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ON PSEUDO-UNIFORM MODULES

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Abstract. In this article we introduce and study the concept of pseudo-uniform modules. An R-module M is called pseudo-uniform if each non-finitely generated submodule of M is essential in M. We show that each pseudo-uniform module M has finite Goldie dimension. If M is a pseudo-uniform module which is not uniform, then there exists a non-zero Noetherian submodule N which is essential in M. We also introduce and study the concept of essentially Noetherian submodules. We provide some basic facts for these modules.

1 Introduction

Lemonnier [25] introduced the concept of deviation and codeviation of an arbitrary poset, which in particular, when applied to the lattice of all submodules of a module M_R give the concepts of Krull dimension (in the sense of Rentschler and Gabriel, see [18]) and dual Krull dimension of M, respectively. The dual Krull dimension in [20], [21], [23] and [24] is called Noetherian dimension whereas in [5] it is called N-dimension. Let R be a ring and let M be an R-module. A submodule L of M is called essential if $L \cap N \neq 0$ for every non-zero submodule N of M, we write $L \subseteq_e M$ to denote this situation. Otherwise L is a non-essential submodule of M. We recall that a uniform module is a nonzero module M such that the intersection of any two nonzero submodules of M is nonzero, or, equivalently, such that every nonzero submodule of M is essential in M. The socle of M, denoted by $Soc(M)$, is the sum of all simple submodules of M. Recall that $Soc(M)$ is the intersection of all essential submodules of M. We also recall that M has finite Goldie dimension if it does not contain a direct sum of an infinite number of non-zero submodules of M. More recently, the partially ordered set (shortly poset) of all non-finitly generated submodules of an R-module M, has been studied, see [12, 14, 15, 13]. The purpose of this article is to extend the notion of uniform modules in view of this poset. Let us give a brief outline of this paper. Section 1 is the introduction. In Section 2, we investigate the concepts of pseudo-uniform and almost uniform modules. An R-module M is called pseudo-uniform if each non-finitely generated submodule of M is essential in M. An R-module M is called almost uniform, if for each two non-finitely generated submodules M_1 and M_2 of M, we get $M_1 \cap M_2 \neq 0$. It is manifest that any pseudo-uniform module is almost uniform. We observe that each pseudo-uniform module M has finite Goldie dimension. If M is a pseudo-uniform module which is not uniform, then there exists a non-zero Noetherian submodule N which is essential in M. We also show that if M is a pseudo-uniform module which satisfying ascending chain condision on essensial submodules, then M has Noetherian dimension and n-dim $M \leq 1$. Section 3 is devoted to a brief study of essentially Noetherian modules. We say that a submodule E of M is essentially Noetherian in M, denoted $E \subseteq_{en} M$, if for each nonzero submodule P of M, $P \cap E$ contains a nonzero Noetherian submodule. We show that if M is an R-module with finite Goldie dimension and it has an essentially Noetherian submodule, then M is λ finitely embedded for some ordinal number λ, see the comment which follows Proposition 3.12. Vedadi and Smith [29], studied modules M which satisfy the ascending chain condition on non-essential modules. We investigate some properties of these modules in view of this terminology. If an R-module M satisfies the ascending chain condition on non-essential submodules, we prove that either M is uniform or M has an essentially Noetherian submodule. Throughout this paper R will always denote an associative ring with a non-zero identity, $1 \neq 0$, and M is a left unital R-module. The notation $N \subseteq M$ (resp., $N \subset M$) means that N is a submodule (resp. proper submodule) of M. The reader is referred to [6, 17, 18, 22, 23], for definitions, concepts, and the necessary background not explicitly given here.

2 Pseudo-uniform modules

In this section we introduce and study the concepts of pseudo-uniform modules and almost uniform modules.

We begin with the following definition.

Definition 2.1. Let M be a module and N a submodule of M. Then N is called non-finitely generated if N can not be a finitely generated submodule of M.

Next, we give our definition of pseudo-uniform modules.

Definition 2.2. Let M be an R-module. M is called pseudo-uniform if each non-finitely generated submodule N of M is essential in M.

The following results are evident.

Remark 2.3. Every Noetherian module is pseudo-uniform.

Remark 2.4. Let M be a uniform R-module. Then M is pseudo-uniform.

