Palestine Journal of Mathematics

Vol. 8(2)(2019) , 169–176

Some commutativity theorems for rings with involution involving generalized derivations

Alsharif Maha, Altassan Alaa and Muthana Najat

Communicated by Prof. Najib Mahdou

MSC 2010 Classifications: Primary 20M99, 13F10; Secondary 13A15, 13M05.

Keywords and phrases: Prime rings; derivations; generalized derivations; involution; involution of the second kind.

Abstract. In this paper, we investigate the commutativity of rings with involution of the second kind in which generalized derivations satisfy certain algebraic identities on ideals. Moreover, we extend some results for generalized derivations of prime rings to ideals. Finally, we give examples to prove that various restrictions imposed in the hypothesis of our result are not superfluous.

1 Introduction

Let R be a ring with center $Z = Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$, and $x \circ y$ stands for the anti-commutator $xy + yx$.

We shall use repeatedly the familiar commutator identities: $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$ for all $x, y, z \in R$. A ring R is called prime if for any $x, y \in R$, $xRy = (0)$ implies $x = 0$ or $y = 0$. A ring R is said to be of characteristic different from 2 if it satisfies: for any $x \in R$, $2x = 0$ implies $x = 0$. An additive mapping $d : R \longrightarrow R$ is said to be a derivation on R if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F : R \longrightarrow R$ is called a generalized derivation on R if it is associated with a derivation $d : R \longrightarrow R$ satisfying $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. An additive map $* : R \longrightarrow R$ is said to be an involution if it satisfies the following conditions: (i) $(xy)^* = y^*x^*$; and (ii) $(x^*)^* = x$ for all $x, y \in R$. A ring with involution or a $*$-ring is a ring equipped with an involution. In a ring with involution $*$, an element x is called hermition if $x^* = x$, and skew-hermition if $x^* = -x$. The set of all hermition and skew-hermition elements denoted by $H(R)$ and $S(R)$ respectively. An element $x \in R$ is said to be normal if $xx^* = x^*x$. A ring R is called a normal ring if every element in R is a normal. For example, the ring of quaternions is normal ring. An involution is said to be of the first kind if $Z(R) \subseteq H(R)$, and of the second kind if $S(R) \cap Z(R) \neq (0)$.

Over the past few years, several authors have studied the commutativity of a ring R that admits certain specific types of derivations (see [1], [3], [8], and [13]). In [4], M. Ashraf and N. Rehman proved that if R is a prime ring, I is a nonzero ideal of R and R admitting a derivation d, then R is commutative if the following conditions holds: $d(xy) + xy \in Z(R)$, or $d(xy) - xy \in Z(R)$ for all $x, y \in I$.

There has been ongoing interest concerning the generalized derivations. For instance, Hvala [7] introduced the first study in this direction, and he extended some results from derivations to generalized derivations since the generalized derivations covers the concept of derivations. Inspired by Ashraf's work, M. Ashraf, et. al. [3] proved the results (and some others) in light of a generalized derivation of a prime ring. Motivated by the above, our aim is to continue the line of investigation regarding the study of commutativity for prime rings with involution admitting a generalized derivation associated with a derivation satisfying certain algebraic identities.

2 Preliminaries

We shall use the following results which are essential to prove our main theorems:

Lemma 2.1. *[[12], Lemma 2.1] Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If $[x, x^*] \in Z(R)$ (or $x \circ x^* \in Z(R)$) for all $x \in R$, then R is commutative.*

Theorem 2.2. *[[12], Theorem] Let R be a prime ring with involution $*$ such that $char(R) \neq 2$ provided with a derivation d. If $d(h) = 0$ for all $h \in H(R) \cap Z(R)$, then $d(z) = 0$ for all $z \in Z(R)$.*

Lemma 2.3. *[[11], Lemma 4] If R is a prime ring and $0 \neq b \in Z(R)$ and $ab \in Z(R)$, then $a \in Z(R)$.*

