}u) + (-\Delta_{\mathbb{H}})^{\frac{\alpha_5}{2}}(a_{32}v) + (-\Delta_{\mathbb{H}})^{\frac{\alpha_6}{2}}(a_{33}w)\right\} = k(\eta,t)|u|^r, \end{cases}$$

where $\Delta_{\mathbb{H}}$ denotes the Kohn-Laplace operator on the $(2N+1)$-dimensional Heisenberg group $\mathbb{H}$. Our method of proof is based on suitable choices of the test functions in the weak formulation of the sought solutions.

1 Introduction and preliminaries

In this article, we are concerned with the nonexistence of global weak solutions of the following system:

$$\begin{cases} u_t + |x|^{\alpha}(-\Delta_{\mathbb{H}})^{\frac{\alpha_1}{2}}(a_{11}u) = f(\eta,t)|v|^p \\ v_t + |x|^{\beta}\left\{(-\Delta_{\mathbb{H}})^{\frac{\alpha_2}{2}}(a_{21}u) + (-\Delta_{\mathbb{H}})^{\frac{\alpha_3}{2}}(a_{22}v)\right\} = h(\eta,t)|w|^q \\ w_t + |x|^{\gamma}\left\{(-\Delta_{\mathbb{H}})^{\frac{\alpha_4}{2}}(a_{31}u) + (-\Delta_{\mathbb{H}})^{\frac{\alpha_5}{2}}(a_{32}v) + (-\Delta_{\mathbb{H}})^{\frac{\alpha_6}{2}}(a_{33}w)\right\} = k(\eta,t)|u|^r, \end{cases} \tag{1.1}$$

where $p,q,r > 1$, with the initial data

$$u(\eta,0) = u_0(\eta), v(\eta,0) = v_0(\eta) \quad , w(\eta,0) = w_0(\eta), \quad \eta = (x,y,\tau).$$

Here $\Delta_{\mathbb{H}}$ is the Kohn-Laplace operator on the $(2N+1)$-dimensional Heisenberg group $\mathbb{H}$. The fractional power of the Laplacian on the Heisenberg group $(-\Delta_{\mathbb{H}})^{\frac{\alpha_i}{2}}$, $0 < \alpha_i < 2$ accounts for anomalous diffusion and is to be defined later. $\alpha, \beta, \gamma \geq 0$, a_{ij} measurable, positive and bounded functions and the functions f, h, k are assumed to satisfy:

$$\left|f\left(R^Q\eta, R^2t\right)\right| \simeq R^{\nu}; \quad \left|h(R^Q\eta, R^2t)\right| \simeq R^{\mu}; \quad \left|k(R^Q\eta, R^2t)\right| \simeq R^{\xi}.$$

Our article is motivated by the recent paper by R. Kellil and M. Kirane [4] which deals with nonexistence of global weak solutions of a system of wave equations on the Heisenberg group. A similar system was investigated by A. Hakem and G. Abdelkader ([7]) on $(0,T)\times\mathbb{R}^n$.

The purpose of this present paper is to investigate the nonexistence of global nontrivial solutions for system(1.1). The method used to prove the blow-up result is the test function method considered by Mitidieri and Pohozaev ([10],[11]), Pohozaev and Tesei [9] and Kirane et al [5]. Before stating our main result and for the reader convenience, some background facts used in the sequel are recalled.

The Heisenberg group $\mathbb{H}$ whose points will be denoted by $\eta = (x, y, \tau)$, is the Lie group $(\mathbb{R}^{2N+1}; \circ)$ with the non-commutative group operation $\circ$ defined by

$$\eta \circ \eta' = (x + x', y + y', \tau + \tau' + 2(x \cdot y' - x' \cdot y)),$$

for all $\eta = (x, y, \tau), \eta' = (x', y', \tau') \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$, where $\cdot$ denotes the standard scalar product in $\mathbb{R}^N$. This group operation endows $\mathbb{H}$ with the structure of a Lie group.

The Laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}$ is obtained from the vectors fields

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial \tau} \quad \text{and} \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial \tau},$$

for $i = 1, 2....., N$ as follows

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left(X_i^2 + Y_i^2 \right).$$

Observe that the vector field $T = \frac{\partial}{\partial \tau}$ does not appear in the equality above. This fact makes us presume a "loss of derivative" in the variable τ. The compensation comes from the relation

$$[X_i, Y_j] = -4T, \quad i, j \in 1, 2, 3, ..., N.$$

The relation above proves that $\mathbb{H}$ is a nilpotent Lie group of order 2. Explicit computation gives the expression

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \tau} - 4x_i \frac{\partial^2}{\partial y_i \partial \tau} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial \tau^2} \right).$$

A natural group of dilatations on $\mathbb{H}$ is given by

$$\delta_\lambda(\eta) = (\lambda x, \lambda y, \lambda^2 \tau), \qquad \lambda > 0,$$

whose Jacobian determinant is λ^Q, where $Q = 2N + 2$ is the homogeneous dimension of $\mathbb{H}$. The operator $\Delta_{\mathbb{H}}$ is a degenerate elliptic operator. It is invariant with respect to the left translation of $\mathbb{H}$ and homogeneous with respect to the dilations δ_λ. More precisely, we have

$$\Delta_{\mathbb{H}}(u(\eta \circ \eta')) = (\Delta_{\mathbb{H}} u)(\eta \circ \eta'), \Delta_{\mathbb{H}}(u \circ \delta_\lambda) = \lambda^2 (\Delta_{\mathbb{H}} u) \circ \delta_\lambda, \eta, \eta' \in \mathbb{H}.$$

The natural distance from η to the origin is introduced by Folland and Stein, see [8].

$$|\eta|_{\mathbb{H}} = \left(\tau^2 + \left(\sum_{i=1}^{N} (x_i^2 + y_i^2) \right)^2 \right)^{\frac{1}{4}}.$$

1.1 Fractional powers of sub-elliptic Laplacians

The representation of the fractional power of $(-\Delta_{\mathbb{H}})^s$ is given by the following theorem:

Theorem 1.1. *The operator $\Delta_{\mathbb{H}}$ is a positive self-adjoint operator with domain $W^{2,2}_{\mathbb{H}}(\mathbb{H})$. Denote now by $E(\lambda)$ the spectral resolution of $\Delta_{\mathbb{H}}$ in $\mathbb{L}^2(\mathbb{H})$. If $\alpha > 0$, then*

$$(-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}} = \int_0^{+\infty} \lambda^{\frac{\alpha}{2}} dE(\lambda),$$

with domain

$$W^{\alpha,2}_{\mathbb{H}}(\mathbb{H}) = \left\{ v \in \mathbb{L}^2(\mathbb{H}); \quad \int_0^{+\infty} \lambda^\alpha d\langle E(\lambda) v, v \rangle < \infty \right\},$$

endowed with graph norm.

Proposition 1.2. *([1]) Assume that the function* $\varphi \in C_0^\infty\left(\mathbb{R}^{2N+1}\right)$ *then*

$$\sigma\varphi^{\sigma-1}(-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}\varphi \geq (-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}\varphi^\sigma, \tag{1.2}$$

holds point-wise.

Lemma 1.3. *([1]) Let* $f \in \mathcal{L}^1\left(\mathbb{R}^{2N+1}\right)$ *and* $\int_{\mathbb{R}^{2N+1}} f d\eta > 0$*. Then there exists a test function* $\varphi,\ 0 \leq \varphi \leq 1$ *such that*

$$\int_{\mathbb{R}^{2N+1}} f\varphi d\eta \geq 0.$$

Let us set $\mathcal{H}_T = \mathbb{H} \times (0,T)$ and $\mathcal{H} = \mathbb{H} \times (0,\infty)$. We also consider

$$L^p_{loc}\bigg(\mathcal{H}_T; f(\eta,t)d\eta dt\bigg) = \bigg\{u : \mathcal{H}_T \to \mathbb{R}/ \int_K f(\eta,t)|u|^p d\eta dt < +\infty \quad for \quad any \quad compact \quad K \subset \mathcal{H}_T\bigg\}.$$

