HYBRID ITERATION METHOD FOR FIXED POINTS OF ASYMPIOTICALLY $\phi$–DEMICONTRACTIVE MAPS IN REAL HILBERT SPACES

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Abstract. A strong convergence theorem of Hybrid iteration method to fixed points of asymptotically $\phi$–demicontractive mapping is proved in real Hilbert spaces. Our results extend, generalize and complement the results of Wang [8], Osilike, Isiogugu and Nwokoro [14], and extend several others from asymptotically demicontractive to the more general class of asymptotically $\phi$–demicontractive maps (see for example [2, 11, 17]).

1 Introduction

Let $K$ be a nonempty subset of a real Hilbert space $H$. A mapping $T : K \to K$ is said to be asymptotically $\phi$–demicontractive with a sequence $\{k_n\}_{n=1}^\infty \subseteq [1, \infty)$, if

$$\lim_{n \to \infty} k_n = 1, \text{ and } \forall x, y \in K, n \geq 1 \exists a \in (0, 1) \ni$$

$$\|T^n x - p\|^2 \leq k_n \|x - p\|^2 + \|x - T^n x\|^2 - \phi(\|x - T^n x\|),$$

(1.1)

where $\phi : [0, \infty) \to [0, \infty)$ is a nonincreasing continuous function with $\phi(0) = 0$ such that

$$\|T^n x - p\|^2 \leq a_n^2 \|x - p\|^2 + k_1 \|x - T^n x\|^2,$$

(1.2)

A mapping $T : K \to K$ is said to be $k$–strictly asymptotically pseudocontractive with a sequence $\{a_n\}_{n=1}^\infty \subseteq [1, \infty)$, if

$$\lim_{n \to \infty} a_n = 1, \text{ and } \forall x, y \in K, n \in N \exists a \in [0, 1) \exists$$

$$\|T^n x - T^n y\|^2 \leq k_n \|x - y\|^2 + k_1 \|(I - T^n)x - (I - T^n)y\|^2,$$

(1.3)

where $I$ is the identity operator. The class of $k$–strictly asymptotically pseudocontractive and asymptotically demicontractive maps were first introduced in Hilbert spaces by Qihou [10]. Observe that a $k$–strictly asymptotically pseudocontractive map with a nonempty fixed point set $F(T)$ is asymptotically demicontractive. An example of a $k$–strictly asymptotically pseudocontractive map is given in Osilike et al. [16]. Furthermore, $T$ is uniformly $L$–Lipschitzian if there exists a constant $L > 0 \ni$

$$\|T^n x - T^n y\| \leq L\|x - y\|,$$

(1.4)

$\forall x, y \in K$ and $n \geq 1$.

The class of asymptotically $\phi$–demicontractive maps was first introduced in arbitrary real Banach spaces by Osilike and Isiogugu [13]. It is shown in [13] that the class of asymptotically demicontractive map is a proper subclass of the class of asymptotically $\phi$–demicontractive maps (see for example [2, 11, 17]).
map while in [4], it is shown that every asymptotically demicontractive map is asymptotically φ–demicontractive with φ : [0, ∞) → [0, ∞) given by
\[ \phi(t) = (1 - k)t^2 - \frac{1}{2}(a_n^2 - 1)\|x - p\|^2. \]

These classes of operators have been studied by several authors (See for example, [2, 3, 4, 5, 10, 11, 13, 17]). In [13] Osilike and Isiogugu proved the convergence of the modified averaging iteration process of Mann [19] to the fixed points of asymptotically φ–demicontractive maps. Specifically they proved the following:

**Theorem 1.1.** ([13], p. 65): Let E be a real Banach space and K a nonempty closed convex subset of E. Let \( T : K \to K \) be a completely continuous uniformly \( L \)-Lipschitzian asymptotically φ–demicontractive mapping with a sequence \( \{k_n\}_{n=1}^{\infty} \subseteq [1, \infty), \sum(a_n^2 - 1) < \infty. \) Let \( \{a_n\} \) be a real sequence satisfying (i) \( 0 < a_n < 1 \) (ii) \( \sum a_n = \infty \) (iii) \( \sum a_n^2 < \infty. \) Then the sequence \( \{x_n\}_{n=1}^{\infty} \) generated from arbitrary \( x_1 \in K \) by the modified averaging Mann iteration process
\[ x_{n+1} = (1 - a_n)x_n + a_nT^n x_n, \quad n \geq 1 \] (1.5)
converges strongly to a common fixed point of \( T. \)