Theorem 2.4. *[[3], Theorem 2.1] Let R be a prime ring and let I be a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(xy) - xy \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Theorem 2.5. *[[3], Theorem 2.3] Let R be a prime ring and let I be a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(xy) - yx \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Theorem 2.6. *[[9], Theorem 2.5] Let R be a prime ring and let I be a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(x)F(y) - xy \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Theorem 2.7. *[[9], Theorem 2.6] Let R be a prime ring and let I be a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(x)F(y) - yx \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Theorem 2.8. *[[10], Theorem 1] Let R be a $*$-prime ring with $char(R) \neq 2$, and F a generalized derivation associated with a nonzero derivation d. If F is centralizing on a nonzero $*$-Jordan ideal J, then R is commutative.*

3 Main results

Theorem 3.1. *Let $(R, *)$ be a prime ring with involution of the second kind such that $char(R) \neq 2$, and I an $*$-ideal. If R admits a generalized derivation F associated with a derivation d such that $F(xx^*) - xx^* \in Z(R)$ (or $F(xx^*) + xx^* \in Z(R)$) for all $x \in I$, then R is commutative.*

Proof. By the hypothesis, we have

$$F(xx^*) - xx^* \in Z(R) \ \text{ for all } \ x \in I. \tag{3.1}$$

If $F = 0$, then $xx^* \in Z(R)$ for all $x \in I$, which implies that $x \circ x^* \in Z(R)$ for all $x \in I$. Hence, R is commutative by Lemma 2.1.
If $F \neq 0$, a linearization of the equation (3.1) gives

$$F(xy^*) + F(yx^*) - xy^* - yx^* \in Z(R) \ \text{ for all } \ x, y \in I. \tag{3.2}$$

Replacing y by yh in (3.2), where $h \in Z(R) \cap H(R) \setminus \{0\}$, we obtain

$$(xy^* + yx^*)d(h) \in Z(R).$$

Then

$$[(xy^* + yx^*)d(h), r] = 0 \ \text{ for all } \ x, y \in I \ \text{ and } \ r \in R.$$

It follows that

$$[xy^* + yx^*, r]d(h) = 0.$$

The primeness of R forces that either $[xy^* + yx^*, r] = 0$ or $d(h) = 0$.
Firstly, suppose that for all $x, y \in I$ and $r \in R$ we have $[xy^* + yx^*, r] = 0$. Take $y = y^*$ to have $[xy + y^*x^*, r] = 0$. That is

$$x[y, r] + [x, r]y + y^*[x^*, r] + [y^*, r]x^* = 0 \ \text{ for all } \ x, y \in I \ \text{ and } \ r \in R. \tag{3.3}$$

Substituting yk for y in (3.3), where $k \in S(R) \cap Z(R)$, we obtain

$$(x[y, r] + [x, r]y - y^*[x^*, r] - [y^*, r]x^*)k = 0.$$

Since R is prime and $S(R) \cap Z(R) \neq (0)$, we get

$$x[y,r] + [x,r]y - y^*[x^*,r] - [y^*,r]x^* = 0 \text{ for all } x,y \in I \text{ and } r \in R. \tag{3.4}$$

Combining (3.3) and (3.4) and since $char(R) \neq 2$, we arrive at $x[y,r] + [x,r]y = 0$. Replacing y by ys in the previous equation, where $s \in R$ forces that $xy[s,r] = 0$. Thus, $xRI[s,r] = (0)$ for all $x \in I$ and $r,s \in R$. In light of the primeness of R together with the fact that $I \neq (0)$, the last equation implies that R is commutative.
Now consider the case when $d(h) = 0$ for all $h \in Z(R) \cap H(R) \setminus \{0\}$. By Theorem 2.2 we get $d(k) = 0$ for all $k \in S(R) \cap Z(R)$. Put $y = yk$ in (3.2) to obtain

$$(-F(xy^*) + F(yx^*) + xy^* - yx^*)k \in Z(R)$$

Since R is prime and $S(R) \cap Z(R) \neq (0)$, Lemma 2.3 yields that

$$F(xy^*) - F(yx^*) - xy^* + yx^* \in Z(R) \tag{3.5}$$

Combining (3.2) and (3.5), we get $2(F(xy^*) - xy^*) \in Z(R)$. Since $char(R) \neq 2$, $F(xy^*) - xy^* \in Z(R)$. Replacing y by y^* in the previous equation, we arrive at $F(xy) - xy \in Z(R)$ for all $x,y \in I$. Therefore, in view of Theorem 2.4, the proof is complete. □