2 Main results

Definition 2.1. A local weak solution of the system (1.1) is a triplet of functions (u,v,w) such that

$$u \in C\left([0,T]\,; L^1_{loc}(\mathcal{H})\right) \cap C\left((0,T); L^r_{loc}(\mathcal{H}) \cap L^r_{loc}(\mathcal{H}; k(\eta,t)d\eta dt)\right),$$
$$v \in C\left([0,T]\,; L^1_{loc}(\mathcal{H})\right) \cap C\left((0,T); L^p_{loc}(\mathcal{H}) \cap L^p_{loc}(\mathcal{H}; f(\eta,t)d\eta dt)\right),$$

and

$$w \in C\left([0,T]\,; L^1_{loc}(\mathcal{H})\right) \cap C\left((0,T); L^q_{loc}(\mathcal{H}) \cap L^q_{loc}(\mathcal{H}; h(\eta,t)d\eta dt)\right),$$

subject to the initial data u_0,v_0, $w_0 \in L^1_{loc}(\mathbb{R}^{2N+1})$ satisfying the equations

$$\begin{aligned}&-\int_{\mathcal{H}_T} u\varphi_t d\eta dt + \int_{\mathcal{H}_T} a_{11}u|x|^\alpha(-\Delta_{\mathbb{H}})^{\frac{\alpha_1}{2}}\varphi d\eta dt = \int_{\mathcal{H}_T} f(\eta;t)|v|^p\varphi d\eta dt\\ &+\int_{\mathbb{H}} u_0(\eta)\varphi(\eta,0)d\eta,\end{aligned} \tag{2.1}$$

$$\begin{aligned}&-\int_{\mathcal{H}_T} v\varphi_t d\eta dt + \int_{\mathcal{H}_T} a_{21}u|x|^\beta(-\Delta_{\mathbb{H}})^{\frac{\alpha_2}{2}}\varphi d\eta dt + \int_{\mathcal{H}_T} a_{22}v|x|^\beta(-\Delta_{\mathbb{H}})^{\frac{\alpha_3}{2}}\varphi d\eta dt\\ &= \int_{\mathcal{H}_T} h(\eta;t)|w|^q\varphi d\eta dt + \int_{\mathbb{H}} v_0(\eta)\varphi(\eta,0)d\eta,\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}&-\int_{\mathcal{H}_T} w\varphi_t d\eta dt + \int_{\mathcal{H}_T} a_{31}u|x|^\gamma(-\Delta_{\mathbb{H}})^{\frac{\alpha_4}{2}}\varphi d\eta dt + \int_{\mathcal{H}_T} a_{32}v|x|^\gamma(-\Delta_{\mathbb{H}})^{\frac{\alpha_5}{2}}\varphi d\eta dt\\ &+\int_{\mathcal{H}_T} a_{33}w|x|^\gamma(-\Delta_{\mathbb{H}})^{\frac{\alpha_6}{2}}\varphi d\eta dt = \int_{\mathcal{H}_T} k(\eta;t)|u|^r\varphi d\eta dt + \int_{\mathbb{H}} w_0(\eta)\varphi(\eta,0)d\eta,\end{aligned} \tag{2.3}$$

for any regular test function φ with $\varphi(.,T) = 0,\ \varphi \geq 0$. The solution is called global if $T = +\infty$.

We now state the main result in this paper.