Similarly, in [4], using the modified averaging implicit iteration scheme \( \{x_n\} \) of Sun [20], generated from an \( x_1 \in K, \) by
\[ x_n = a_n x_{n-1} + (1 - a_n)T^i x_n, \quad n \geq 1 \]
where \( n = (k-1)N + i, i \in I = \{1, 2, 3, ..., N\}, \) Igokwe and Udofia [4] proved that under certain conditions on the iteration sequence \( \{a_n\}, \) the above iteration process \( \{x_n\} \) converges strongly to the common fixed point of the family \( \{T_i\}_{i=1}^{\infty} \) of \( N \) uniformly \( L_i \)-Lipschitzian asymptotically φ–demicontractive self maps of nonempty closed convex subset of a Hilbert space \( H. \)

The hybrid approximation methods below was first introduced by Yamada [7]. Yamada proposed the method in order to reduce the complexity in computation caused by the the projection \( P_K(u^* - \mu F(u^*)) \) in the fixed point equation
\[ u^* = P_K(u^* - \mu F(u^*)) \] (1.6)
where \( P_K \) is a projection from a Hilbert space \( H \) onto a closed convex subset \( K \) of \( H. \) To solve variational inequalities associated with the fixed point equation (1.6), Yamada introduced the following iteration method: For arbitrary \( u_0 \in H; \)
\[ u_{n+1} = T u_n - \lambda_{n+1} F(T u_n), \quad n \geq 0 \]
(where \( T \) is a nonexpansive mapping from \( H \) into itself, \( K \) is the fixed point set of \( T, F \) is \( \eta \)-strongly monotone and \( L \)-Lipschitzian on \( K, \) \( \{\lambda_n\} \) is a sequence in \((0, 1)\) and \( 0 < \mu < \frac{2\eta}{L^2} \)).

Yamada proved strong convergence in the Hilbert space \( H. \)

Motivated by Yamada’s work, Wang [8] proposed a new explicit iteration scheme with a mapping \( F \) to approximate the fixed points of nonexpansive mapping \( T \) in Hilbert spaces and proved strong and weak convergence theorems. The explicit iteration scheme of Wang is given below:

Let \( T : H \to H \) be a nonexpansive mapping, \( F : H \to H \) an \( \eta \)-strongly monotone \( L \)-Lipschitzian mapping on \( K, \) \( \{\alpha_n\} \subset (0, 1), \{\lambda_n\} \subset [0, 1) \) and \( \mu \) a fixed constant in \((0, \frac{2\eta}{L^2})\).

For arbitrary initial point \( x_1 \in H, \) the explicit iteration scheme with mapping \( F \) is defined as follows
\[ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(T x_n - \lambda_n + 1 F(T x_n)), \quad n \geq 0. \] (1.7)

In the sequel we shall need the mapping \( T^\lambda : E \to E \) defined by
\[ T^\lambda = T x - \lambda F(T x), \quad \forall x \in H. \] (1.8)
With (1.8), we observe that (1.7) becomes
\[ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T^{\lambda_{n+1}}x_n, \quad n \geq 0. \tag{1.9} \]

Recently Osilike et al. [14] extend the work of Wang to arbitrary Banach spaces without the strong monotonicity assumption on the hybrid operator \( F \). Specifically, they proved the following theorem.

**Theorem 1.2.** Let \( E \) be an arbitrary real Banach space, \( T : K \rightarrow K \) a nonexpansive mapping with \( F(T) \neq \emptyset \), and \( F : E \rightarrow E \) an \( L \)-Lipschitzian mapping. Let \( \{x_n\} \) be the sequence generated from an arbitrary \( x_1 \in E \) by (1.9) and \( \{\alpha_n\} \) and \( \lambda_n \) are real sequences in \([0, 1)\) satisfying the conditions:

(i) \( 0 < \alpha \leq \alpha_n < 1, \forall n \geq 1 \) and for some \( \alpha \in (0, 1) \),

(ii) \( \sum_{n=1}^{\infty} (1 - \alpha_n) = \infty \),

(iii) \( \sum_{n=1}^{\infty} \lambda_n < \infty \).

