Palestine Journal of Mathematics

Vol. 8(2)(2019) , 159–168

Some generalizations of second submodules

H. Ansari-Toroghy and F. Farshadifar

Communicated by Ayman Badawi

MSC 2010 Classifications: 13C13, 13C99.

Keywords and phrases: 2-absorbing second, completely irreducible, strongly 2-absorbing second.

Abstract. In this paper, we will introduce two generalizations of second submodules of a module over a commutative ring and explore some basic properties of these classes of modules.

1 Introduction

Throughout this paper, R will denote a commutative ring with identity and "$\subset$" will denote the strict inclusion. Further, $\mathbb{Z}$ will denote the ring of integers.

Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [14]. A non-zero submodule S of M is said to be *second* if for each $a \in R$, the homomorphism $S \stackrel{a}{\to} S$ is either surjective or zero [19]. In this case $Ann_R(S)$ is a prime ideal of R.

Badawi gave a generalization of prime ideals in [9] and said such ideals 2- absorbing ideals. A proper ideal I of R is a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. He proved that I is a 2-absorbing ideal of R if and only if whenever I_1, I_2, and I_3 are ideals of R with $I_1I_2I_3 \subseteq I$, then $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$. The authors in [12] and [17] extended 2-absorbing ideals to 2-absorbing submodules. A proper submodule N of M is called a *2-absorbing submodule* of M if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$. Several authors investigated properties of 2-absorbing submodules, for example see [12, 17, 18].

A submodule N of an R-module M is called *strongly 2-absorbing* if $IJL \subseteq N$ for some ideals I, J of R and a submodule L of M, then $IL \subseteq N$ or $JL \subseteq N$ or $IJ \in (N :_R M)$ [13].

The purpose of this paper is to introduce the dual notions of 2-absorbing and strongly 2-absorbing submodules and obtain some related results.

2 2-absorbing second submodules

Let M be an R-module. A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [15].

We frequently use the following basic fact without further comment.

Remark 2.1. Let N and K be two submodules of an R-module M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$.

Definition 2.2. Let N be a non-zero submodule of an R-module M. We say that N is a *2-absorbing second submodule of* M if whenever $a, b \in R$, L is a completely irreducible submodule of M, and $abN \subseteq L$, then $aN \subseteq L$ or $bN \subseteq L$ or $ab \in Ann_R(N)$. This can be regarded as a dual notion of the 2-absorbing submodule.

Example 2.3. (a) The $\mathbb{Z}$-module $\mathbb{Z}_n$ is a 2-absorbing second submodule of $\mathbb{Z}_n$ if $n = p$ or $n = pq$, where p, q are prime integers.

[0]This research was in part supported by a grant from IPM (No. 94130048)

(b) Consider $n\mathbb{Z}$ as a submodule of the $\mathbb{Z}$-module $\mathbb{Z}$. Then $n = p_1^{\alpha_1}...p_t^{\alpha_t}$, where $p_i^{\alpha_i}$ $(1 \leq i \leq t)$ are distinct prime integers. For $p_1 \in \mathbb{Z}$ and completely irreducible submodule $p_1^{\alpha_1+2}\mathbb{Z}$ of $\mathbb{Z}$, we have $p_1.p_1.(n\mathbb{Z}) \subseteq p_1^{\alpha_1+2}\mathbb{Z}$ but $p_1.(n\mathbb{Z}) \not\subseteq p_1^{\alpha_1+2}\mathbb{Z}$ and $p_1.p_1 = p_1^2 \notin Ann_{\mathbb{Z}}(n\mathbb{Z}) = (0)$. Therefore, the $\mathbb{Z}$-module $\mathbb{Z}$ has no 2-absorbing second submodule.

A non-zero R-module M is said to be *secondary* if for each $a \in R$ the endomorphism of M given by multiplication by a is either surjective or nilpotent [16].

Theorem 2.4. Let M be an R-module. Then we have the following.

(a) If either N is a second submodule of M or N is a sum of two second submodules of M, then N is 2-absorbing second.

(b) If N is a secondary submodule of M and $R/Ann_R(N)$ has no non-zero nilpotent element, then N is 2-absorbing second.

Proof. (a) The first assertion is clear. To see the second assertion, let N_1 and N_2 be two second submodules of M. We show that $N_1 + N_2$ is a 2-absorbing second submodule of M. Assume that $a, b \in R$, L is a completely irreducible submodule of M, and $ab(N_1 + N_2) \subseteq L$. Since N_1 is second, $abN_1 = 0$ or $N_1 \subseteq L$ by [3, 2.10]. Similarly, $abN_2 = 0$ or $N_2 \subseteq L$. If $abN_1 = 0 = abN_2$ (resp. $N_1 \subseteq L$ and $N_2 \subseteq L$), then we are done. Now let $abN_1 = 0$ and $N_2 \subseteq L$. Then $aN_1 = 0$ or $bN_1 = 0$ because $Ann_R(N_1)$ is a prime ideal of R. If $aN_1 = 0$, then $a(N_1 + N_2) \subseteq aN_1 + N_2 \subseteq N_2 \subseteq L$. Similarly, if $bN_1 = 0$, we get $b(N_1 + N_2) \subseteq L$ as desired.

(b) Let $a, b \in R$, L be a completely irreducible submodule of M, and $abN \subseteq L$. Then if $aN \subseteq L$ or $bN \subseteq L$, we are done. Let $aN \not\subseteq L$ and $bN \not\subseteq L$. Then $a, b \in \sqrt{Ann_R(N)}$. Thus, $(ab)^s \in Ann_R(N)$ for some positive integer s. Therefore, $ab \in Ann_R(N)$ because $R/Ann_R(N)$ has no non-zero nilpotent element. □

Lemma 2.5. Let I be an ideal of R and N be a 2-absorbing second submodule of M. If $a \in R$, L is a completely irreducible submodule of M, and $IaN \subseteq L$, then $aN \subseteq L$ or $IN \subseteq L$ or $Ia \in Ann_R(N)$.