Corollary 3.2. *Let $(R,*)$ be a prime ring with involution of the second kind such that $char(R) \neq 2$, and I an $*$-ideal. If R admits a generalized derivation F associated with a derivation d such that $F(xy) - xy \in Z(R)$ (or $F(xy) + xy \in Z(R)$) for all $x,y \in I$, then R is commutative.*

Theorem 3.3. *Let $(R,*)$ be a prime ring with involution of the second kind such that $char(R) \neq 2$, and I an $*$-ideal. If R admits a generalized derivation F associated with a derivation d such that $F(xx^*) - x^*x \in Z(R)$ (or $F(xx^*) + x^*x \in Z(R)$) for all $x \in I$, then R is commutative.*

Proof. We are given that

$$F(xx^*) - x^*x \in Z(R) \text{ for all } x \in I. \tag{3.6}$$

We can suppose that $F \neq 0$, so a linearization of (3.6) yields that

$$F(xy^*) + F(yx^*) - x^*y - y^*x \in Z(R) \text{ for all } x,y \in I. \tag{3.7}$$

Substitute yh for y in (3.7), where $h \in Z(R) \cap H(R) \setminus \{0\}$ to obtain

$$[xy^* + yx^*, r]d(h) = 0 \text{ for all } x,y \in I \text{ and } r \in R. \tag{3.8}$$

Since R is prime, the equation (3.8) yields that either $[xy^* + yx^*, r] = 0$ or $d(h) = 0$. For the case $[xy^* + yx^*, r] = 0$, the result follows by the same technique as in the proof of Theorem 3.1. Now assume that $d(h) = 0$ for all $h \in Z(R) \cap H(R) \setminus \{0\}$. By Theorem 2.2, we have $d(k) = 0$ for all $k \in S(R) \cap Z(R)$. Replacing y by yk in (3.7), we get

$$(-F(xy^*) + F(yx^*) - x^*y + y^*x)k \in Z(R). \tag{3.9}$$

Since R is prime and $S(R) \cap Z(R) \neq (0)$, Lemma 2.3 forces that

$$F(xy^*) - F(yx^*) + x^*y - y^*x \in Z(R) \text{ for all } x,y \in I. \tag{3.10}$$

Adding (3.7) to (3.10) and since $char(R) \neq 2$, we arrive at $F(xy) - yx \in Z(R)$ for all $x,y \in I$. Hence, by Theorem 2.5, R is commutative. □

Corollary 3.4. *Let $(R,*)$ be a prime ring with involution of the second kind such that $char(R) \neq 2$, and I an $*$-ideal. If R admits a generalized derivation F associated with a derivation d such that $F(xy) - yx \in Z(R)$ (or $F(xy) + yx \in Z(R)$) for all $x,y \in I$, then R is commutative.*

Theorem 3.5. *Let $(R,*)$ be a prime ring with involution of the second kind such that $char(R) \neq 2$, and I an $*$-ideal. If R admits a generalized derivation F associated with a derivation d such that $F(x)F(x^*) - xx^* \in Z(R)$ (or $F(x)F(x^*) + xx^* \in Z(R)$) for all $x \in I$, then R is commutative.*

Proof. By the hypothesis, we have

$$F(x)F(x^*) - xx^* \in Z(R) \ \text{ for all } \ x \in I. \tag{3.11}$$

As above we can suppose that $F \neq 0$, and by linearizing the last relation, we find that

$$F(x)F(y^*) + F(y)F(x^*) - xy^* - yx^* \in Z(R) \ \text{ for all } \ x, y \in I. \tag{3.12}$$

