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HYBRID ITERATION METHOD FOR FIXED POINTS OF ASYMPTOTICALLY $\phi-$DEMICONTRACTIVE MAPS IN REAL HILBERT SPACES

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Abstract. A strong convergence theorem of Hybrid iteration method to fixed points of asymptotically $\phi-$demicontractive mapping is proved in real Hilbert spaces. Our results extend, generalize and complement the results of Wang [8], Osilike, Isiogugu and Nwokoro [14], and extend several others from asymptotically demicontractive to the more general class of asymptotically $\phi-$demicontractive maps (see for example [2, 11, 17]).

1 Introduction

Let K be a nonempty subset of a real Hilbert space H. A mapping $T : K \longrightarrow K$ is said to be **asymptotically** $\phi-$**demicontractive** with a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty)$, $\lim\limits_{n\to\infty} k_n = 1$,(see for example, [3, 4]), if $F(T) = \{x \in K : Tx = x\} \neq \emptyset$ and there exists an increasing continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\|T^n x - p\|^2 \leq k_n\|x - p\|^2 + \|x - T^n x\|^2 - \phi(\|x - T^n x\|), \tag{1.1}$$

$\forall x \in K, p \in F(T)$ and $n \geq 1$.
A mapping $T : K \longrightarrow K$ is said to be **asymptotically demicontractive** with a sequence $\{a_n\}_{n=1}^{\infty} \subseteq [1, \infty)$, $\lim\limits_{n\to\infty} a_n = 1$, if $F(T) \neq \emptyset$ and $\forall x \in K, p \in F(T), \exists$ a $k \in [0, 1) \ni$

$$\|T^n x - p\|^2 \leq a_n^2\|x - p\|^2 + k\|(I - T^n)x\|^2, \tag{1.2}$$

A mapping $T : K \longrightarrow K$ is said to be $k-$**strictly asymptotically pseudocontractive** with a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty)$,
$\lim\limits_{n\to\infty} k_n = 1$ if $\forall x, y \in K, n \in N, \exists$ a $k \in [0, 1) \ni$

$$\|T^n x - T^n y\|^2 \leq k_n\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2, \tag{1.3}$$

where I is the identity operator. The class of $k-$strictly asymptotically pseudocontractive and asymptotically demicontractive maps were first introduced in Hilbert spaces by Qihou [10]. Observe that a $k-$strictly asymptotically pseudocontractive map with a nonempty fixed point set $F(T)$ is asymptotically demicontractive. An example of a $k-$strictly asymptotically pseudocontractive map is given in Osilike *et al.* [16].
Furthermore, T is uniformly $L-$Lipschitzian if there exists a constant $L > 0 \ni$

$$\|T^n x - T^n y\| \leq L\|x - y\|, \tag{1.4}$$

$\forall x, y \in K$ and $n \geq 1$.
The class of asymptotically $\phi-$demicontractive maps was first introduced in arbitrary real Banach spaces by Osilike and Isiogugu [13]. It is shown in [13] that the class of asymptotically demicontractive map is a proper subclass of the class of asymptotically $\phi-$demicontractive

map while in [4], it is shown that every asymptotically demicontractive map is asymptotically $\phi-$demicontractive with $\phi : [0, \infty) \to [0, \infty)$ given by

$$\phi(t) = (1-k)t^2 - \frac{1}{2}(a_n^2 - 1)\|x-p\|^2.$$

These classes of operators have been studied by several authors (See for example, [2, 3, 4, 5, 10, 11, 13, 17]). In [13] Osilike and Isiogugu proved the convergence of the modified averaging iteration process of Mann [19] to the fixed points of asymptotically $\phi-$demicontractive maps. Specifically they proved the following:

Theorem 1.1. *([13], p. 65): Let E be a real Banach space and K a nonempty closed convex subset of E. Let $T : K \to K$ be a completely continuous uniformly $L-$Lipschitzian asymptotically $\phi-$demicontractive mapping with a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty), \ni \sum(a_n^2 - 1) < \infty$. Let $\{a_n\}$ be a real sequence satisfying (i) $0 < \alpha_n < 1$ (ii) $\sum \alpha_n = \infty$ (iii) $\sum a_n^2 < \infty$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated from arbitrary $x_1 \in K$ by the modified averaging Mann iteration process*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1 \tag{1.5}$$

converges strongly to a common fixed point of T.