Proof. Let $aN \not\subseteq L$ and $Ia \notin Ann_R(N)$. Then there exists $b \in I$ such that $abN \neq 0$. Now as N is a 2-absorbing second submodule of M, $baN \subseteq L$ implies that $bN \subseteq L$. We show that $IN \subseteq L$. To see this, let c be an arbitrary element of I. Then $(b + c)aN \subseteq L$. Hence, either $(b+c)N \subseteq L$ or $(b+c)a \in Ann_R(N)$. If $(b+c)N \subseteq L$, then since $bN \subseteq L$ we have $cN \subseteq L$. If $(b + c)a \in Ann_R(N)$, then $ca \notin Ann_R(N)$, but $caN \subseteq L$. Thus $cN \subseteq L$. Hence, we conclude that $IN \subseteq L$. □

Lemma 2.6. Let I and J be two ideals of R and N be a 2-absorbing second submodule of M. If L is a completely irreducible submodule of M and $IJN \subseteq L$, then $IN \subseteq L$ or $JN \subseteq L$ or $IJ \subseteq Ann_R(N)$.

Proof. Let $IN \not\subseteq L$ and $JN \not\subseteq L$. We show that $IJ \subseteq Ann_R(N)$. Assume that $c \in I$ and $d \in J$. By assumption there exists $a \in I$ such that $aN \not\subseteq L$ but $aJN \subseteq L$. Now Lemma 2.6 shows that $aJ \subseteq Ann_R(N)$ and so $(I \setminus (L :_R N))J \subseteq Ann_R(N)$. Similarly there exists $b \in (J \setminus (L :_R N))$ such that $Ib \subseteq Ann_R(N)$ and also $I(J \setminus (L :_R N)) \subseteq Ann_R(N)$. Thus we have $ab \in Ann_R(N)$, $ad \in Ann_R(N)$ and $cb \in Ann_R(N)$. As $a+c \in I$ and $b+d \in J$, we have $(a+c)(b+d)N \subseteq L$. Therefore, $(a + c)N \subseteq L$ or $(b + d)N \subseteq L$ or $(a + c)(b + d) \in Ann_R(N)$. If $(a + c)N \subseteq L$, then $cN \not\subseteq L$. Hence $c \in I \setminus (L :_R N)$ which implies that $cd \in Ann_R(N)$. Similarly if $(b + d)N \subseteq L$, we can deduce that $cd \in Ann_R(N)$. Finally if $(a + c)(b + d) \in Ann_R(N)$, then $ab + ad + cb + cd \in Ann_R(N)$ so that $cd \in Ann_R(N)$. Therefore, $IJ \subseteq Ann_R(N)$. □

Corollary 2.7. Let M be an R-module and N be a 2-absorbing second submodule of M. Then IN is a 2-absorbing second submodules of M for all ideals I of R with $I \not\subseteq Ann_R(N)$.

Proof. Let I be an ideal of R with $I \not\subseteq Ann_R(N)$, $a, b \in R$, L be a completely irreducible submodule of M, and $abIN \subseteq L$. Then $aN \subseteq L$ or $bIN \subseteq L$ or $abIN = 0$ by Lemma 2.5. If $bIN \subseteq L$ or $abIN = 0$, then we are done. If $aN \subseteq L$, then $aIN \subseteq aN$ implies that $aIN \subseteq L$, as needed. □

An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [10].

Corollary 2.8. Let M be a multiplication 2-absorbing second R-module. Then every non-zero submodule of M is a 2-absorbing second submodule of M.

Proof. This follows from Corollary 2.7. □

The following example shows that the condition "M is a multiplication module" in Corollary 2.8 can not be omitted.

Example 2.9. For any prime integer p, let $M = \mathbb{Z}_{p^\infty}$ and $N = \langle 1/p^3 + \mathbb{Z}\rangle$. Then clearly, M is a 2-absorbing second $\mathbb{Z}$-module but $p^2\langle 1/p^3 + \mathbb{Z}\rangle \subseteq \langle 1/p + \mathbb{Z}\rangle$ implies that N is not a 2-absorbing second submodule of M.

We recall that an R-module M is said to be a *cocyclic module* if $Soc_R(M)$ is a large and simple submodule of M [21]. (Here $Soc_R(M)$ denotes the sum of all minimal submodules of M.). A submodule L of M is a completely irreducible submodule of M if and only if M/L is a cocyclic R-module [15].

Proposition 2.10. Let N be a 2-absorbing second submodule of an R-module M. Then we have the following.

(a) If L is a completely irreducible submodule of M such that $N \not\subseteq L$, then $(L :_R N)$ is a 2-absorbing ideal of R.

(b) If M is a cocyclic module, then $Ann_R(N)$ is a 2-absorbing ideal of R.

(c) If $a \in R$, then $a^n N = a^{n+1}N$, for all $n \geq 2$.

(d) If $Ann_R(N)$ is a prime ideal of R, then $(L :_R N)$ is a prime ideal of R for all completely irreducible submodules L of M such that $N \not\subseteq L$.

Proof. (a) Since $N \not\subseteq L$, we have $(L :_R N) \neq R$. Let $a, b, c \in R$ and $abc \in (L :_R N)$. Then $abN \subseteq (L :_M c)$. Thus $aN \subseteq (L :_M c)$ or $bN \subseteq (L :_M c)$ or $abN = 0$ because by [8, 2.1], $(L :_M c)$ is a completely irreducible submodule of M. Therefore, $ac \in (L :_R N)$ or $bc \in (L :_R N)$ or $ab \in (L :_R N)$.

(b) Since M is cocyclic, the zero submodule of M is a completely irreducible submodule of M. Thus the result follows from part (a).