Let us recall that the codeviation of a partially ordered set $E = (E, \leq)$, (shortly poset), denoted by co-dev(E) is defined as follows: co-dev $(E) = -1$ if and only if E is a trivial poset, i.e., E has no two distinct comparable elements. If E is nontrivial but satisfies the ascending chain condition on its elements, then co-dev$(E) = 0$. For a general ordinal α, we define co-dev$(E) = \alpha$ provided:

(i) co-dev$(E) \neq \beta < \alpha$;

(ii) for any ascending chain

$$x_1 \leq x_2 \leq ... \leq x_n \leq ...$$

of elements of E there is some $n_0 \in N$ for all $n \geq n_0$ the codeviation of the poset $\frac{x_{n+1}}{x_n} = \{x \in E : x_n \leq x \leq x_{n+1}\}$ is already defined and satisfies

$$co\text{-dev}\,(\frac{x_{n+1}}{x_n}) < \alpha.$$

If no ordinal α exists such that co-dev$(E) = \alpha$, we say E does not have codeviation. In particular, if we apply this concept to $L(M)$, the lattice of all submodules of a module M, we obtain the concept of Noetherian dimension of M, denoted by n-dim M, see [20, 25, 26]. We also recall that the name of dual Krull dimension is also used by some authors, see [1] and [2]. If an R-module M has Noetherian dimension and α is an ordinal number, then M is called α-atomic if n-dim $M = \alpha$ and n-dim $N < \alpha$, for all proper submodules N of M. An R-module M is called atomic if it is α-atomic for some ordinal α, see [23] (note, atomic modules are also called conotable, N-critical and dual critical in some other articles for example, see [26], [5], and [1], respectively).

Remark 2.5. Let M be an R-module. If M is 1-atomic, then it is pseudo-uniform.

Lemma 2.6. *Let M be an R-module. Then M is a pseudo-uniform module if and only if each non-essential submodule of M is Noetherian.*

Proof. Let X be any proper submodule of M. If there exists a non-finitely generated submodule N of M such that $N \subseteq X$, then $X \subseteq_e M$. Otherwise each submodule of X is finitely generated, hence X is Noetherian. The converse is obvious. □

Corollary 2.7. *Let M be a pseudo-uniform module, then M has finite Goldie dimension.*

Proof. Let $N_1 \oplus N_2 \oplus N_3 \oplus ...$ be an infinite direct sum of submodules of M. Then $X = N_2 \oplus N_3 \oplus ...$ is a non-finitely generated submodule of M and $N_1 \cap X = 0$ which is a contradiction □

In view of Corollary 2.7 and [17, Corollary 5.21], we have the following results.

Corollary 2.8. *Let M be a pseudo-uniform R-module, then for each non-finitely generated submodule N of M, we have* $G\text{-dim}\, N = G\text{-dim}\, M$.

Corollary 2.9. *Let M be an R-module with finite Goldie dimension. Then M is pseudo-uniform if and only if for each non-finitely generated submodule N of M, we have* $G\text{-dim}\, N = G\text{-dim}\, M$.

Lemma 2.10. *Let M be a pseudo-uniform module. If M is not uniform, then each submodule of M has a non-zero Noetherian submodule.*

Proof. In view of Corollary 2.7, we infer that there exists an integer number n and submodules $N_1, ..., N_n$ of M such that $N_1 \oplus N_2 \oplus ... \oplus N_n \subseteq_e M$. By our hypothesis $n > 1$. Since M is pseudo-uniform each N_i is Noetherian and we are done. □

Next, we recall the following result from [19, Lemma 3].

Proposition 2.11. *A module M satisfies ACC on essential submodules if and only if $\frac{M}{Soc(M)}$ is Noetherian.*

Let us recall that an R-module M is called α-short if for each submodule N of M either $n\text{-dim}\, N \leq \alpha$ or $n\text{-dim}\, \frac{M}{N} \leq \alpha$ and α is the least ordinal number with this property. In [7, Proposition 1.12] it is observed that if M is an α-short module, then $n\text{-dim}\, M = \alpha$ or $n\text{-dim}\, M = \alpha+1$.

Corollary 2.12. *Let M be a pseudo-uniform module. If M satisfies the ascending chain condition on essential submodules, then M has Noetherian dimension and* $n\text{-dim}\, M \leq 1$.