Theorem 2.2. *Let* $(u_0,v_0,w_0) \in \mathbb{L}^1(\mathbb{H}) \times \mathbb{L}^1(\mathbb{H}) \times \mathbb{L}^1(\mathbb{H})$ *suppose that*

$$\int_{\mathbb{H}} u_0(\eta)d\eta > 0, \quad \int_{\mathbb{H}} v_0(\eta)d\eta > 0, \quad \int_{\mathbb{H}} w_0(\eta)d\eta > 0. \tag{2.4}$$

If

$$Q < min\{\lambda_i\} - 2, \qquad i = 1, ..., 6 \tag{2.5}$$

where

$$\lambda_1=\frac{\xi+r(\alpha_1-\alpha)}{r-1},\quad \lambda_4=\frac{\mu+q(\alpha_6-\gamma)}{q-1},\quad \lambda_2=\frac{\nu+p(\alpha_3-\beta)}{p-1},$$
$$\lambda_5=\frac{\xi+r(\alpha_4-\gamma)}{r-1},\quad \lambda_3=\frac{\xi+r(\alpha_2-\beta)}{r-1},\quad \lambda_6=\frac{\nu+p(\alpha_5-\gamma)}{p-1},$$

then the system (1.1) does not have a nontrivial weak solution.

Proof. The proof is by contradiction. For that, let (u,v,w) be a solution and φ be a smooth nonnegative test function such that

$$\mathcal{A}(k,r)=\left(\int_{\mathcal{H}}|k|^{\frac{-r'}{r}}\varphi^{\sigma-r'}|\varphi_t|^{r'}d\eta dt\right)^{\frac{1}{r'}}, \tag{2.6}$$

$$\mathcal{B}(k,\alpha,\alpha_1,r)=\left(\int_{\mathcal{H}}|x|^{\alpha r'}|k|^{\frac{-r'}{r}}\varphi^{\sigma-r'}|(-\Delta)^{\frac{\alpha_1}{2}}\varphi|^{r'}d\eta dt\right)^{\frac{1}{r'}}. \tag{2.7}$$

Taking φ^σ, $\sigma >> 1$ instead of φ in (2.1), we have

$$\begin{aligned}&-\int_{\mathcal{H}}u(\varphi^\sigma)_t d\eta dt+\int_{\mathcal{H}}a_{11}u|x|^\alpha(-\Delta_{\mathbb{H}})^{\frac{\alpha_1}{2}}\varphi^\sigma d\eta dt=\int_{\mathcal{H}}f(\eta,t)|v|^p\varphi^\sigma d\eta dt\\&+\int_{\mathbb{H}}u_0(\eta)\varphi^\sigma(\eta,0)d\eta.\end{aligned} \tag{2.8}$$

Invoking (1.2) and (2.5), we get

$$\begin{aligned}\int_{\mathcal{H}}f(\eta,t)|v|^p\varphi^\sigma d\eta dt\le\sigma\Bigg[&\int_{\mathcal{H}}|u|\varphi^{\sigma-1}|\varphi_t|d\eta dt\\&+\int_{\mathcal{H}}|x|^\alpha a_{11}|u|\varphi^{\sigma-1}|(-\Delta_{\mathbb{H}})^{\frac{\alpha_1}{2}}\varphi|d\eta dt\Bigg].\end{aligned} \tag{2.9}$$

Using the Holder inequality, we obtain

$$\int_{\mathcal{H}}f(\eta,t)|v|^p\varphi^\sigma d\eta dt\le C\left[(\mathcal{A}(k,r)+\mathcal{B}(k,\alpha,\alpha_1,r))\left(\int_{\mathcal{H}}|k||u|^r\varphi^\sigma d\eta dt\right)^{\frac{1}{r}}\right]. \tag{2.10}$$