(c) It is enough to show that $a^2N = a^3N$. It is clear that $a^3N \subseteq a^2N$. Let L be a completely irreducible submodule of M such that $a^3N \subseteq L$. Then $a^2N \subseteq (L :_R a)$. Since N is 2-absorbing second submodule and $(L :_R a)$ is a completely irreducible submodule of M by [8, 2.1], $aN \subseteq (L :_R a)$ or $a^2N = 0$. Therefore, $a^2N \subseteq L$. This implies that $a^2N \subseteq a^3N$.

(d) Let $a, b \in R$, L be a completely irreducible submodule of M such that $N \not\subseteq L$, and $ab \in (L :_R N)$. Then $aN \subseteq L$ or $bN \subseteq L$ or $abN = 0$. If $abN = 0$, then by assumption, $aN = 0$ or $bN = 0$. Thus in any cases we get that, $aN \subseteq L$ or $bN \subseteq L$. □

Theorem 2.11. Let N be a 2-absorbing second submodule of an R-module M. Then we have the following.

(a) If $\sqrt{Ann_R(N)} = P$ for some prime ideal P of R and L is a completely irreducible submodule of M such that $N \not\subseteq L$, then $\sqrt{(L :_R N)}$ is a prime ideal of R containing P.

(b) If $\sqrt{Ann_R(N)} = P \cap Q$ for some prime ideals P and Q of R, L is a completely irreducible submodule of M such that $N \not\subseteq L$, and $P \subseteq \sqrt{(L :_R N)}$, then $\sqrt{(L :_R N)}$ is a prime ideal of R.

Proof. (a) Assume that $a, b \in R$ and $ab \in \sqrt{(L :_R N)}$. Then there is a positive integer t such that $a^t b^t N \subseteq L$. By hypotheses, N is a 2-absorbing second submodule of M, thus $a^t N \subseteq L$ or $b^t N \subseteq L$ or $a^t b^t \in Annn_R(N)$. If either $a^t N \subseteq L$ or $b^t N \subseteq L$, we are done. So assume that $a^t b^t \in Ann_R(N)$. Then $ab \in \sqrt{Ann_R(N)} = P$ and so $a \in P$ or $b \in P$. It is clear that $P = \sqrt{Ann_R(N)} \subseteq \sqrt{(L :_R N)}$. Therefore, $a \in \sqrt{(L :_R N)}$ or $b \in \sqrt{(L :_R N)}$.

(b) The proof is similar to that of part (a). □

Proposition 2.12. Let M be an R-module and let $\{K_i\}_{i\in I}$ be a chain of 2-absorbing second submodules of M. Then $\cup_{i\in I}K_i$ is a 2-absorbing second submodule of M.

Proof. Let $a, b \in R$, L be a completely irreducible submodule of M, and $ab(\cup_{i\in I}K_i) \subseteq L$. Assume that $a(\cup_{i\in I}K_i) \not\subseteq L$ and $b(\cup_{i\in I}K_i) \not\subseteq L$. Then there are $m, n \in I$, where $aK_n \not\subseteq L$ and $bK_m \not\subseteq L$. Hence, for every $K_n \subseteq K_s$ and $K_m \subseteq K_d$ we have $aK_s \not\subseteq L$ and $bK_d \not\subseteq L$. Therefore, for each submodule K_h such that $K_n \subseteq K_h$ and $K_m \subseteq K_h$ we have $abK_h = 0$. Hence $ab(\cup_{i\in I}K_i) = 0$, as needed. □

Definition 2.13. We say that a 2-absorbing second submodule N of an R-module M is a *maximal 2-absorbing second submodule* of a submodule K of M, if $N \subseteq K$ and there does not exist a 2-absorbing second submodule H of M such that $N \subset H \subset K$.

Lemma 2.14. Let M be an R-module. Then every 2-absorbing second submodule of M is contained in a maximal 2-absorbing second submodule of M.

Proof. This is proved easily by using Zorn's Lemma and Proposition 2.12. □

Theorem 2.15. Every Artinian R-module M has only a finite number of maximal 2-absorbing second submodules.

Proof. Suppose that the result is false. Let Σ denote the collection of non-zero submodules N of M such that N has an infinite number of maximal 2-absorbing second submodules. The collection Σ is non-empty because $M \in \Sigma$ and hence has a minimal member, S say. Then S is not 2-absorbing second submodule. Thus there exist $a, b \in R$ and a completely irreducible submodule L of M such that $abS \subseteq L$ but $aS \not\subseteq L$, $bS \not\subseteq L$, and $abS \neq 0$. Let V be a maximal 2-absorbing second submodule of M contained in S. Then $aV \subseteq L$ or $bV \subseteq L$ or $abV = 0$. Thus $V \subseteq (L :_M a)$ or $V \subseteq (L :_M b)$ or $V \subseteq (0 :_M ab)$. Therefore, $V \subseteq (L :_S a)$ or $V \subseteq (L :_S b)$ or $V \subseteq (0 :_S ab)$. By the choice of S, the modules $(L :_S a)$, $(L :_S b)$, and $(0 :_S ab)$ have only finitely many maximal 2-absorbing second submodules. Therefore, there is only a finite number of possibilities for the module S which is a contradiction. □

3 Strongly 2-absorbing second submodules

Definition 3.1. Let N be a non-zero submodule of an R-module M. We say that N is a *strongly 2-absorbing second submodule of* M if whenever $a, b \in R$, L_1, L_2 are completely irreducible submodules of M, and $abN \subseteq L_1 \cap L_2$, then $aN \subseteq L_1 \cap L_2$ or $bN \subseteq L_1 \cap L_2$ or $ab \in Ann_R(N)$. This can be regarded as a dual notion of the strongly 2-absorbing submodule.

Clearly every strongly 2-absorbing second submodule is a 2-absorbing second submodule. In [18, 2.3], the authors show that N is a 2-absorbing submodule of an R-module M if and only if N is a strongly 2-absorbing submodule of M. Dually, this motivates the following question.