Similarly, in [4], using the modified averaging implicit iteration scheme $\{x_n\}$ of Sun [20], generated from an $x_1 \in K$, by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_i^k x_n, \quad n \geq 1$$

where $n = (k-1)N + i, i \in \mathcal{I} = \{1, 2, 3, ..., N\}$, Igbokwe and Udofia [4] proved that under certain conditions on the iteration sequence $\{\alpha_n\}$, the above iteration process $\{x_n\}$ converges strongly to the common fixed point of the family $\{T_i\}_{i=1}^{\infty}$ of N uniformly L_i-Lipschitzian asymptotically $\phi-$demicontractive self maps of
nonempty closed convex subset of a Hilbert space H.

The hybrid approximation methods below was first introduced by Yamada [7]. Yamada proposed the method in order to reduce the complexity in computation caused by the the projection $P_K(u^* - \mu F(u^*))$ in the fixed point equation

$$u^* = P_K(u^* - \mu F(u^*)) \tag{1.6}$$

(where P_K is a projection from a Hilbert space H onto a closed convex subset K of H). To solve variational inequalities associated with the fixed point equation (1.6), Yamada introduced the following iteration method: For arbitrary $u_0 \in H$;

$$u_{n+1} = Tu_n - \lambda_{n+1}F(Tu_n), \quad n \geq 0$$

(where T is a nonexpansive mapping from H into itself, K is the fixed point set of T, F is $\eta-$strongly monotone and $L-$Lipschitzian on K, $\{\lambda_n\}$ is a sequence in $(0, 1)$ and $0 < \mu < \frac{2\eta}{L^2}$). Yamada proved strong convergence in the Hilbert space H.

Motivated by Yamada's work, Wang [8] proposed a new explicit iteration scheme with a mapping F to approximate the fixed points of nonexpansive mapping T in Hilbert spaces and proved strong and weak convergence theorems. The explicit iteration scheme of Wang is given below: Let $T : H \longrightarrow H$ be a nonexpansive mapping, $F : H \longrightarrow H$ an $\eta-$strongly monotone $L-$Lipschitzian mapping on K, $\{\alpha_n\} \subset (0, 1), \{\lambda_n\} \subset [0, 1)\}$ and μ a fixed constant in $(0, \frac{2\eta}{L^2})$. For arbitrary initial point $x_1 \in H$, the explicit iteration scheme with mapping F is defined as follows

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)(Tx_n - \lambda_{n+1}\mu F(Tx_n)),\ n \geq 0. \tag{1.7}$$

In the sequel we shall need the mapping $T^{\lambda} : E \longrightarrow E$ defined by

$$T^{\lambda} = Tx - \lambda\mu F(Tx),\ \forall\, x \in H. \tag{1.8}$$

With (1.8), we observe that (1.7) becomes

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T^{\lambda_{n+1}}x_n,\ n \geq 0. \tag{1.9}$$

Recently Osilike *et al.* [14] extend the work of Wang to arbitrary Banch spaces without the strong monotonicity assumption on the hybrid operator F. Specifically, they proved the following theorem.

Theorem 1.2. *Let E be an arbitrary real Banach space, $T : K \longrightarrow K$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $F : E \longrightarrow E$ an $L-$Lipschitzian mapping. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in E$ by (1.9) and $\{\alpha_n\}$ and λ_n are real sequences in $[0,1)$ satisfying the conditions:*
(i) $0 < \alpha \leq \alpha_n < 1, \forall n \geq 1$ and for some $\alpha \in (0,1)$,
(ii) $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, (iii) $\sum_{n=1}^{\infty}\lambda_n < \infty$.
Then (a) $\lim_{n\to\infty}\|x - x^*\|$ exists for each $x^* \in F(T)$,
(b) $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$,
(c) $\{x_n\}$ converges strongly to a fixed point of T if and only if
$\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

The main purpose of this paper is to modify (1.9) and proved that the modified hybrid iteration process converges to fixed points of N uniformly $L-$asymptotically $\phi-$demicontractive mappings in Hilbert space. Our results extend, generalize and complement the results of Wang [8], Osilike, Isiogugu and Nwokoro [14], and extend several others in literature (see for example, [2, 5, 11, 13, 17]). In the sequel we shall make use of the following lemma.