Then (a) \( \lim_{n \rightarrow \infty} \|x_n - x^*\| \) exists for each \( x^* \in F(T) \),

(b) \( \lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0 \),

(c) \( \{x_n\} \) converges strongly to a fixed point of \( T \) if and only if \( \lim \inf_{n \rightarrow \infty} d(x_n, F(T)) = 0 \).

The main purpose of this paper is to modify (1.9) and proved that the modified hybrid iteration process converges to fixed points of \( N \) uniformly \( L \)-asymptotically \( \phi \)-demicontractive mappings in Hilbert space. Our results extend, generalize and complement the results of Wang [8], Osilike, Isiogugu and Nwokoro [14], and extend several others in literature (see for example, [2, 5, 11, 13, 17]). In the sequel we shall make use of the following lemma.

**Lemma 1.3.** (17, p. 80): Let \( \{a_n\} \), \( \{b_n\} \) and \( \{\delta_n\} \) be sequences of nonnegative real numbers satisfying the inequality
\[ a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad n \geq 1. \]

If \( \sum_{n=1}^{\infty} \delta_n < \infty \) and \( \sum_{n=1}^{\infty} b_n < \infty \) then \( \lim_{n \rightarrow \infty} a_n \) exists. In particular, if \( \{a_n\} \) has a subsequence which converges strongly to zero, then \( \lim_{n \rightarrow \infty} a_n = 0 \).

**2 MAIN RESULTS**

**Theorem 2.1** (Main Theorem). Let \( H \) be a real Hilbert space and \( K \) be a nonempty closed convex subset of \( H \). Let \( T : K \rightarrow K \) be \( N \) uniformly \( L_1 \)-Lipschitzian asymptotically \( \phi \)-demicontractive self maps of \( K \) with sequence \( \{a_n\}_{n=1}^{\infty} \subseteq [1, \infty) \) such that \( \sum_{n=1}^{\infty} (a_n - 1) < \infty \) \( \forall i \in I = \{1, 2, ..., N\} \) and \( F(T) = \{x \in K : Tx = x\} \neq \emptyset \). Suppose \( F : K \rightarrow K \) be an \( L_2 \)-Lipschitzian mappings. Let \( \{x_n\}_{n=1}^{\infty} \) be the sequence generated from an arbitrary \( x_1 \in K \) by
\[ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T^{\lambda_n+1}x_n, \quad n \geq 1 \tag{2.1} \]
where \( T^{\lambda_n+1}x_n := T\lambda_nx_n - \lambda_{n+1}x_n = \lambda_{n+1}F(T)x_n, \quad \mu > 0 \) and \( n = (k-1)N \). Let \( \{a_n\}_{n=1}^{\infty} \) and \( \{\lambda_n\}_{n=1}^{\infty} \) be two real sequences in \([0, 1)\) satisfying the conditions:

(i) \( 0 < \alpha \leq a_n \leq \beta < 1 \) and for some \( \alpha, \beta \in (0, 1) \),

(ii) \( \sum_{n=1}^{\infty} (1 - \alpha_n) = \infty \),

(iii) \( \sum_{n=1}^{\infty} (1 - \alpha_n)^2 < \infty \),

(iv) \( \sum_{n=1}^{\infty} \lambda_n < \infty \).

Then,

(a) \( \lim_{n \rightarrow \infty} \|x_n - p\| \) exists for each \( p \in F(T) \),

(b) \( \lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0 \),

(c) \( \{x_n\}_{n=1}^{\infty} \) converges strongly to a fixed point of \( T \) if and only if there exists a subsequence \( \{x_{n_j}\}_{j=1}^{\infty} \) of \( \{x_n\}_{n=1}^{\infty} \) which converges strongly to \( p \) of \( T \).
Proof. We use result of Reinerman [9] (see also [1, 12]) and the fact that \( T \) and \( F \) are \( L \)–Lipschitzian.

\[ ||tx + (1 - t)y||^2 = t||x||^2 + (1 - t)||y||^2 - t(1 - t)||x - y||^2 \]  

(2.2)

which holds \( \forall x, y \in H \). Let \( p \in F \), then using (2.1) and (2.2), we have

\[ \|x_{n+1} - p\|^2 = \| (\alpha_n x_n + (1 - \alpha_n) T^{\lambda_n+1} x_n) - p \|^2 \]

\[ \leq \alpha_n \| x_n - p \|^2 + (1 - \alpha_n) \| T^{\lambda_n+1} x_n - p \|^2 \]

\[ - \alpha_n (1 - \alpha_n) \| x_n - T^{\lambda_n+1} x_n \|^2 \]  