Question 3.2. Let M be an R-module. Is every 2-absorbing second submodule of M a strongly 2-absorbing second submodule of M?

Theorem 3.3. Let N be a submodule of an R-module M. The following statements are equivalent:

(a) N is a strongly 2-absorbing second submodule of M;

(b) If $N \neq 0$, $IJN \subseteq K$ for some ideals I, J of R and a submodule K of M, then $IN \subseteq K$ or $JN \subseteq K$ or $IJ \in Ann_R(N)$;

(c) $N \neq 0$ and for each $a, b \in R$, we have $abN = aN$ or $abN = bN$ or $abN = 0$.

Proof. $(a) \Rightarrow (b)$. Assume that $IJN \subseteq K$ for some ideals I, J of R, a submodule K of M, and $IJ \not\subseteq Ann_R(N)$. Then by Lemma 2.6, for all completely irreducible submodules L of M with $K \subseteq L$ either $IN \subseteq L$ or $JN \subseteq L$. If $IN \subseteq L$ (resp. $JN \subseteq L$) for all completely irreducible submodules L of M with $K \subseteq L$, we are done. Now suppose that L_1 and L_2 are two completely irreducible submodules of M with $K \subseteq L_1$, $K \subseteq L_2$, $IN \not\subseteq L_1$, and $JN \not\subseteq L_2$. Then $IN \subseteq L_2$

and $JN \subseteq L_1$. Since $IJN \subseteq L_1 \cap L_2$, we have either $IN \subseteq L_1 \cap L_2$ or $JN \subseteq L_1 \cap L_2$. As $IN \subseteq L_1 \cap L_2$, we have $IN \subseteq L_1$ which is a contradiction. Similarly from $JN \subseteq L_1 \cap L_2$ we get a contradiction.

$(b) \Rightarrow (a)$. This is clear.

$(a) \Rightarrow (c)$. By part (a), $N \neq 0$. Let $a, b \in R$. Then $abN \subseteq abN$ implies that $aN \subseteq abN$ or $bN \subseteq abN$ or $abN = 0$. Thus $abN = aN$ or $abN = bN$ or $abN = 0$.

$(c) \Rightarrow (a)$. This is clear. □

Lemma 3.4. Let M be an R-module, $N \subset K$ be two submodules of M, and K be a strongly 2-absorbing second submodule of M. Then K/N is a strongly 2-absorbing second submodule of M/N.

Proof. This is straightforward. □

Proposition 3.5. Let N be a strongly 2-absorbing second submodule of an R-module M. Then we have the following.

(a) $Ann_R(N)$ is a 2-absorbing ideal of R.

(b) If K is a submodule of M such that $N \not\subseteq K$, then $(K :_R N)$ is a 2-absorbing ideal of R.

(c) If I is an ideal of R, then $I^n N = I^{n+1} N$, for all $n \geq 2$.

(d) If $(L_1 \cap L_2 :_R N)$ is a prime ideal of R for all completely irreducible submodules L_1 and L_2 of M such that $N \not\subseteq L_1 \cap L_2$, then $Ann_R(N)$ is a prime ideal of R.

Proof. (a) Let $a, b, c \in R$ and $abc \in Ann_R(N)$. Then $abN \subseteq abN$ implies that $aN \subseteq abN$ or $bN \subseteq abN$ or $abN = 0$ by Theorem 3.3. If $abN = 0$, then we are done. If $aN \subseteq abN$, then $caN \subseteq cabN = 0$. In other case, we do the same.

(b) Let $a, b, c \in R$ and $abc \in (K :_R N)$. Then $acN \subseteq K$ or $bcN \subseteq K$ or $abcN = 0$. If $acN \subseteq K$ or $bcN \subseteq K$, then we are done. If $abcN = 0$, then the result follows from part (a).

(c) It is enough to show that $I^2 N = I^3 N$. It is clear that $I^3 N \subseteq I^2 N$. Since N is strongly 2-absorbing second submodule, $I^3 N \subseteq I^3 N$ implies that $I^2 N \subseteq I^3 N$ or $IN \subseteq I^3 N$ or $I^3 N = 0$ by Theorem 3.3. If $I^2 N \subseteq I^3 N$ or $IN \subseteq I^3 N$, then we are done. If $I^3 N = 0$, then the result follows from part (a).

(d) Suppose that $a, b \in R$ and $abN = 0$. Assume contrary that $aN \neq 0$ and $bN \neq 0$. Then there exist completely irreducible submodules L_1 and L_2 of M such that $aN \not\subseteq L_1$ and $bN \not\subseteq L_2$. Now since $(L_1 \cap L_2 :_R N)$ is a prime ideal of R, $0 = abN \subseteq L_1 \cap L_2$ implies that $bN \subseteq L_1 \cap L_2$ or $aN \subseteq L_1 \cap L_2$. In any cases, we have a contradiction. □

Remark 3.6. ([9, Theorem 2.4]). If I is a 2-absorbing ideal of R, then one of the following statements must hold:

(a) $\sqrt{I} = P$ is a prime ideal of R such that $P^2 \subseteq I$;

(b) $\sqrt{I} = P \cap Q$, $PQ \subseteq I$, and $\sqrt{I}^2 \subseteq I$ where P and Q are the only distinct prime ideals of R that are minimal over I.

Theorem 3.7. If N is a strongly 2-absorbing second submodule of M and $N \not\subseteq K$, then either $(K :_R N)$ is a prime ideal of R or there exists an element $a \in R$ such that $(K :_R aN)$ is a prime ideal of R.

Proof. By Preposition 3.5 and Remark 3.6, we have one of the following two cases.