Lemma 1.3. *([17], p. 80): Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$ then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

2 MAIN RESULTS

Theorem 2.1 (Main Theorem). *Let H be a real Hilbert space and K be a nonempty closed convex subset of H. Let $T : K \to K$ be N uniformly L_1-Lipschitzian asymptotically $\phi-$demicontractive self maps of K with sequence $\{a_{in}\}_{n=1}^{\infty} \subseteq [1,\infty)$ such that $\sum\limits_{n=1}^{\infty}(a_{in}-1) < \infty \quad \forall i \in \mathcal{I} = \{1,2,...,N\}$ and $F(T) = \{x \in K : Tx = x\} \neq \emptyset$. Suppose $F : K \to K$ be an L_2-Lipschitzian mappings. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated from an arbitrary $x_1 \in K$ by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T^{\lambda_{n+1}}x_n, \quad n \geq 1 \tag{2.1}$$

where $T^{\lambda_{n+1}}x_n := T^k x_n - \lambda_{n+1}\mu F(T^k)$, $\mu > 0$ and $n = (k-1)N$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$ be two real sequences in $[0,1)$ satisfying the conditions;
(i) $0 < \alpha \leq a_n \leq \beta < 1$ and for some $\alpha, \beta \in (0,1)$,
(ii) $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, (iii) $\sum\limits_{n=1}^{\infty}(1-\alpha_n)^2 < \infty$,
(iv) $\sum\limits_{n=1}^{\infty}\lambda_n < \infty$.
Then,

(a) $\lim\limits_{n\to\infty}\|x_n - p\|$ *exists for each $p \in F(T)$,*

(b) $\lim\limits_{n\to\infty}\|x_n - Tx_n\| = 0$,

(c) $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T if and only if there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which converges strongly to p of T.

Proof. We use result of Reinerman [9] (see also [1, 12]) and the fact that T and F are $L-$Lipschitzian.

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2 \tag{2.2}$$

which holds $\forall x, y \in H$. Let $p \in F$, then using (2.1) and (2.2), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(\alpha_n x_n + (1-\alpha_n)T^{\lambda_{n+1}}x_n) - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|T^{\lambda_{n+1}}x_n - p\|^2 \\ &\quad -\alpha_n(1-\alpha_n)\|x_n - T^{\lambda_{n+1}}x_n\|^2 \end{aligned} \tag{2.3}$$

Observe that $(1-\alpha_n)\|x_n - T^{\lambda_{n+1}}x_n\| = \|x_{n+1} - x_n\|$ and

$$(1-\alpha_n)^2\|x_n - T^{\lambda_{n+1}}x_n\|^2 = \|x_{n+1} - x_n\|^2$$

$$(1-\alpha_n)\|x_n - T^{\lambda_{n+1}}x_n\|^2 = \frac{1}{(1-\alpha_n)}\|x_{n+1} - x_n\|^2 \tag{2.4}$$

$$\begin{aligned} \|T^{\lambda_{n+1}}x_n - p\| &= \|(T^k x_n - \lambda_{n+1}\mu F(T^k x_n) - p\| \\ &\le \|T^k x_n - p\| + \lambda_{n+1}\mu\|F(T^k x_n)\| \end{aligned}$$

$$\begin{aligned} \|T^{\lambda_{n+1}}x_n - p\|^2 &= \|T^k x_n - p\|^2 + 2\lambda_{n+1}\mu\|F(T^k x_n)\|\|(T^k x_n - p\| \\ &\quad +\lambda_{n+1}^2\mu^2\|F(T^k x_n)\|^2 \end{aligned} \tag{2.5}$$