(2.3)

Observe that \((1 - \alpha_n) \| x_n - T^{\lambda_n+1} x_n \| = \| x_{n+1} - x_n \| \) and

\[ \| x_{n+1} - x_n \|^2 = \| x_{n+1} - x_n \|^2 \]

\[ \| T^{\lambda_n+1} x_n - p \|^2 = \| (T^k x_n - \lambda_{n+1} \mu F(T^k x_n) - p) \|^2 \]

\[ \leq \| T^k x_n - p \|^2 + \lambda_{n+1} \mu \| F(T^k x_n) \|^2 \]

\[ + \lambda_{n+1}^2 \mu^2 \| F(T^k x_n) \|^2 \]  

(2.5)

since \( 2 \| F(T^k x_n) \|^2 \| T^k x_n - p \|^2 \leq \| F(T^k x_n) \|^2 + \| (T^k x_n - p) \|^2, \) then

\[ \| T^{\lambda_n+1} x_n - p \|^2 = \| T^k x_n - p \|^2 + \lambda_{n+1} \mu \| F(T^k x_n) \|^2 \]

\[ + \lambda_{n+1}^2 \mu^2 \| F(T^k x_n) \|^2 \]

\[ \leq \| 1 + \alpha_n \|^2 + \lambda_{n+1} \mu \| F(T^k x_n) \|^2 \]

\[ + \lambda_{n+1}^2 \mu^2 \| F(T^k x_n) \|^2 \]

(2.6)

Substitute (2.5), (2.6) into (2.4)

\[ \| x_{n+1} - p \|^2 \leq \alpha_n \| x_n - p \|^2 + (1 - \alpha_n) \| T^k x_n - p \|^2 \]

\[ + \lambda_{n+1} \mu \| F(T^k x_n) \|^2 \]

\[ - \alpha_n (1 - \alpha_n) \| x_n - T^{\lambda_n+1} x_n \|^2 \]

\[ = \alpha_n \| x_n - p \|^2 + (1 - \alpha_n) \| 1 + \alpha_n \|^2 + \lambda_{n+1} \mu \| F(T^k x_n) \|^2 \]

\[ + \lambda_{n+1}^2 \mu^2 \| F(T^k x_n) \|^2 \]

\[ - \alpha_n (1 - \alpha_n) \| x_n - T^{\lambda_n+1} x_n \|^2 \]

(2.7)

since \( \| F(T^k x_n) \| = L_1 L_2 \| x_n - p \| + ||F(p)||, \) then

\[ \| F(T^k x_n) \|^2 = \{ L_1 L_2 \| x_n - p \| + ||F(p)|| \}^2 \]

\[ = L_1^2 L_2^2 \| x_n - p \|^2 + 2 L_1 L_2 \| x_n - p \| ||F(p)|| + ||F(p)||^2 \]

\[ \leq L_1 L_2 (L_1 L_2 + 1) \| x_n - p \|^2 + (L_1 L_2 + 1) ||F(p)||^2 \]

(2.8)

Substitute (2.8) into (2.7)

\[ \| x_{n+1} - p \|^2 \leq \alpha_n \| x_n - p \|^2 + (1 - \alpha_n) \| 1 + \alpha_n \|^2 + \lambda_{n+1} \mu \| F(T^k x_n) \|^2 \]

\[ + (1 - \alpha_n) \lambda_{n+1} \mu \| 1 + \alpha_n \|^2 \| L_1 L_2 (L_1 L_2 + 1) \| x_n - p \|^2 \]

\[ + (L_1 L_2 + 1) ||F(p)||^2 \]

\[ - \alpha_n (1 - \alpha_n) \| x_n - T^{\lambda_n+1} x_n \|^2 \]

(2.9)
Now estimating (2.9) using (1.1)

$$
\|T^k x_n - T^k p\|^2 \leq \left[1 + \frac{1}{2}(a_n - 1)\right] \|x_n - p\|^2 + \|x_n - T^k x\|^2 - \phi(\|x - T^k x_n\|).\tag{2.10}
$$

Also \(\|x_n - T^k x_n\| = \|x_n - p + T^k x_n\| \leq \|x_n - p\| + \|T^k x_n - p\| = (L + 1)\|x_n - p\|.