Similarly, we have the estimates

$$\begin{aligned}\int_{\mathcal{H}}h(\eta,t)|w|^q\varphi^\sigma d\eta dt\le C&\left[\mathcal{A}(f,p)+\mathcal{B}(f,\beta,\alpha_3,p))\left(\int_{\mathcal{H}}|f||v|^p\varphi^\sigma d\eta dt\right)^{\frac{1}{p}}\right]\\&+C\left[\mathcal{B}(k,\beta,\alpha_2,r))\left(\int_{\mathcal{H}}|k||u|^r\varphi^\sigma d\eta dt\right)^{\frac{1}{r}}\right],\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\int_{\mathcal{H}}k(\eta,t)|u|^r\varphi^\sigma d\eta dt\quad&\le C(\mathcal{A}(h,q)+\mathcal{B}(h,\gamma,\alpha_6,q))\left(\int_{\mathcal{H}}|h||w|^q\varphi^\sigma d\eta dt\right)^{\frac{1}{q}}\\&+C\mathcal{B}(k,\gamma,\alpha_4,r))\left(\int_{\mathcal{H}}|k||u|^r\varphi^\sigma d\eta dt\right)^{\frac{1}{r}}\\&+C\mathcal{B}(f,\gamma,\alpha_5,p))\left(\int_{\mathcal{H}}|f||v|^p\varphi^\sigma d\eta dt\right)^{\frac{1}{p}},\end{aligned} \tag{2.12}$$

for some constant $C>0$. For the simplicity let us set

$$\mathcal{I}=\left(\int_{\mathcal{H}}|f||v|^p\varphi^\sigma d\eta dt\right)^{\frac{1}{p}}, \tag{2.13}$$

$$\mathcal{J} = \left(\int_{\mathcal{H}} |h||w|^q \varphi^\sigma d\eta dt \right)^{\frac{1}{q}}, \tag{2.14}$$

$$\mathcal{L} = \left(\int_{\mathcal{H}} |k||u|^r \varphi^\sigma d\eta dt \right)^{\frac{1}{r}}. \tag{2.15}$$

Here C denotes a constant that may change in different occurrences. Then the above inequalities may be written as

$$\mathcal{I}^p \leq C\Big[\mathcal{A}(k,r) + \mathcal{B}(k,\alpha,\alpha_1,r)\Big]\mathcal{L} \tag{2.16}$$

$$\mathcal{J}^q \leq C\Big[(\mathcal{A}(f,p) + \mathcal{B}(f,\beta,\alpha_3,p))\mathcal{I} + \mathcal{B}(k,\beta,\alpha_2,r))\mathcal{L}\Big] \tag{2.17}$$

$$\mathcal{L}^r \leq C\Big[(\mathcal{A}(h,q) + \mathcal{B}(h,\gamma,\alpha_6,q))\mathcal{J} + \mathcal{B}(k,\gamma,\alpha_4,r))\mathcal{L} + \mathcal{B}(f,\gamma,\alpha_5,p))\mathcal{I}\Big]. \tag{2.18}$$

Using the ε-Young inequality and the inequality

$$(a+b)^\theta \leq 2^{\theta-1}(a^\theta + b^\theta); \quad \theta \geq 1,$$

to the right hand side of (2.16), (2.17) and (2.18), we obtain

$$\mathcal{I}^p \leq C\Big[C_\varepsilon(\mathcal{A}^{r'}(k,r) + \mathcal{B}^{r'}(k,\alpha,\alpha_1,r)) + \varepsilon\mathcal{L}^r\Big] \tag{2.19}$$

$$\mathcal{J}^q \leq C\Big[C_\varepsilon(\mathcal{A}^{p'}(f,p) + \mathcal{B}^{p'}(f,\beta,\alpha_3,p)) + \varepsilon\mathcal{I}^p + C_\varepsilon\mathcal{B}^{r'}(k,\beta,\alpha_2,r)) + \varepsilon\mathcal{L}^r\Big] \tag{2.20}$$

and

$$\begin{aligned} \mathcal{L}^r \quad &\leq C\Big[C_\varepsilon(\mathcal{A}^{q'}(h,q) + \mathcal{B}^{q'}(h,\gamma,\alpha_6,q)) + \varepsilon\mathcal{J}^q + C_\varepsilon\mathcal{B}^{r'}(k,\gamma,\alpha_4,r)\Big] \\ &+C\Big[\varepsilon\mathcal{L}^r + C_\varepsilon\mathcal{B}^{p'}(f,\gamma,\alpha_5,p)) + \epsilon\mathcal{I}^p\Big]. \end{aligned} \tag{2.21}$$