Write yh instead of y in (3.12), where $h \in Z(R) \cap H(R) \setminus \{0\}$ to get

$$[F(x)y^* + yF(x^*), r]d(h) = 0 \ \text{ for all } \ r \in R. \tag{3.13}$$

The primeness of R forces that either $[F(x)y^* + yF(x^*), r] = 0$ or $d(h) = 0$. Consider the case when $[F(x)y^* + yF(x^*), r] = 0$, that is

$$[F(x)y^*, r] + [yF(x^*), r] = 0 \tag{3.14}$$

Replacing y by yz, where $z \in Z(R)$, we get

$$[F(x)y^*, r]z^* + [yF(x^*), r]z = 0 \tag{3.15}$$

Multiply (3.14) by z^* from the right and substract the result from (3.15) to obtain $[yF(x^*), r](z - z^*) = 0$, that is

$$(y[F(x), r] + [y, r]F(x))(z - z^*) = 0 \tag{3.16}$$

Since R is prime, either $z - z^* = 0$ or $y[F(x), r] + [y, r]F(x) = 0$. If $z - z^* = 0$, then $z = z^*$, a contradiction since the involution $*$ is of the second kind. Thus, $y[F(x), r] + [y, r]F(x) = 0$. Substitute ty for y in the last equation, where $t \in R$, to get $[t, r]yF(x) = 0$ that is $[t, r]IRF(x) = (0)$ for all $x \in I$ and $r, t \in R$. By using the primeness of R together with the fact that $F \neq 0$, we get $[t, r]I = (0)$. That is $[t, r]RI = (0)$. Since R is prime and $I \neq (0)$, we conclude that $[t, r] = 0$ for all $r, t \in R$. Hence, R is commutative. □

Corollary 3.6. *Let $(R, *)$ be a prime ring with involution of the second kind such that $char(R) \neq 2$, and I an $*$-ideal. If R admits a generalized derivation F associated with a derivation d such that $F(x)F(y) - xy \in Z(R)$ (or $F(x)F(y) + xy \in Z(R)$) for all $x, y \in I$, then R is commutative.*

Theorem 3.7. *Let $(R, *)$ be a prime ring with involution of the second kind such that $char(R) \neq 2$, and I an $*$-ideal. If R admits a generalized derivation F associated with a derivation d such that $F(x)F(x^*) - x^*x \in Z(R)$ (or $F(x)F(x^*) + x^*x \in Z(R)$) for all $x \in I$, then R is commutative.*

Proof. We can suppose that $F \neq 0$, by the hypothesis we have

$$F(x)F(x^*) - x^*x \in Z(R) \ \text{ for all } \ x \in I. \tag{3.17}$$

Linearizing (3.17), we get

$$F(x)F(y^*) + F(y)F(x^*) - x^*y - y^*x \in Z(R) \ \text{ for all } \ x, y \in I. \tag{3.18}$$

Replacing y by y^*, we have

$$F(x)F(y) + F(y^*)F(x^*) - x^*y^* - yx \in Z(R) \ \text{ for all } \ x, y \in I. \tag{3.19}$$

Substituting yh for y where $h \in Z(R) \cap H(R) \setminus \{0\}$, we obtain

$$(F(x)F(y) + F(y^*)F(x^*) - x^*y^* - yx)h + (F(x)y + y^*F(x^*))d(h) \in Z(R). \tag{3.20}$$

By using (3.19), we have

$$(F(x)y + y^*F(x^*))d(h) \in Z(R) \ \text{ for all } \ x, y \in I. \tag{3.21}$$

By invoking the primeness of R and from Lemma 2.3, the last relation implies that either $d(h) = 0$ or $F(x)y + y^*F(x^*) \in Z(R)$.

If $d(h) = 0$ for all $h \in Z(R) \cap H(R) \setminus \{0\}$, then Theorem 2.2 implies that $d(Z(R)) = \{0\}$. Substituting ys for y in (3.19), where $s \in Z(R) \cap S(R) \setminus \{0\}$, we get

$$(F(x)F(y) - F(y^*)F(x^*) + x^*y^* - yx)s \in Z(R) \text{ for all } x, y \in I. \tag{3.22}$$