Hence

$$
\|x_n - T^k x_n\|^2 = (L + 1)^2\|x_n - p\|^2. \tag{2.11}
$$

Substitute (2.8) into (2.7)

$$
\|T^k x_n - T^k p\|^2 \leq \left[1 + \frac{1}{2}(a_n - 1)\right] \|x_n - p\|^2 + (L + 1)^2\|x_n - p\|^2 - \phi(\|x_n - T^k x_n\|).\tag{2.12}
$$

Substitute (2.12) into (2.9)

$$
\|x_{n+1} - p\|^2 \leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)\left[1 + \lambda_n + \mu\right] \left[1 + \frac{1}{2}(a_n - 1)\right] \|x_n - p\|^2 + (L + 1)^2\|x_n - p\|^2 + (1 - \alpha_n)\lambda_n + \mu\|x_n - p\|^2 - 2(1 - \alpha_n)\|x_{n+1} - p\|^2 - 2(1 - \alpha_n)^2\alpha_n \|x_{n+1} - p\|^2 - 2(1 - \alpha_n)^2\alpha_n \lambda_n + \mu\|L_1 L_2 + 1\|\|F(p)\|^2. \tag{2.13}
$$

(1 - \alpha_n)

$$
\|x_{n+1} - p\|^2 \leq \{1 - \alpha_n\} \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)^2\left[1 + \lambda_n + \mu\right] \left[1 + \frac{1}{2}(a_n - 1)\right] \|x_n - p\|^2 + (L + 1)^2\|x_n - p\|^2 + (1 - \alpha_n)^2\lambda_n + \mu\|x_n - p\|^2 - 2(1 - \alpha_n)^2\alpha_n \|x_{n+1} - p\|^2 - 2(1 - \alpha_n)^2\alpha_n \lambda_n \lambda_n + \mu\|L_1 L_2 + 1\|\|F(p)\|^2. \tag{2.14}
$$

(1 - \alpha_n)

$$
\|x_{n+1} - p\|^2 \leq \{1 - \alpha_n\} \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)^2\left[1 + \lambda_n + \mu\right] \left[1 + \frac{1}{2}(a_n - 1)\right] \|x_n - p\|^2 + (L + 1)^2\|x_n - p\|^2 + (1 - \alpha_n)^2\lambda_n + \mu\|x_n - p\|^2 - 2(1 - \alpha_n)^2\alpha_n \|x_{n+1} - p\|^2 - 2(1 - \alpha_n)^2\alpha_n \lambda_n \lambda_n + \mu\|L_1 L_2 + 1\|\|F(p)\|^2. \tag{2.15}
$$
Setting

\[ \Psi_n = (1 - \alpha_n)\alpha_n + (1 - \alpha_n)^2[1 + \lambda_{n+1}\mu]\left[1 + \frac{1}{2}(a_{in} - 1)\right] \]
\[ + (1 - \alpha_n)^2[1 + \lambda_{n+1}\mu](L + 1)^2 \]
\[ + (1 - \alpha_n)^2L_1L_2(L_1L_2 + 1)\lambda_{n+1}\mu[1 + \lambda_{n+1}\mu] \]
\[ - 2(1 - \alpha_n)^2\alpha_n\lambda_{n+1}\mu L_1L_2(L_1L_2 + 1) \]  \hspace{1cm} (2.16)

and

\[ \eta_n = (1 - \alpha_n)^2\lambda_{n+1}\mu[1 + \lambda_{n+1}\mu](L_1L_2 + 1) \]
\[ - 2(1 - \alpha_n)^2\alpha_n\lambda_{n+1}\mu(L_1L_2 + 1) \]

\[ \|x_{n+1} - p\|^2 \leq \left\{ 1 + \frac{\Psi_n - (1 - \alpha_n)}{(1 - \alpha_n)}\right\}\|x_n - p\|^2 \]
\[ - (1 - \alpha_n)^2[1 + \lambda_{n+1}\mu]\phi(\|x_n - T^kx_n\|) \]
\[ + \frac{\eta_n}{(1 - \alpha_n)}\|F(p)\|^2 - \frac{2(1 - \alpha_n)^2\alpha_n}{(1 - \alpha_n)}\|T^kx_n - x_n\|^2 \]  \hspace{1cm} (2.17)