Combining the above inequalities, we deduce the estimates

$$\begin{aligned} \mathcal{I}^p \quad &\leq C\Big[\mathcal{A}^{r'}(k,r) + \mathcal{B}^{r'}(k,\alpha,\alpha_1,r) + \mathcal{A}^{q'}(h,q)\Big] \\ &+C\Big[\mathcal{B}^{q'}(h,\gamma,\alpha_6,q) + \mathcal{A}^{p'}(f,p) + \mathcal{B}^{p'}(f,\beta,\alpha_3,p)\Big] \\ &+C\Big[\mathcal{B}^{r'}(k,\beta,\alpha_2,r) + \mathcal{B}^{r'}(k,\gamma,\alpha_4,r) + \mathcal{B}^{p'}(f,\gamma,\alpha_5,p)\Big], \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} \mathcal{J}^q \quad &\leq C\Big[\mathcal{A}^{r'}(k,r) + \mathcal{B}^{r'}(k,\alpha,\alpha_1,r) + \mathcal{A}^{q'}(h,q)\Big] \\ &+C\Big[\mathcal{B}^{q'}(h,\gamma,\alpha_6,q) + \mathcal{A}^{p'}(f,p) + \mathcal{B}^{p'}(f,\beta,\alpha_3,p)\Big] \\ &+C\Big[\mathcal{B}^{r'}(k,\beta,\alpha_2,r) + \mathcal{B}^{r'}(k,\gamma,\alpha_4,r) + \mathcal{B}^{p'}(f,\gamma,\alpha_5,p)\Big], \end{aligned} \tag{2.23}$$

also

$$\begin{aligned}\mathcal{L}^r &\le C\Big[\mathcal{A}^{r'}(k,r)+\mathcal{B}^{r'}(k,\alpha,\alpha_1,r)+\mathcal{A}^{q'}(h,q)\Big]\\ &+C\Big[\mathcal{B}^{q'}(h,\gamma,\alpha_6,q)+\mathcal{A}^{p'}(f,p)+\mathcal{B}^{p'}(f,\beta,\alpha_3,p)\Big]\\ &+C\Big[\mathcal{B}^{r'}(k,\beta,\alpha_2,r)+\mathcal{B}^{r'}(k,\gamma,\alpha_4,r)+\mathcal{B}^{p'}(f,\gamma,\alpha_5,p)\Big].\end{aligned} \tag{2.24}$$

Now, let us consider the test function

$$\varphi(\eta,t)=\Phi\left(\frac{\tau^2+|x|^4+|y|^4+t^2}{R^4}\right),$$

where $R>0$ and $\Phi\in\mathcal{D}([0,+\infty[)$ is the standard cut-off function

$$\Phi(r)=\begin{cases}1\,, & 0\le r\le 1\\ \searrow\,, & 1\le r\le 2\\ 0\,, & r\ge 2.\end{cases}$$

Set

$$\Omega=\Big\{(\eta,t)\in\mathbb{H}\times(0,\infty),\quad 0\le\tau^2+|x|^4+|y|^4+t^2\le 2R^4\Big\}.$$

At this stage, we use the scaled variables

$$\tilde\tau=R^{-2}\tau,\quad \tilde x=R^{-1}x,\quad \tilde y=R^{-1}y,\quad \tilde t=R^{-2}t. \tag{2.25}$$