Since R prime and $s \in Z(R) \cap S(R) \setminus \{0\}$, we get

$$F(x)F(y) - F(y^*)F(x^*) + x^*y^* - yx \in Z(R).$$

Combining the previous relation with (3.19), and since $char(R) \neq 2$, we deduce

$$F(x)F(y) - yx \in Z(R) \text{ for all } x, y \in I. \tag{3.23}$$

By using Theorem 2.7, we conclude that R is commutative.
If $F(x)y + y^*F(x^*) \in Z(R)$ for all $x, y \in I$, then by replacing y by ys, where $s \in Z(R) \cap S(R) \setminus \{0\}$, we obtain $F(x)y - y^*F(x^*) \in Z(R)$. Thus, $F(x)y \in Z(R)$ which implies that $[F(x)y, r] = 0$ for all $r \in R$. Therefore,

$$[F(x), r]y + F(x)[y, r] = 0.$$

Write $yF(x)$ instead of y in the last expression to obtain $F(x)y[F(x), r] = 0$. Thus, $[F(x), r]RI[F(x), r] = (0)$. In light of the primeness of R, either $I[F(x), r] = (0)$ or $[F(x), r] = 0$. If the initial case holds, then $IR[F(x), r] = (0)$. Since R is prime and $I \neq (0)$, both cases yield that $[F(x), r] = 0$ for all $x \in I$ and $r \in R$. This forces that

$$F(x) \in Z(R) \text{ for all } x \in I. \tag{3.24}$$

Since an ideal is a Jordan ideal, then in view of Theorem 2.8, R must be commutative. □

Corollary 3.8. *Let $(R, *)$ be a prime ring with involution of the second kind such that $char(R) \neq 2$, and I an $*$-ideal. If R admits a generalized derivation F associated with a derivation d such that $F(x)F(y) - yx \in Z(R)$ (or $F(x)F(y) + yx \in Z(R)$) for all $x, y \in I$, then R is commutative.*

Theorem 3.9. *Let $(R, *)$ be a prime ring with involution of the second kind such that $char(R) \neq 2$, and I an $*$-ideal. If R admits a non zero generalized derivation F associated with a nonzero derivation d such that $F(xx^*) + d(xx^*) \in Z(R)$ for all $x \in I$, then R is commutative.*

Proof. We are given that

$$F(xx^*) + d(xx^*) \in Z(R) \text{ for all } x \in I. \tag{3.25}$$

A linearization of the last relation gives

$$F(xy^*) + F(yx^*) + d(xy^*) + d(yx^*) \in Z(R) \text{ for all } x, y \in I. \tag{3.26}$$

Replacing y by yh in the above expression, where $h \in Z(R) \cap H(R) \setminus \{0\}$, we get

$$(xy^* + yx^*)d(h) \in Z(R) \text{ for all } x, y \in I. \tag{3.27}$$

Then by Lemma 2.3, we get $xy^* + yx^* \in Z(R)$ or $d(h) = 0$.
If $xy^* + yx^* \in Z(R)$, then this yields that R is commutative (Reasoning as in the proof of Theorem 3.1.)
If $d(h) = 0$ for all $h \in Z(R) \cap H(R) \setminus \{0\}$, then in view of Theorem 2.2 we have $d(Z(R)) = \{0\}$. Replacing y by ys where $s \in Z(R) \cap S(R) \setminus \{0\}$, we obtain

$$-F(xy^*) + F(yx^*) - d(xy^*) + d(yx^*) \in Z(R) \text{ for all } x, y \in I. \tag{3.28}$$

Comparing (3.26) and (3.28), we arrive at $F(xy^*) + d(xy^*) \in Z(R)$. That is

$$F(xy) + d(xy) \in Z(R) \text{ for all } x, y \in I. \tag{3.29}$$

This can be written as $(F(x) + d(x))y + 2xd(y) \in Z(R)$. Substitute yr for y in the last relation, where $r \in R$ to obtain $[xyd(r), r] = 0$ for all $x, y \in I$ and $r \in R$. Thus, $xy[d(r), r] + x[y, r]d(r) +$

$[x,r]yd(r) = 0$ for all $x, y \in I$ and $r \in R$. Put wx instead of x in the previous equation, where $w \in R$, and use it again to obtain $[w,r]xyd(r) = 0$ for all $x, y \in I$ and $r, w \in R$. That is, $[w,r]xRId(r) = (0)$ for all $x \in I$ and $r, w \in R$. By invoking the primeness of R, either $[w,r]x = 0$ or $Id(r) = (0)$. The latter case gives that $IRd(r) = (0)$. Since R is prime and by our hypothesis for I and d, we get a contradiction.
Therefore, $[w,r]x = 0$ for all $x \in I$ and $r, w \in R$. Then $[w,r]RI = (0)$ for all $r, w \in R$. Since $I \neq (0)$ and R is prime, we conclude that $[w,r] = 0$ for all $r, w \in R$, and hence R is commutative. □