since $2\|F(T^k x_n)\|\|T^k x_n - p\| \le \|F(T^k x_n)\|^2 + \|(T^k x_n - p\|^2$, then

$$\begin{aligned} \|T^{\lambda_{n+1}}x_n - p\|^2 &= \|T^k x_n - p\|^2 + \lambda_{n+1}\mu\|F(T^k x_n)\|^2 \\ &\quad +\lambda_{n+1}\mu\|(T^k x_n - p\| + \lambda_{n+1}^2\mu^2\|F(T^k x_n)\|^2 \\ &\le [1 + \lambda_{n+1}\mu]\|T^k x_n - p\|^2 \\ &\quad +\lambda_{n+1}\mu[1 + \lambda_{n+1}\mu]\|F(T^k x_n)\|^2 \end{aligned} \tag{2.6}$$

Substitute (2.5), (2.6) into (2.4)

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\{[1 + \lambda_{n+1}\mu]\|T^k x_n - p\|^2 \\ &\quad +\lambda_{n+1}\mu[1 + \lambda_{n+1}\mu]\|F(T^k x_n)\|^2\} \\ &\quad -\frac{\alpha_n}{(1-\alpha_n)}\|x_{n+1} - x_n\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[1 + \lambda_{n+1}\mu]\|T^k x_n - p\|^2 \\ &\quad +(1-\alpha_n)\lambda_{n+1}\mu[1 + \lambda_{n+1}\mu]\|F(T^k x_n)\|^2 \\ &\quad -\frac{\alpha_n}{(1-\alpha_n)}\|x_{n+1} - x_n\|^2 \end{aligned} \tag{2.7}$$

since $\|F(T^k x_n)\| = L_1L_2\|x_n - p\| + \|F(p)\|$, then

$$\begin{aligned} \|F(T^k x_n)\|^2 &= \{L_1L_2\|x_n - p\| + \|F(p)\|\}^2 \\ &= L_1^2L_2^2\|x_n - p\|^2 + 2L_1L_2\|x_n - p\|\|F(p)\| + \|F(p)\|^2 \\ &\le L_1L_2(L_1L_2 + 1)\|x_n - p\|^2 + (L_1L_2 + 1)\|F(p)\|^2 \end{aligned} \tag{2.8}$$

Substitute (2.8) into (2.7)

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[1 + \lambda_{n+1}\mu]\|T^k x_n - p\|^2 \\ &\quad +(1-\alpha_n)\lambda_{n+1}\mu[1 + \lambda_{n+1}\mu]\{L_1L_2(L_1L_2 + 1)\|x_n - p\|^2 \\ &\quad +(L_1L_2 + 1)\|F(p)\|^2\} - \frac{\alpha_n}{(1-\alpha_n)}\|x_{n+1} - x_n\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[1 + \lambda_{n+1}\mu]\|T^k x_n - p\|^2 \\ &\quad +(1-\alpha_n)\lambda_{n+1}\mu[1 + \lambda_{n+1}\mu]L_1L_2(L_1L_2 + 1)\|x_n - p\|^2 \\ &\quad +(1-\alpha_n)\lambda_{n+1}\mu[1 + \lambda_{n+1}\mu](L_1L_2 + 1)\|F(p)\|^2 \\ &\quad -\frac{\alpha_n}{(1-\alpha_n)}\|x_{n+1} - x_n\|^2 \end{aligned} \tag{2.9}$$

Now estimating (2.9) using (1.1)

$$\begin{aligned}\|T^k x_n - T^k p\|^2 &\leq \left[1+\frac{1}{2}(a_{in}-1)\right]\|x_n-p\|^2 + \|x_n - T^k x\|^2 \\ &-\phi(\|x-T^k x_n\|). \end{aligned} \tag{2.10}$$