(a) Let $\sqrt{Ann_R(N)} = P$, where P is a prime ideal of R. We show that $(K :_R N)$ is a prime ideal of R when $P \subseteq (K :_R N)$. Assume that $a, b \in R$ and $ab \in (K :_R N)$. Hence $aN \subseteq K$ or $bN \subseteq K$ or $ab \in Ann_R(N)$. If either $aN \subseteq K$ or $bN \subseteq K$, we are done. Now assume that $ab \in Ann_R(N)$. Then $ab \in P$ and so $a \in P$ or $b \in P$. Thus, $a \in (K :_R N)$ or $b \in (K :_R N)$ and the assertion follows. If $P \not\subseteq (K :_R N)$, then there exists $a \in P$ such that $aN \not\subseteq K$. By Remark 3.6, $P^2 \subseteq Ann_R(N) \subseteq (K :_R N)$, thus $P \subseteq (K :_R aN)$. Now a similar argument shows that $(K :_R aN)$ is a prime ideal of R.

(b) Let $\sqrt{Ann_R(N)} = P \cap Q$, where P and Q are distinct prime ideals of R. If $P \subseteq (K :_R N)$, then the result follows by a similar proof to that of part (a). Assume that $P \not\subseteq (K :_R N)$. Then there exists $a \in P$ such that $aN \not\subseteq K$. By Remark 3.6, we have $PQ \subseteq Ann_R(N) \subseteq (K :_R N)$. Thus, $Q \subseteq (K :_R aN)$ and the result follows by a similar proof to that of part (a).

□

Let M be an R-module. A prime ideal P of R is said to be a *coassociated prime* of M if there exists a cocyclic homomorphic image T of M such that $P = Ann_R(T)$. The set of all coassociated prime ideals of M is denoted by $Coass_R(M)$ [20].

Theorem 3.8. Let N be a strongly 2-absorbing second submodule of an R-module M. Then we have the following.

(a) If $\sqrt{Ann_R(N)} = P$ for some prime ideal P of R, L_1 and L_2 are completely irreducible submodules of M such that $N \not\subseteq L_1$, and $N \not\subseteq L_2$, then either $\sqrt{(L_1 :_R N)} \subseteq \sqrt{(L_2 :_R N)}$ or $\sqrt{(L_2 :_R N)} \subseteq \sqrt{(L_1 :_R N)}$. Hence, $Coass_R(N)$ is a totally ordered set.

(b) If $\sqrt{Ann_R(N)} = P \cap Q$ for some prime ideals P and Q of R, L_1 and L_2 are completely irreducible submodules of M such that $N \not\subseteq L_1$ and $N \not\subseteq L_2$, and $P \subseteq \sqrt{(L_1 :_R N)} \cap \sqrt{(L_2 :_R N)}$, then either $\sqrt{(L_1 :_R N)} \subseteq \sqrt{(L_2 :_R N)}$ or $\sqrt{(L_2 :_R N)} \subseteq \sqrt{(L_1 :_R N)}$. Hence, $Coass_R(N)$ is the union of two totally ordered sets.

Proof. (a) Assume that $\sqrt{(L_1 :_R N)} \not\subseteq \sqrt{(L_2 :_R N)}$. We show that $\sqrt{(L_2 :_R N)} \subseteq \sqrt{(L_2 :_R N)}$. Suppose that $a \in \sqrt{(L_1 :_R N)}$ and $b \in \sqrt{(L_2 :_R N)}$. Then there exists a positive integer s such that $a^sN \subseteq L_1$, $b^sN \subseteq L_2$, and $b^sN \not\subseteq L_1$. If $a^sN \subseteq L_1 \cap L_2$, then $a^sN \subseteq L_2$ and so $a \in \sqrt{(L_2 :_R N)}$. Now assume that $a^sN \not\subseteq L_1 \cap L_2$. Then $a^sb^s \in Ann_R(N)$ because $a^sb^sN \subseteq L_1 \cap L_2$, $a^sN \not\subseteq L_1 \cap Ł_2$, and $b^sN \not\subseteq L_1 \cap L_2$. Thus, $ab \in P$. If $b \in P$, then $b^sN \subseteq L_1$ which is a contradiction. Hence $a \in P$ and so $a \in \sqrt{(L_2 :_R N)}$. Let $P, Q \in Coass_R(N)$. Then there exist completely irreducible submodules L_1 and L_2 of M such that $P = (L_1 :_R N)$ and $Q = (L_2 :_R N)$. Thus, $P = \sqrt{(L_1 :_R N)}$ and $Q = \sqrt{(L_2 :_R N)}$. Hence, either $P \subseteq Q$ or $Q \subseteq P$ and this completes the proof.

(b) The proof is similar to that of part (a). □

In [17, 2.10], it is shown that, if R be a Noetherian ring, M a finitely generated multiplication R-module, N a proper submodule of M such that $Ass_R(M/N)$ is a totally ordered set, and $(N :_R M)$ is a 2-absorbing ideal of R, then N is a 2-absorbing submodule of M. In the following theorem we see that some of this conditions are redundant.

Theorem 3.9. Let N be a submodule of a multiplication R-module M such that $(N :_R M)$ is a 2-absorbing ideal of R. Then N is a 2-absorbing submodule of M.

Proof. As $(N :_R M) \neq R$, $N \neq M$. Let $a, b \in R$, $m \in M$, and $abm \in N$. Since M is a multiplication R-module, there exists an ideal I of R such that $Rm = IM$. Thus $abIM \subseteq N$. Hence, $abI \subseteq (N :_R M)$. Now by assumption, $ab \in (N :_R M)$ or $aI \subseteq (N :_R M)$ or $bI \subseteq (N :_R M)$. Therefore, $ab \in (N :_R M)$ or $aIM \subseteq N$ or $bIM \subseteq N$. Thus $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. □

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$, equivalently, for each submodule N of M, we have $N = (0 :_M Ann_R(N))$ [7].

Theorem 3.10. Let N be a submodule of a comultiplication R-module M. Then we have the following.

(a) If $Ann_R(N)$ is a 2-absorbing ideal of R, then N is a strongly 2-absorbing second submodule of M. In particular, N is a 2-absorbing second submodule of M.