Corollary 3.10. *Let $(R,*)$ be a prime ring with involution of the second kind such that $char(R) \neq 2$, and I an $*$-ideal. If R admits a generalized derivation F associated with a derivation d such that $F(xy) + d(xy) \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Theorem 3.11. *Let $(R,*)$ be a prime ring with involution of the second kind such that $char(R) \neq 2$, and I an $*$-ideal. If R admits a generalized derivation F associated with a derivation d such that $F(xx^*) + d(x^*x) \in Z(R)$ for all $x \in I$, then R is commutative.*

Proof. We are given that

$$F(xx^*) + d(x^*x) \in Z(R) \text{ for all } x \in I. \tag{3.30}$$

A linearization of the last relation gives

$$F(xy^*) + F(yx^*) + d(x^*y) + d(y^*x) \in Z(R) \text{ for all } x, y \in I. \tag{3.31}$$

Replacing y by yh, where $h \in Z(R) \cap H(R) \setminus \{0\}$, we get

$$(xy^* + yx^* + x^*y + y^*x)d(h) \in Z(R) \text{ for all } x, y \in I. \tag{3.32}$$

As above we obtain R is commutative or $d(Z(R)) = \{0\}$. For the last case, replacing y by ys in (3.31), where $s \in Z(R) \cap S(R) \setminus \{0\}$, we obtain

$$-F(xy^*) + F(yx^*) + d(x^*y) - d(y^*x) \in Z(R) \text{ for all } x, y \in I. \tag{3.33}$$

Combining (3.31) with (3.33), we arrive at

$$F(xy^*) + d(y^*x) \in Z(R) \text{ for all } x, y \in I. \tag{3.34}$$

Replacing y by y^*, we see that

$$F(xy) + d(yx) \in Z(R) \text{ for all } x, y \in I. \tag{3.35}$$

That is

$$F(x)y + xd(y) + d(y)x + yd(x) \in Z(R) \text{ for all } x, y \in I. \tag{3.36}$$

We have $I \cap Z(R) \neq (0)$. Otherwise, $F(x)y + xd(y) + d(y)x + yd(x) = 0$, and replacing y by $yd(x)$, we get $xyd^2(x) + d(y)d(x)x - d(y)xd(x) + yd^2(x)x = 0$. Write xy instead y to obtain $d(x)y[d(x), x] = 0$ which implies that $[d(x), x] = 0$ for all $x \in I$. Since an ideal is a Jordan ideal and a derivation is a generalized derivation, then Theorem 2.8 forces that R is a commutative ring. This contradict our supposition, then $I \cap Z(R) \neq (0)$. Let now $0 \neq z \in I \cap Z(R)$ and replacing y by z, we find that

$$(F(x) + d(x))z \in Z(R) \text{ for all } x \in I. \tag{3.37}$$

By using Lemma 2.3, we obtain

$$F(x) + d(x) \in Z(R) \text{ for all } x \in I.$$

Replacing x by xy in the last relation, we arrive at

$$F(xy) + d(xy) \in Z(R) \text{ for all } x, y \in I. \tag{3.38}$$

Since (3.38) is the same as (3.29), then arguing as above, we conclude that R is commutative. □

Corollary 3.12. *Let $(R,*)$ be a prime ring with involution of the second kind such that $char(R)\neq 2$, and I an $*$-ideal. If R admits a generalized derivation F associated with a derivation d such that $F(xy)+d(yx)\in Z(R)$ for all $x,y\in I$, then R is commutative.*

The follwing example proves that $*$ is the second kind is not superfluous in Theorems 3.1, 3.3, 3.5, 3.7, 3.9 and 3.11.

Example1.