Also $\|x_n - T^k x_n\| = \|x_n - p + p + T^k x_n\| \leq \|x_n - p\| + \|T^k x_n - p\| = (L+1)\|x_n - p\|$. Hence

$$\|x_n - T^k x_n\|^2 = (L+1)^2\|x_n - p\|^2 \tag{2.11}$$

Substitute (2.8) into (2.7)

$$\begin{aligned}\|T^k x_n - T^k p\|^2 &\leq \left[1+\frac{1}{2}(a_{in}-1)\right]\|x_n-p\|^2 \\ &+(L+1)^2\|x_n-p\|^2 - \phi(\|x_n - T^k x_n\|). \end{aligned} \tag{2.12}$$

Substitute (2.12) into (2.9)

$$\begin{aligned}\|x_{n+1} \quad - \quad & p\|^2 \leq \alpha_n\|x_n-p\|^2 \\ &+(1-\alpha_n)[1+\lambda_{n+1}\mu]\{\left[1+\frac{1}{2}(a_{in}-1)\right]\|x_n-p\|^2 \\ &+(L+1)^2\|x_n-p\|^2 - \phi(\|x_n-T^k x_n\|)\} \\ &+(1-\alpha_n)\lambda_{n+1}\mu[1+\lambda_{n+1}\mu]L_1L_2(L_1L_2+1)\|x_n-p\|^2 \\ &+(1-\alpha_n)\lambda_{n+1}\mu[1+\lambda_{n+1}\mu](L_1L_2+1)\|F(p)\|^2 \\ &-\frac{\alpha_n}{(1-\alpha_n)}\|x_{n+1}-x_n\|^2 \end{aligned} \tag{2.13}$$

$$\begin{aligned}(1-\alpha_n) \qquad & \|x_{n+1}-p\|^2 \leq (1-\alpha_n)\alpha_n\|x_n-p\|^2 \\ &+(1-\alpha_n)^2[1+\lambda_{n+1}\mu][1+\frac{1}{2}(a_{in}-1)]\|x_n-p\|^2 \\ &+(1-\alpha_n)^2[1+\lambda_{n+1}\mu](L+1)^2\|x_n-p\|^2 \\ &-(1-\alpha_n)^2[1+\lambda_{n+1}\mu]\phi(\|x_n-T^k x_n\|) \\ &+(1-\alpha_n)^2L_1L_2(L_1L_2+1)\lambda_{n+1}\mu[1+\lambda_{n+1}\mu]\|x_n-p\|^2 \\ &+(1-\alpha_n)^2\lambda_{n+1}\mu[1+\lambda_{n+1}\mu](L_1L_2+1)\|F(p)\|^2 \\ &-2(1-\alpha_n)^2\alpha_n\|T^k x_n - x_n\|^2 \\ &-2(1-\alpha_n)^2\alpha_n\lambda_{n+1}\mu L_1L_2(L_1L_2+1)\|x_n-p\|^2 \\ &-2(1-\alpha_n)^2\alpha_n\lambda_{n+1}\mu(L_1L_2+1)\|F(p)\|^2 \end{aligned} \tag{2.14}$$

$$\begin{aligned}(1-\alpha_n) \qquad & \|x_{n+1}-p\|^2 \leq \{(1-\alpha_n)\alpha_n \\ &+(1-\alpha_n)^2[1+\lambda_{n+1}\mu][1+\frac{1}{2}(a_{in}-1)] \\ &+(1-\alpha_n)^2[1+\lambda_{n+1}\mu](L+1)^2 \\ &+(1-\alpha_n)^2L_1L_2(L_1L_2+1)\lambda_{n+1}\mu[1+\lambda_{n+1}\mu] \\ &-2(1-\alpha_n)^2\alpha_n\lambda_{n+1}\mu(L_1L_2+1)(1-\alpha_n)\}\|x_n-p\|^2 \\ &-(1-\alpha_n)^2[1+\lambda_{n+1}\mu]\phi(\|x_n-T^k x_n\|) \\ &+\{(1-\alpha_n)^2\lambda_{n+1}\mu[1+\lambda_{n+1}\mu](L_1L_2+1) \\ &-2(1-\alpha_n)^2\alpha_n\lambda_{n+1}\mu(L_1L_2+1)\}\|F(p)\|^2 \\ &-2(1-\alpha_n)^2\alpha_n\|T^k x_n - x_n\|^2 \end{aligned} \tag{2.15}$$