Proof. Let N be any submodule of M. By Lemma 2.6, N is Noetherian or essential. If N is Noetherian, then $n\text{-dim}\, N = 0$. Now let N be an essential submodule of M. Then by Proposition 2.11, $\frac{M}{N}$ is Noetherian. This shows that M is a short module and by [7, Proposition 1.12], $n\text{-dim}\, M \leq 1$. □

In the following we introduce the concept of almost-uniform modules.

Definition 2.13. Let M be an R-module. M is called almost uniform, if for each two non-finitely generated submodules M_1 and M_2 of M, we get $M_1 \cap M_2 \neq 0$.

It is manifest that each pseudo-uniform module is also almost uniform, but the converse is not true. For example, the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Q}$ is not pseudo-uniform, but it is almost uniform.

The following result is now immediate.

Corollary 2.14. *Let N be a Noetherian R-module and U be a uniform R-module. Then, $N \oplus U$ is an almost uniform module.*

Lemma 2.15. *Let M an R-module. If M is an almost uniform module, then so does each non-zero proper submodule of M.*

Lemma 2.16. *Let M be an almost uniform module. Then M has finite Goldie dimension and there exists a Noetherian submodule N and a submodule X of M, where X is zero or it is non-finitely generated and $X \oplus N \subseteq_e M$.*

Proof. Let $N_1 \oplus N_2 \oplus N_3 \oplus N_4 \oplus ....$ be a submodule of M. Then $X = N_1 \oplus N_3 \oplus N_5 \oplus ...$ and $X' = N_2 \oplus N_4 \oplus N_6 \oplus ...$ are non-finitely generated submodules of M and $X \cap X' = 0$ which is a contradiction. Hence M has a finite Goldie dimension. Thus there exists an integer number n and uniform submodules $N_1, ..., N_n$ of M such that $N_1 \oplus N_2 \oplus ... \oplus N_n \subseteq_e M$. If for each i, N_i is Noetherian, then $N = N_1 \oplus N_2 \oplus ... \oplus N_n$ is a Noetherian submodule of M which is essential in M, (note, in this case X is zero). Otherwise for some integer number i, N_i is not Noetherian. Without less of generality we may assume that N_1 is not Noetherian. Thus N_1 has a non-finitely generated submodule say it X_1. Therefore $X_1 \oplus (N_2 \oplus ... \oplus N_n) \subseteq_e M$. Since M is almost uniform, we infer that $(N_2 \oplus ... \oplus N_n) = 0$ or it is Noetherian and we are done. □

3 Essentially Noetherian modules

We begin with the following definition.

Definition 3.1. Let M be an R-module and E be a submodule of M. We say that E is an essentially Noetherian submodule of M, denoted by $E \subseteq_{en} M$, if for each nonzero submodule P of M, $P \cap E$ contains a nonzero Noetherian submodule.

The proof of the next result is elementary and is omitted.

Proposition 3.2. *Let A, B, and C be modules with $A \subseteq B \subseteq C$. Then:*

(i) If $A \subseteq_{en} C$, then both $A \subseteq_{en} B$ and $B \subseteq_{en} C$.

(ii) If $A \subseteq_{e} B$ and $B \subseteq_{en} C$, then $A \subseteq_{en} C$.

(iii) If $A \subseteq_{en} B$ and $B \subseteq_{e} C$, then $A \subseteq_{en} C$.

The proof of the following three facts are standard.

Lemma 3.3. *If $A \subseteq_{e} B$ and A is Noetherian, then $A \subseteq_{en} B$.*

Lemma 3.4. *Let R be a Noetherian ring. Then $A \subseteq_{e} B$ if and only if $A \subseteq_{en} B$.*

Lemma 3.5. *Let $A \subseteq_{e} B$ and A has a Noetherian submodule such as N. If $N \subseteq_{e} A$, then $N \subseteq_{e} B$ and therefore $A \subseteq_{en} B$.*

Lemma 3.6. *Let A_1, A_2, B_1 and B_2 be submodules of a module C. If $A_1 \subseteq_{en} B_1$ and $A_2 \subseteq_{e} B_2$, then $A_1 \cap A_2 \subseteq_{en} B_1 \cap B_2$.*

Proof. Let $0 \neq X \subseteq B_1 \cap B_2$, then $A_1 \cap X$ contains a nonzero Noetherian submodule such as N_1. Now N_1 is a nonzero submodule of B_2 and $A_2 \subseteq_{e} B_2$, therefore $N_1 \cap A_2 \neq 0$. But we know that $0 \neq N_1 \cap A_2$ is Noetherian. Thus $X \cap A_1 \cap A_2$ contains a nonzero Noetherian submodule and we are done. □

In view of previous lemma we have the following corollary.