Let us consider R the ring of the 2×2 matrices with coefficients in $\mathbb{Z}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Let d and F defined as: $F\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = d\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$. It is straightforward to check that $(R,*)$ is a prime ring with involution of the first kind such that

$$XX^*\in Z(R) \text{ and } F(X)F(X^*)\in Z(R) \text{ for all } X=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in R.$$

Clearly, the conditions of Theorems 3.1, 3.3, 3.5, 3.7, 3.9 and 3.11 are satisfied but R is not commutative.

Now, we show that in Theorems 3.1, 3.3, 3.5, 3.7, 3.9, and 3.11 the hypothesis of $*$-ideal is crucial.

Example 2.

Let R be a noncommutative prime ring which admits a nonzero derivation d and let $R=R\times R^0$. If we set $I=R\times\{0\}$; then I is a nonzero ideal of the $*_{ex}$-prime ring R. Furthermore, if we defined $F(x,y)=(d(x),0)$; then F is a derivation of R so F is a generalized derivation associated itself which satisfies conditions of Theorems 3.1, 3.3, 3.5, 3.7, 3.9 and 3.11. But R is a noncommutative ring.

References

[1] S. Ali, N. A. Dar, On $*$-centralizing mappings in rings with involution, *Georgian Math. J.*, **21**, 25–28 (2014).

[2] A. Ashraf, A. Ali, and R. Rani, On generalized derivations of prime rings, *Southeast Asian Bulletin of Mathematics*, **29**(4), 669–675 (2005).

[3] M. Ashraf, A. Ali, and S. Ali, Some commutativity theorems for rings with generalized derivations, *Southeast Asian Bulletin of Mathematics*, **31**, 415–421 (2007).

[4] M. Ashraf and N. Rehman, On derivations and commutativity in prime rings, *East-west J. Math.*, **3**(1), 87–91 (2001).

[5] J. Bergen, I. N. Herstein, and J. W. Kerr, Lie ideals and derivations of prime rings, *Journal of Algebra*, **7**(1), 259–267 (1981).

[6] I. N. Herstein, Rings with Involution, *University of Chicago Press*, Chicago, Ill, USA (1976).

[7] B. Hvala, Generalized derivations in rings, *Comm. Algebra*, **26**(4), 1147–1166 (1998).

[8] M. A. Idrissi, A. Mamouni, and L. Oukhtite, On generalized derivations and commutativity of prime rings with involution, DOI: 10.1515/9783110542400-008, August 2018, In book: Algebra and Its Applications, ICAA16 Aligarh, India.

[9] A. Mamouni, Some results on semiprime rings with generalized derivations, *Int. J. Open Problems Compt. Math.*, **6**(2), June 2013.

[10] A. Mamouni, and L. Oukhtite, Generalized derivations centralizing on Jordan ideals of rings with involution, *Turk. J. Math.* 225–232 (2014). , http://dx.doi.org/10.3906/mat-1203-14.

[11] J. H. Mayne, Centralizing mappings of prime rings, *Canad. Math. Bull.*, **27**(1), 122–126 (1984).

[12] B. Nejjar, A. Kacha, A. Mamouni, and L. Oukhtite, Commutativity theorems in rings with involution, *Comm. Alg.*, **45**(2), 698–708 (2017).

[13] E. C. Posner, Derivations in prime rings, *Proc. Amer. Math. Soc.*, (8), 1093–1100 (1957).

Author information

Alsharif Maha, Department of Mathematics, King Abdulaziz University, P.O.Box 80203, Jeddah, 21589, and Department of Mathematics, Umm Al-Qura University, P.O.Box 715, Makkah, Saudi Arabia..
E-mail: malsharif0109@stu.kau.edu.sa and masharef@uqu.edu.sa.

Altassan Alaa, Department of Mathematics, King Abdulaziz University, P.O.Box 80203, Jeddah, 21589, Saudi Arabia..
E-mail: aaltassan@kau.edu.sa.

Muthana Najat, Department of Mathematics, King Abdulaziz University, P.O.Box 80203, Jeddah, 21589, and Department of Mathematics, Jeddah University, P. O. Box 80200, Jeddah, 21589, Saudi Arabia..
E-mail: najat_muthana@hotmail.com and nmuthana@kau.edu.sa

Received: November 22, 2018.
Accepted: January 29, 2019.