Setting

$$\begin{aligned}\Psi_n &= (1-\alpha_n)\alpha_n + (1-\alpha_n)^2[1+\lambda_{n+1}\mu][1+\frac{1}{2}(a_{in}-1)] \\ &\quad +(1-\alpha_n)^2[1+\lambda_{n+1}\mu](L+1)^2 \\ &\quad +(1-\alpha_n)^2L_1L_2(L_1L_2+1)\lambda_{n+1}\mu[1+\lambda_{n+1}\mu] \\ &\quad -2(1-\alpha_n)^2\alpha_n\lambda_{n+1}\mu L_1L_2(L_1L_2+1) \qquad (2.16)\end{aligned}$$

and

$$\begin{aligned}\eta_n &= (1-\alpha_n)^2\lambda_{n+1}\mu[1+\lambda_{n+1}\mu](L_1L_2+1) \\ &\quad -2(1-\alpha_n)^2\alpha_n\lambda_{n+1}\mu(L_1L_2+1)\end{aligned}$$

$$\begin{aligned}&\|x_{n+1}-p\|^2 \le \{1+\frac{\Psi_n-(1-\alpha_n)}{(1-\alpha_n)}\|x_n-p\|^2 \\ &-\frac{(1-\alpha_n)^2[1+\lambda_{n+1}\mu]}{(1-\alpha_n)}\phi(\|x_n-T^kx_n\|) \\ &+\frac{\eta_n}{(1-\alpha_n)}\|F(p)\|^2-\frac{2(1-\alpha_n)^2\alpha_n}{(1-\alpha_n)}\|T^kx_n-x_n\|^2 \qquad (2.17)\end{aligned}$$

$$\begin{aligned}&\|x_{n+1}-p\|^2 \le \{1-\frac{\Psi_n-(1-\alpha_n)}{-(1-\alpha_n)}\|x_n-p\|^2 \\ &-\frac{(1-\alpha_n)^2[1+\lambda_{n+1}\mu]}{(1-\alpha_n)}\phi(\|x_n-T^kx_n\|) \\ &+\frac{\eta_n}{(1-\alpha_n)}\|F(p)\|^2-\frac{2(1-\alpha_n)^2\alpha_n}{(1-\alpha_n)}\|T^kx_n-x_n\|^2 \qquad (2.18)\end{aligned}$$

since $-\frac{1}{(1-\alpha_n)} \le -1$, we have

$$\begin{aligned}\|x_{n+1} &- p\|^2 \le \{1+\delta_{in}\}\|x_n-p\|^2 \\ &-(1-\alpha_n)^2[1+\lambda_{n+1}\mu]\phi(\|x_n-T^kx_n\|) \\ &-2(1-\alpha_n)^2\alpha_n\|T^kx_n-x_n\|^2+\sigma_n \qquad (2.19)\end{aligned}$$

where

$$\begin{aligned}\delta_{in} &= \Psi_n-(1-\alpha_n) \\ &= (1-\alpha_n)\alpha_n + (1-\alpha_n)^2[1+\lambda_{n+1}\mu][1+\frac{1}{2}(a_{in}-1)] \\ &\quad +(1-\alpha_n)^2[1+\lambda_{n+1}\mu](L+1)^2 \\ &\quad +(1-\alpha_n)^2L_1L_2(L_1L_2+1)\lambda_{n+1}\mu[1+\lambda_{n+1}\mu] \\ &\quad -2(1-\alpha_n)^2\alpha_n\lambda_{n+1}\mu L_1L_2(L_1L_2+1)-(1-\alpha_n) \\ &= (1-\alpha_n)^2\lambda_{n+1}\mu+\frac{1}{2}(1-\alpha_n)^2[1+\lambda_{n+1}\mu](a_{in}-1) \\ &\quad +(1-\alpha_n)^2[1+\lambda_{n+1}\mu](L+1)^2 \\ &\quad +(1-\alpha_n)^2L_1L_2(L_1L_2+1)\lambda_{n+1}\mu[1+\lambda_{n+1}\mu] \\ &\quad -2(1-\alpha_n)^2\alpha_n\lambda_{n+1}\mu L_1L_2(L_1L_2+1) \qquad (2.20)\end{aligned}$$