Corollary 3.7. *Let M be an R-module. Then $\bigcap_{N \subseteq_{en} M} N = M$ or $\bigcap_{N \subseteq_{en} M} N = Soc(M)$.*

Proof. If M does not have any essentially Noetherian submodule, then $\bigcap_{N \subseteq_{en} M} N = M$. Otherwise M has an essentially Noetherian submodule such as N. Let E be an essential submodule of M, then by Lemma 3.6 we infer that $N \cap E$ is an essentially Noetherian submodule of M and $E \cap N \subseteq E$. Therefore $\bigcap_{N \subseteq_{en} M} N \subseteq \bigcap_{E \subseteq_{e} M} E$. Conversely it is clear that each essentially Noetherian submodule is an essential submodule of M. Hence $\bigcap_{E \subseteq_{e} M} E \subseteq \bigcap_{N \subseteq_{en} M} N$. Therefore $\bigcap_{N \subseteq_{en} M} N = \bigcap_{E \subseteq_{e} M} E = Soc(M)$. □

Proposition 3.8. *Let A be a submodule of a module C and let $f : B \to C$ be a monomorphism. If $A \subseteq_{en} C$, then $f^{-1}(A) \subseteq_{en} B$.*

Proof. Let M be any nonzero submodule of B. Then $f(M) \neq 0$ and $A \cap f(M)$ contains a nonzero Noetherian submodule such as N. Hence $f^{-1}(N) \cap M$ is nonzero Noetherian module, it follows that $f^{-1}(A) \subseteq_{en} B$. □

Lemma 3.9. *Given a right module A over a domain R, the set*

$$ZN(A) = \{x \in A : xI = 0 for\ some\ I \subseteq_{en} R_R\}$$

If $ZN(A)$ is a non-empty set, then it is a submodule of A.

Proof. Given any $x, y \in ZN(A)$ there are essentially Noetherian right ideals I, J in R such that $xI = yJ = 0$. By Lemma 3.6, we infer that $I \cap J$ is an essentially Noetherian right ideals of R and $(x + y)(I \cap J) = 0$, we obtain $x + y \in ZN(A)$. For any $t \in R$, the right ideal $K = \{r \in R : tr \in I\}$ is essentially Noetherian by Proposition 3.8, and $xtK \subseteq xI = 0$, whence $xt \in ZN(A)$. Thus $ZN(A)$ is a submodule of A. □

We recall that an R-module M is called α-critical, where α is an ordinal number, if k-dim $M = \alpha$ and k-dim $\frac{M}{N} < \alpha$ for all nonzero submodules N of M. An R-module M is called critical if M is α-critical for some ordinal number α.

Note the following well-known result from [18].

Proposition 3.10. *Let M be an R-module with Krull dimension; then it has a critical submodule.*

Next, we recall the following definition from [22].

Definition 3.11. Let M be an R-module. For each ordinal α, we define $S_\alpha = \sum_{i\in I} \oplus C_i$, where $\{C_i\}_{i\in I}$ is a maximal independent set of α-critical submodules of M. S_α is called an α-critical socle of M. Now a critical socle of M is defined to be a submodule S of M with $S = \sum_{\alpha<\lambda} S_\alpha$, where λ is the least ordinal such that each critical submodule is α-critical for some $\alpha \leq \lambda$. If for some ordinal α, there is no α-critical submodule, then we put $S_\alpha = 0$. Clearly, the sum of any maximal independent family of critical submodules of M is a critical socle of M.

We cite the following result from[22].

Proposition 3.12. *If S is a critical socle of an R-module M, then $S = \sum_{\alpha\leq\lambda} S_\alpha = \sum_{\alpha\leq\lambda} \oplus S_\alpha$.*

Proof. See [22, Proposition 2.3]. □

We recall that an R-module M is called λ-finitely embedded (λ-f.e.) if λ is the least ordinal such that each critical submodule of M is α-critical for some $\alpha \leq \lambda$ and M contains a $f.g.$ essential critical socle (equivalently, M contains an essential critical socle with Krull dimension λ), see [22].