(b) If M is a cocyclic module and N is a 2-absorbing second submodule of M, then N is a strongly 2-absorbing second submodule of M.

Proof. (a) Let $a, b \in R$, K be a submodule of M, and $abN \subseteq K$. Then we have $Ann_R(K)abN = 0$. So by assumption, $Ann_R(K)aN = 0$ or $Ann_R(K)bN = 0$ or $abN = 0$. If $abN = 0$, we are done. If $Ann_R(K)aN = 0$ or $Ann_R(K)bN = 0$, then $Ann_R(K) \subseteq Ann_R(aN)$ or $Ann_R(K) \subseteq Ann_R(bN)$. Hence, $aN \subseteq K$ or $bN \subseteq K$ since M is a comultiplication R-module.

(b) By Proposition 2.10, $Ann_R(N)$ is a 2-absorbing ideal of R. Thus the result follows from part (a). □

The following example shows that Theorem 3.10 (a) is not satisfied in general.

Example 3.11. By [7, 3.9], the $\mathbb{Z}$-module $\mathbb{Z}$ is not a comultiplication $\mathbb{Z}$-module. The submodule $N = p\mathbb{Z}$ of $\mathbb{Z}$, where p is a prime number, is not strongly 2-absorbing second submodule. But $Ann_{\mathbb{Z}}(p\mathbb{Z}) = 0$ is a 2-absorbing ideal of R.

For a submodule N of an R-module M the the *second radical* (or second socle) of N is defined as the sum of all second submodules of M contained in N and it is denoted by $sec(N)$ (or $soc(N)$). In case N does not contain any second submodule, the second radical of N is defined to be (0) (see [11] and [2]).

Theorem 3.12. Let M be a finitely generated comultiplication R-module. If N is a strongly 2-absorbing second submodule of M, then $sec(N)$ s a strongly 2-absorbing second submodule of M.

Proof. Let N be a strongly 2-absorbing second submodule of M. By Proposition 3.5 (a), $Ann_R(N)$ is a 2-absorbing ideal of R. Thus by [9, 2.1], $\sqrt{Ann_R(N)}$ is a 2-absorbing ideal of R. By [5, 2.12], $Ann_R(sec(N)) = \sqrt{Ann_R(N)}$. Therefore, $Ann_R(sec(N))$ is a 2-absorbing ideal of R. Now the result follows from Theorem 3.10 (a). □

Lemma 3.13. Let $f : M \to \acute{M}$ be a monomorphism of R-modules. If $\acute{L}$ is a completely irreducible submodule of $f(M)$, then $f^{-1}(\acute{L})$ is a completely irreducible submodule of M.

Proof. This is strighatforward. □

Lemma 3.14. Let $f : M \to \acute{M}$ be a monomorphism of R-modules. If L is a completely irreducible submodule of M, then $f(L)$ is a completely irreducible submodule of $f(M)$.

Proof. Let $\{\acute{N}_i\}_{i\in I}$ be a family of submodules of $f(M)$ such that $f(L) = \cap_{i\in I}\acute{N}_i$. Then $L = f^{-1}f(L) = f^{-1}(\cap_{i\in I}\acute{N}_i) = \cap_{i\in I}f^{-1}(\acute{N}_i)$. This implies that there exists $i \in I$ such that $L = f^{-1}(\acute{N}_i)$ since L is a completely irreducible submodule of M. Therefore, $f(L) = ff^{-1}(\acute{N}_i) = f(M) \cap \acute{N}_i = \acute{N}_i$, as requested. □

Theorem 3.15. Let $f : M \to \acute{M}$ be a monomorphism of R-modules. Then we have the following.

(a) If N is a strongly 2-absorbing second submodule of M, then $f(N)$ is a 2-absorbing second submodule of $\acute{M}$.

(b) If N is a 2-absorbing second submodule of M, then $f(N)$ is a 2-absorbing second submodule of $f(M)$.

(c) If $\acute{N}$ is a strongly 2-absorbing second submodule of $\acute{M}$ and $\acute{N} \subseteq f(M)$, then $f^{-1}(\acute{N})$ is a 2-absorbing second submodule of M.

(d) If $\acute{N}$ is a 2-absorbing second submodule of $f(M)$, then $f^{-1}(\acute{N})$ is a 2-absorbing second submodule of M.

Proof. (a) Since $N \neq 0$ and f is a monomorphism, we have $f(N) \neq 0$. Let $a, b \in R$, $\acute{L}$ be a completely irreducible submodule of $\acute{M}$, and $abf(N) \subseteq \acute{L}$. Then $abN \subseteq f^{-1}(\acute{L})$. As N is strongly 2-absorbing second submodule, $aN \subseteq f^{-1}(\acute{L})$ or $bN \subseteq f^{-1}(\acute{L})$ or $abN = 0$. Therefore,

$$af(N) \subseteq f(f^{-1}(\acute{L})) = f(M) \cap \acute{L} \subseteq \acute{L}$$

or

$$bf(N) \subseteq f(f^{-1}(\acute{L})) = f(M) \cap \acute{L} \subseteq \acute{L}$$

or $abf(N) = 0$, as needed.

(b) This is similar to the part (a).

(c) If $f^{-1}(\acute{N}) = 0$, then $f(M) \cap \acute{N} = ff^{-1}(\acute{N}) = f(0) = 0$. Thus $\acute{N} = 0$, a contradiction. Therefore, $f^{-1}(\acute{N}) \neq 0$. Now let $a, b \in R$, L be a completely irreducible submodule of M, and $abf^{-1}(\acute{N}) \subseteq L$. Then

$$ab\acute{N} = ab(f(M) \cap \acute{N}) = abff^{-1}(\acute{N}) \subseteq f(L).$$

As $\acute{N}$ is strongly 2-absorbing second submodule, $a\acute{N} \subseteq f(L)$ or $b\acute{N} \subseteq f(L)$ or $ab\acute{N} = 0$. Hence $af^{-1}(\acute{N}) \subseteq f^{-1}f(L) = L$ or $bf^{-1}(\acute{N}) \subseteq f^{-1}f(L) = L$ or $abf^{-1}(\acute{N}) = 0$, as desired.