We obtain easily the estimates

$$\begin{aligned}&|\mathcal{A}(k,r)|\le CR^{\sigma_1},\quad |\mathcal{A}(f,p)|\le CR^{\sigma_2},\quad |\mathcal{A}(h,q)|\le CR^{\sigma_3},\\ &|\mathcal{B}(k,\alpha;\alpha_1,r)|\le CR^{\theta_1},\ |\mathcal{B}(f,\beta,\alpha_3,p)|\le CR^{\theta_2},\ |\mathcal{B}(k,\beta,\alpha_2,r)|\le CR^{\theta_3},\\ &|\mathcal{B}(h,\gamma,\alpha_6,q)|\le CR^{\theta_4},\ |\mathcal{B}(k,\gamma,\alpha_4,r)|\le CR^{\theta_5},\ |\mathcal{B}(f,\gamma,\alpha_5,p)|\le CR^{\theta_6},\end{aligned} \tag{2.26}$$

with

$$\sigma_1=\frac{1}{r'}\left(\frac{-\xi r'}{r}-2r'+2N+4\right),\quad \sigma_2=\frac{1}{p'}\left(\frac{-\nu p'}{p}-2p'+2N+4\right),$$

$$\sigma_3=\frac{1}{q'}\left(\frac{-\mu q'}{q}-2q'+2N+4\right),$$

$$\theta_1=\frac{1}{r'}\left(\alpha r'-\frac{\xi r'}{r}-\alpha_1 r'+2N+4\right),\quad \theta_2=\frac{1}{p'}\left(\beta p'-\frac{\nu p'}{p}-\alpha_3 p'+2N+4\right),$$

$$\theta_3=\frac{1}{r'}\left(\beta r'-\frac{\xi r'}{r}-\alpha_2 r'+2N+4\right),\quad \theta_4=\frac{1}{q'}\left(\gamma q'-\frac{\mu q'}{q}-\alpha_6 q'+2N+4\right),$$

$$\theta_5=\frac{1}{r'}\left(\gamma r'-\frac{\xi r'}{r}-\alpha_4 r'+2N+4\right),\quad \theta_6=\frac{1}{p'}\left(\gamma p'-\frac{\nu p'}{p}-\alpha_5 p'+2N+4\right).$$

From (2.22)-(2.24) and the above estimates, we get

$$\mathcal{I}^p\le C\Big[R^{\sigma_1 r'}+R^{\theta_1 r'}+R^{\sigma_3 q'}+R^{\theta_4 q'}+R^{\sigma_2 p'}+R^{\theta_2 p'}+R^{\theta_3 r'}+R^{\theta_5 r'}+R^{\theta_6 p'}\Big], \tag{2.27}$$

$$\mathcal{J}^q\le C\Big[R^{\sigma_1 r'}+R^{\theta_1 r'}+R^{\sigma_3 q'}+R^{\theta_4 q'}+R^{\sigma_2 p'}+R^{\theta_2 p'}+R^{\theta_3 r'}+R^{\theta_5 r'}+R^{\theta_6 p'}\Big], \tag{2.28}$$

and

$$\mathcal{L}^r \leq C\left[R^{\sigma_1 r'} + R^{\theta_1 r'} + R^{\sigma_3 q'} + R^{\theta_4 q'} + R^{\sigma_2 p'} + R^{\theta_2 p'} + R^{\theta_3 r'} + R^{\theta_5 r'} + R^{\theta_6 p'}\right]. \tag{2.29}$$

From the condition (2.5) and by letting $R \to \infty$ in (2.27)-(2.29) and using the dominated convergence theorem, we arrive at

$$\int_{\mathcal{H}} |f||v|^p \varphi^\sigma d\eta dt = 0, \quad \text{whereupon} \quad v \equiv 0,$$

and

$$\int_{\mathcal{H}} |h||w|^q \varphi^\sigma d\eta dt = 0, \quad \text{whereupon} \quad w \equiv 0,$$

also

$$\int_{\mathcal{H}} |k||u|^r \varphi^\sigma d\eta dt = 0, \quad \text{whereupon} \quad u \equiv 0.$$

This is a contradiction. This completes the proof.

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