\[ \|x_{n+1} - p\|^2 \leq \left\{ 1 + \frac{\Psi_n - (1 - \alpha_n)}{(1 - \alpha_n)}\right\}\|x_n - p\|^2 \]
\[ - (1 - \alpha_n)^2[1 + \lambda_{n+1}\mu]\phi(\|x_n - T^kx_n\|) \]
\[ + \frac{\eta_n}{(1 - \alpha_n)}\|F(p)\|^2 - \frac{2(1 - \alpha_n)^2\alpha_n}{(1 - \alpha_n)}\|T^kx_n - x_n\|^2 \]
\[ - 2(1 - \alpha_n)^2\alpha_n\|T^kx_n - x_n\|^2 + \sigma_n \]  \hspace{1cm} (2.18)

since \(-\frac{1}{(1-\alpha_n)} \leq -1\), we have

\[ \|x_{n+1} - p\|^2 \leq \left\{ 1 + \delta_{in}\right\}\|x_n - p\|^2 \]
\[ - (1 - \alpha_n)^2[1 + \lambda_{n+1}\mu]\phi(\|x_n - T^kx_n\|) \]
\[ - 2(1 - \alpha_n)^2\alpha_n\|T^kx_n - x_n\|^2 + \sigma_n \]  \hspace{1cm} (2.19)

where

\[ \delta_{in} = \Psi_n - (1 - \alpha_n) \]
\[ = (1 - \alpha_n)\alpha_n + (1 - \alpha_n)^2[1 + \lambda_{n+1}\mu]\left[1 + \frac{1}{2}(a_{in} - 1)\right] \]
\[ + (1 - \alpha_n)^2[1 + \lambda_{n+1}\mu](L + 1)^2 \]
\[ + (1 - \alpha_n)^2L_1L_2(L_1L_2 + 1)\lambda_{n+1}\mu[1 + \lambda_{n+1}\mu] \]
\[ - 2(1 - \alpha_n)^2\alpha_n\lambda_{n+1}\mu L_1L_2(L_1L_2 + 1) - (1 - \alpha_n) \]
\[ = (1 - \alpha_n)^2\lambda_{n+1}\mu + \frac{1}{2}(1 - \alpha_n)^2[1 + \lambda_{n+1}\mu](a_{in} - 1) \]
\[ + (1 - \alpha_n)^2[1 + \lambda_{n+1}\mu](L + 1)^2 \]
\[ + (1 - \alpha_n)^2L_1L_2(L_1L_2 + 1)\lambda_{n+1}\mu[1 + \lambda_{n+1}\mu] \]
\[ - 2(1 - \alpha_n)^2\alpha_n\lambda_{n+1}\mu L_1L_2(L_1L_2 + 1) \]  \hspace{1cm} (2.20)

and

\[ \sigma_n = \eta_n\|F(p)\|^2 - \{ (1 - \alpha_n)^2\lambda_{n+1}\mu[1 + \lambda_{n+1}\mu](L_1L_2 + 1) \]
\[ - 2(1 - \alpha_n)^2\alpha_n\lambda_{n+1}\mu(L_1L_2 + 1)\} \|F(p)\|^2 \]
\[ \|x_{n+1} - p\|^2 \leq \left\{ 1 + \delta_{in}\right\}\|x_n - p\|^2 \]
\[ - (1 - \alpha_n)^2[1 + \lambda_{n+1}\mu]\phi(\|x_n - T^kx_n\|) \]
\[ + \frac{\eta_n}{(1 - \alpha_n)}\|F(p)\|^2 - \frac{2(1 - \alpha_n)^2\alpha_n}{(1 - \alpha_n)}\|T^kx_n - x_n\|^2 \]  \hspace{1cm} (2.21)
From conditions \((ii)\) \(- (v)\), \(\sum_{n=1}^{\infty} \delta_n < \infty\) and \(\sum_{n=1}^{\infty} \sigma_n < \infty\). Thus using Lemma 1.3, it follows that \(\lim_{n \to \infty} ||x_n - p||\) exists and \(\{x_n\}\) is bounded, thus completing the proof of (a). Since \(\{x_n\}\) is bounded, then there exists \(M > 0\) such that \(||x_n - p|| \leq M \forall n \geq 1\). It follows from (2.19) that