(d) By using Lemma 3.14, this is similar to the part (c). □

Corollary 3.16. Let M be an R-module and $N \subseteq K$ be two submodules of M. Then we have the following.

(a) If N is a strongly 2-absorbing second submodule of K, then N is a 2-absorbing second submodule of M.

(b) If N is a strongly 2-absorbing second submodule of M, then N is a 2-absorbing second submodule of K.

Proof. This follows from Theorem 3.15 by using the natural monomorphism $K \to M$. □

A non-zero submodule N of an R-module M is said to be a *weakly second submodule* of M if $rsN \subseteq K$, where $r, s \in R$ and K is a submodule of M, implies either $rN \subseteq K$ or $sN \subseteq K$ [4].

Proposition 3.17. Let N ba a non-zero submodule of an R-module M. Then N is a weakly second submodule of M if and only if N is a strongly 2-absorbing second submodule of M and $Ann_R(N)$ is a prime ideal of R.

Proof. Clearly, if N is a weakly second submodule of M, then N is a strongly 2-absorbing second submodule of M and by [4, 3.3], $Ann_R(N)$ is a prime ideal of R. For the converse, let $abN \subseteq H$ for some $a, b \in R$ and submodule K of M such that neither $aN \subseteq H$ nor $bN \subseteq H$. Then $ab \in Ann_R(N)$ and so either $a \in Ann_R(N)$ or $b \in Ann_R(N)$. This contradiction shows that N is weakly second. □

The following example shows that the two concepts of strongly 2-absorbing second submodule and weakly second submodule are different in general.

Example 3.18. Let p, q be two prime numbers, $N =< 1/p + \mathbb{Z} >$, and $K =< 1/q + \mathbb{Z} >$. Then $N \oplus K$ is not a weakly second submodule of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty}$. But $N \oplus K$ is a strongly 2-absorbing second submodule of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty}$.

Proposition 3.19. Let M be an R-module and $\{K_i\}_{i \in I}$ be a chain of strongly 2-absorbing second submodules of M. Then $\cup_{i \in I} K_i$ is a strongly 2-absorbing second submodule of M.

Proof. Let $a, b \in R$, H be a submodule of M, and $ab(\cup_{i \in I} K_i) \subseteq H$. Assume that $a(\cup_{i \in I} K_i) \not\subseteq H$ and $b(\cup_{i \in I} K_i) \not\subseteq H$. Then there are $m, n \in I$, where $aK_n \not\subseteq H$ and $bK_m \not\subseteq H$. Hence, for every $K_n \subseteq K_s$ and $K_m \subseteq K_d$, we have that $aK_s \not\subseteq H$ and $bK_d \not\subseteq H$. Therefore, for each submodule K_h such that $K_n \subseteq K_h$ and $K_m \subseteq K_h$ we have $abK_h = 0$. Hence $ab(\cup_{i \in I} K_i) = 0$, as needed. □

Definition 3.20. We say that a 2-absorbing second submodule N of an R-module M is a *maximal strongly 2-absorbing second submodule* of a submodule K of M, if $N \subseteq K$ and there does not exist a strongly 2-absorbing second submodule H of M such that $N \subset H \subset K$.

Lemma 3.21. Let M be an R-module. Then every strongly 2-absorbing second submodule of M is contained in a maximal strongly 2-absorbing second submodule of M.

Proof. This is proved easily by using Zorn's Lemma and Proposition 3.19. □

Definition 3.22. Let N be a submodule of an R-module M. We define the *strongly 2-absorbing second radical* of N as the sum of all strongly 2-absorbing second submodules of M contained in N and we denote it by $s.2.sec(N)$. In case N does not contain any strongly 2-absorbing second submodule, the strongly 2-absorbing second radical of N is defined to be (0). We say that $N \neq 0$ is a *strongly 2-absorbing second radical submodule of* M if $s.2.sec(N) = N$.

Proposition 3.23. Let N and K be two submodules of an R-module M. Then we have the following.

(a) If $N \subseteq K$, then $s.2.sec(N) \subseteq s.2.sec(K)$.

(b) $s.2.sec(N) \subseteq N$.

(c) $s.2.sec(s.2.sec(N)) = s.2.sec(N)$.

(d) $s.2.sec(N) + s.2.sec(K) \subseteq s.2.sec(N + K)$.

(e) $s.2.sec(N \cap K) = s.2.sec(s.2.sec(N) \cap s.2.sec(K))$.

(g) If $N + K = s.2.sec(N) + s.2.sec(K)$, then $s.2.sec(N + K) = N + K$.

Proof. These are straightforward. □

Corollary 3.24. Let N be a submodule of an R-module M. If $s.2.sec(N) \neq 0$, then $s.2.sec(N)$ is a strongly 2-absorbing second radical submodule of M.

Proof. This follows from Proposition 3.23 (c). □

Theorem 3.25. Let M be an R-module. If M satisfies the descending chain condition on strongly 2-absorbing second radical submodules, then every non-zero submodule of M has only a finite number of maximal strongly 2-absorbing second submodules.