Corollary 3.13. *Let R-module M has finite Goldie dimension. If M has an essentially Noetherian submodule, then M is λ-f.e., for some ordinal number λ.*

Proof. Since M has an essentially Noetherian submodule, each non-zero submodule of M has a non-zero Noetherian submodule. Hence each non-zero submodule of M has a non-zero submodule with Krull dimension, see [23, Proposition 1.1]. By Proposition 3.10, we infer that each non-zero submodule of M has a critical submodule. Therefore M is λ-f.e., for some ordinal number λ. □

We should remind the reader that by a quotient finite dimensional module M we mean for each submodule N of M, $\frac{M}{N}$ has finite Goldie dimension.

In view of previous corollary and [22, Proposition 2.20], we have the following result.

Corollary 3.14. *Let M be a quotient finite dimensional module. If each nonzero factor module of M has an essentially Noetherian submodule, then M has Krull dimension.*

Proof. By previous corollary each non-zero factor module of M is λ-f.e., for some ordinal number λ. By [22, Proposition 2.20], we infer that M has Krull dimension. □

In view of previous corollary and Lemma 2.10, we have the following result.

Corollary 3.15. *Let M be a quotient finite dimensional R-module. If for each proper submodule N of M, $\frac{M}{N}$ is pseudo-uniform module which is not uniform, then M has Krull dimension.*

Note the following fact. The proof is standard but we include it for completeness.

Lemma 3.16. *Let R-module M has finite Goldie dimension. If N is an essentially Noetherian submodule of M, then there exists a Noetherian submodule U such that $U \subseteq_e M$.*

Proof. Since M has finite Goldie dimension, we infer that there exists an integer number n such that $U_1 \oplus U_2 \oplus .... \oplus U_n \subseteq_e M$, where each U_i is a uniform submodule of M. For each integer number i, $U_i \cap N$ contains a nonzero Noetherian submodule, U_i' say. It is clear that $U_i' \subseteq_e U_i$ for each i. Therefore $0 \neq U_1' \oplus ... \oplus U_n' \subseteq_e U_1 \oplus ... \oplus U_n \subseteq_e M$. This shows that $U = U_1' \oplus ... \oplus U_n'$ a non-zero Noetherian submodule of M which is essential in M, see [17, Proposition 5.6]. □

Vedadi and Smith in [29], studied modules M which satisfy the ascending chain condition on non-essential submodules. Now we investigate some properties of these modules.

Proposition 3.17. *Let R-module M satisfy the ascending chain condition on non-essential submodules. Then M is uniform or it has a Noetherian submodule N such that $N \subseteq_e M$, i.e., M is uniform or M has an essentially Noetherian submodule.*

Proof. By [29, Theorem 1.8], we infer that M has finite Goldie dimension. If M is not uniform, then $N_1 \oplus N_2 \subseteq_e M$, for some non-zero submodules N_1 and N_2 of M. If N_1 is not Noetherian, then there exists the chain

$$N_1' \subset N_2' \subset N_3' \subset ...$$

of submodules of N_1. Hence $N_1' \subset N_2' \subset ...$ is a chain of submodules of M such that for each i, N_i' is not essential in M, which is a contradiction. Therefore N_1 is Noetherian. Similarly we can show that N_2 is Noetherian and we are done. □

In view of Proposition 3.17 and Corollary 3.13, we have the following result.

Proposition 3.18. *Let R-module M satisfy the ascending chain condition on non-essential submodules. If M is not uniform, then M is λ-f.e. for some ordinal number λ.*

Finally we conclude this section by providing some examples of essential submodules of an R-module M which are not essentially Noetherian. Let M be an R-module. If there exists an R-module $X \subseteq E(M)$ such that $M \subseteq_{en} X$, then $M \subseteq_{en} E(M)$, see Proposition 3.2. If $E(M)$ has finite Goldie dimension and it is not λ-f.e., for each ordinal number λ, then $M \subseteq_e X$ for each $X \subseteq E(M)$ but M is not a Noetherian essential submodule of X, see Corollary 3.13.

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