\[
(1 - \alpha_n)^2[1 + \lambda_{n+1}\mu \phi(||x_n - T^k x_n||)] \leq \{1 + \delta_n\} ||x_n - p||^2
- ||x_{n+1} - p||^2 + \sigma_n
\]

\[
\sum_{j=N}^{\infty} (1 - \alpha_j)^2[1 + \lambda_j\mu \phi(||x_j - T^k x_j||)] \leq \sum_{j=N}^{\infty} \{(1 + \delta_j) ||x_j - p||^2
- ||x_{j+1} - p||^2 + \sigma_j\}
\]

\[
= \sum_{j=N}^{\infty} \{||x_j - p||^2 - ||x_{j+1} - p||^2 + \delta_j ||x_j - p||^2 + \sigma_j\}
\]

\[
\sum_{n=1}^{\infty} (1 - \alpha_n)^2[1 + \lambda_{n+1}\mu \phi(||x_n - T^k x_n||)] \leq ||x_N - p||^2
+ M^2 \sum_{n=1}^{\infty} \delta_n + \sum_{n=1}^{\infty} \sigma_n < \infty
\]

Conditions (iii) and (iv) imply that \(\lim_{n \to \infty} \phi(||x_n - T^k x_n||) = 0\). Since \(\phi\) is an increasing and continuous, then \(\lim_{n \to \infty} ||x_n - T^k x_n|| = 0\). Observe that,

\[
||x_n - T^k x_n|| = ||x_n - T^k x_n + T^k x_n - T x_n|| \leq ||x_n - T^k x_n||
+ ||T^k x_n - T x_n||
\]

\[
\leq ||x_n - T^k x_n|| + ||T^k x_n - T x_n|| = ||x_n - T x_n||
+ ||TT^{k-1} x_n - T x_n||
\]

\[
\leq ||x_n - T^k x_n|| + L ||T^{k-1} x_n - x_n||
= ||x_n - T^k x_n|| + L ||T^{k-1} x_n - T^{k-1} x_{n-1} + T^{k-1} x_{n-1} - x_n||
\]

\[
\leq ||x_n - T^k x_n|| + L ||T^{k-1} x_{n-1} - x_{n-1}|| + L ||x_n - x_{n-1}||
= ||x_n - T^k x_n|| + L(L + 1)||x_n - x_{n-1}||
+ L ||T^{k-1} x_{n-1} - x_{n-1}||
\]

(2.22)

Observe that,

\[
||x_n - x_{n-1}|| = ||x_n - x_{n-1} - (\alpha_{n-1} x_{n-1} + (1 - \alpha_{n-1}) T^{k-1} x_{n-1} - \lambda_{n-1} \mu F(T^{k-1} x_{n-1}))||
- \lambda_{n-1} \mu F(T^{k-1} x_{n-1}))||
\]

\[
\leq ||x_n - T^{k-1} x_{n-1}|| + \lambda_{n-1} \mu ||F(T^{k-1} x_{n-1})||
\]

(2.23)

\[
||x_n - T x_n|| \leq ||x_n - T^k x_n|| + L(L + 2) ||x_n - T^{k-1} x_{n-1}||
\]

\[
+ \lambda_{n-1} \mu (L + 1) L_1^2 L_2 ||x_{n-1} - p|| + \lambda_{n-1} \mu (L + 1) ||F(p)||.
\]

(2.24)

Hence \(\lim_{n \to \infty} ||x_n - T x_n|| = 0\). Thus completing the proof of (b). Since \(\{x_n\}_{n=1}^{\infty}\) has a subsequence \(\{x_n\}_{j=1}^{\infty}\) which converges strongly to \(p\) and \(\lim_{n \to \infty} ||x_n - p|| = 0\) exists, by Lemma 1.3. Thus completing the proof. □
3 CONCLUDING REMARKS

Remark 3.1. If $T$ is in addition completely continuous or demicompact, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of $T$. Furthermore, if $T$ satisfies condition (A), then $\liminf_{n \to +\infty} d(x_n, F(T)) = 0$, so under the conditions of Theorem 2.1, if $T$ satisfies condition (A), then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of $T$.

Remark 3.2. The strong monotonicity condition imposed on $F$ in [8] is not required in our results.

Remark 3.3. Our results extend, generalize and complement the results of Wang [8], Osilike, Isiogugu and Nwokoro [14] and others in literature.

References


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