Proof. Suppose that there exists a non-zero submodule N of M such that it has an infinite number of maximal strongly 2-absorbing second submodules. Then $s.2.sec(N)$ is a strongly 2-absorbing second radical submodule of M and $s.2.sec(N)$ has an infinite number of maximal strongly 2-absorbing second submodules. Let S be a strongly 2-absorbing second radical submodule of M chosen minimal such that S has an infinite number of maximal strongly 2-absorbing second submodules. Then S is not strongly 2-absorbing second. Thus there exist $r, t \in R$ and a submodule L of M such that $rtS \subseteq L$ but $rS \nsubseteq L$, $tS \nsubseteq L$, and $rtS \neq 0$. Let V be a maximal strongly 2-absorbing second submodule of M contained in S. Then $V \subseteq (L :_S r)$ or $V \subseteq (L :_S t)$ or $V \subseteq (0 :_S rt)$ so that $V \subseteq s.2.sec((K :_S r))$ or $V \subseteq s.2.sec((K :_S t))$ or $V \subseteq s.2.sec((0 :_S rt))$. By the choice of S, the modules $s.2.sec((K :_S r))$, $s.2.sec((K :_S t))$, and $s.2.sec((0 :_S rt))$ have only finitely many maximal strongly 2-absorbing second submodules. Therefore, there is only a finite number of possibilities for the module S, which is a contradiction. □

Corollary 3.26. Every Artinian R-module has only a finite number of maximal strongly 2-absorbing second submodules.

Theorem 3.27. Let M be an R-module. If E is an injective R-module and N is a 2-absorbing submodule of M such that $Hom_R(M/N, E) \neq 0$, then $Hom_R(M/N, E)$ is a strongly 2-absorbing second R-module.

Proof. Let $r, s \in R$. Since N is a 2-absorbing submodule of M, we can assume that $(N :_M rs) = (N :_M r)$ or $(N :_M rs) = M$. Since E is an injective R-module, by replacing M with M/N in [4, 3.13 (a)], we have $Hom_R(M/(N :_M r), E) = rHom_R(M/N, E)$. Therefore,

$$rsHom_R(M/N, E) = Hom_R(M/(N :_M rs), E) =$$
$$Hom_R(M/(N :_M r), E) = rHom_R(M/N, E)$$

or

$$rsHom_R(M/N, E) = Hom_R(M/(N :_M rs), E) =$$
$$Hom_R(M/M, E) = 0,$$

as needed □

Theorem 3.28. Let M be a strongly 2-absorbing second R-module and F be a right exact linear covariant functor over the category of R-modules. Then $F(M)$ is a strongly 2-absorbing second R-module if $F(M) \neq 0$.

Proof. This follows from [4, 3.14] and Theorem 3.3 $(c) \Leftrightarrow (d)$. □

References

[1] W. Anderson and K.R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York-Heidelberg-Berlin, 1974.

[2] H. Ansari-Toroghy and F. Farshadifar, *On the dual notion of prime submodules*, Algebra Colloq., **19** (Spec 1), 1109-1116 (2012).

[3] H. Ansari-Toroghy and F. Farshadifar, *On the dual notion of prime submodules (II)*, Mediterr. J. Math., **9**, no. 2, 329-338 (2012).

[4] H. Ansari-Toroghy and F. Farshadifar, *The dual notion of some generalizations of prime submodules*, Comm. Algebra, **39**, 2396-2416 (2011).

[5] H. Ansari-Toroghy and F. Farshadifar, *On the dual notion of prime radicals of submodules*, Asian Eur. J. Math., **6**, no. 2, 1350024 (11 pages) (2013).

[6] H. Ansari-Toroghy, F. Farshadifar, S.S. Pourmortazavi, and F. Khaliphe, *On secondary modules*, International Journal of Algebra, **6**, no. 16, 769-774 (2012).

[7] H. Ansari-Toroghy and F. Farshadifar, *The dual notion of multiplication modules*, Taiwanese J. Math., **11**, no. 4, 1189–1201 (2007).

[8] H. Ansari-Toroghy, F. Farshadifar, and S.S. Pourmortazavi, *On the P-interiors of submodules of Artinian modules*, Hacet. J. Math. Stat., **45**, no. 3, 675-682 (2016).

[9] A. Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc., **75**, 417-429 (2007).

[10] A. Barnard, *Multiplication modules*, J. Algebra, **71**, 174-178 (1981).

[11] S. Ceken, M. Alkan, and P.F. Smith, *The dual notion of the prime radical of a module*, J. Algebra, **392**, 265-275 (2013).

[12] A. Y. Darani and F. Soheilnia, *2-absorbing and weakly 2-absorbing submoduels*, Thai J. Math., **9**, no. 3, 577–584 (2011).

[13] A.Y. Darani and F. Soheilnia, *On n-absorning submodules*, Math. Commun., **17**, 547-557 (2012).

[14] J. Dauns, *Prime submodules*, J. Reine Angew. Math., **298**, 156–181 (1978).

[15] L. Fuchs, W. Heinzer, and B. Olberding, *Commutative ideal theory without finiteness conditions: Irreducibility in the quotient filed*, in : Abelian Groups, Rings, Modules, and Homological Algebra, Lect. Notes Pure Appl. Math., **249**, 121–145 (2006).

[16] I. G. Macdonald, *Secondary representation of modules over a commutative ring*, Sympos. Math. XI, 23-43 (1973).

[17] Sh. Payrovi and S. Babaei, *On 2-absorbing submodules*, Algebra Collq., **19**, 913-920 (2012).

[18] Sh. Payrovi and S. Babaei, *On the 2-absorbing submodules*, Iran. J. Math. Sci. Inform., **10**, no. 1, 131-137 (2015).

[19] S. Yassemi, *The dual notion of prime submodules*, Arch. Math. (Brno), **37**, 273–278 (2001).

[20] S. Yassemi, *Coassociated primes*, Comm. Algebra, **23**, 1473-1498 (1995).

[21] S. Yassemi, *The dual notion of the cyclic modules*, Kobe. J. Math., **15**, 41–46 (1998).

Author information

H. Ansari-Toroghy, Department of pure Mathematics, Faculty of mathematical Sciences, University of Guilan, P. O. Box 41335-19141, Rasht, Iran.
E-mail: `ansari@guilan.ac.ir`

F. Farshadifar, Department of Mathematics, Farhangian University, Tehran, Iran
School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran.
E-mail: `f.farshadifar@cfu.ac.ir`

Received: May 4, 2017.
Accepted: April 21, 2018.