and

$$\begin{aligned}\sigma_n &= \eta_n\|F(p)\|^2 = \{(1-\alpha_n)^2\lambda_{n+1}\mu[1+\lambda_{n+1}\mu](L_1L_2+1) \\ &\quad -2(1-\alpha_n)^2\alpha_n\lambda_{n+1}\mu(L_1L_2+1)\}\|F(p)\|^2 \qquad (2.21)\end{aligned}$$

From conditions $(ii)-(v)$, $\sum_{n=1}^{\infty}\delta_{in} < \infty$ and $\sum_{n=1}^{\infty}\sigma_n < \infty$. Thus using Lemma 1.3, it follows that $\lim_{n\to\infty}\|x_n - p\|$ exists and $\{x_n\}$ is bounded, thus completing the proof of (a). Since $\{x_n\}$ is bounded, then there exists $M > 0$ such that $\|x_n - p\| \leq M\ \forall n \geq 1$. It follows from (2.19) that

$$\begin{aligned}
(1-\alpha_n)^2[1 &+ \lambda_{n+1}\mu]\phi(\|x_n - T^k x_n\|) \leq \{1+\delta_{in}\}\|x_n - p\|^2 \\
&\quad -\|x_{n+1} - p\|^2 + \sigma_n \\
\sum_{j=N}^{\infty}(1-\alpha_j)^2[1 &+ \lambda_{j+1}\mu]\phi(\|x_j - T^k x_j\|) \leq \sum_{j=N}^{\infty}[\{1+\delta_{ij}\}\|x_j - p\|^2 \\
&\quad -\|x_{j+1} - p\|^2 + \sigma_j] \\
&= \sum_{j=N}^{\infty}[\|x_j - p\|^2 - \|x_{j+1} - p\|^2 + \delta_{ij}\}\|x_j - p\|^2 + \sigma_j] \\
\sum_{n=1}^{\infty}(1-\alpha_n)^2[1 &+ \lambda_{n+1}\mu]\phi(\|x_n - T^k x_n\|) \leq \|x_N - p\|^2 \\
&\quad +M^2\sum_{n=1}^{\infty}\delta_{in} + \sum_{n=1}^{\infty}\sigma_n < \infty
\end{aligned}$$

Conditions (iii) and (iv) imply that $\lim_{n\to\infty}\phi(\|x_n - T^k x_n\|) = 0$. Since ϕ is an increasing and continuous, then $\lim_{n\to\infty}\|x_n - T^k x_n\| = 0$. Observe that,

$$\begin{aligned}
\|x_n - Tx_n\| &= \|x_n - T^k x_n + T^k x_n - Tx_n\| \leq \|x_n - T^k x_n\| \\
&\quad +\|T^k x_n - Tx_n\| \\
&\leq \|x_n - T^k x_n\| + \|T^k x_n - Tx_n\| = \|x_n - T^k x_n\| \\
&\quad +\|TT^{k-1}x_n - Tx_n\| \\
&\leq \|x_n - T^k x_n\| + L\|T^{k-1}x_n - x_n\| \\
&= \|x_n - T^k x_n\| + L\|T^{k-1}x_n - T^{k-1}x_{n-1} \\
&\quad +T^{k-1}x_{n-1} - x_n\| \\
&\leq \|x_n - T^k x_n\| + L\|T^{k-1}x_n - T^{k-1}x_{n-1}\| \\
&\quad +L\|T^{k-1}x_{n-1} - x_{n-1}\| + L\|x_{n-1} - x_n\| \\
&= \|x_n - T^k x_n\| + L(L+1)\|x_{n-1} - x_n\| \\
&\quad +L\|T^{k-1}x_{n-1} - x_{n-1}\|
\end{aligned} \tag{2.22}$$

Observe that,

$$\begin{aligned}
\|x_n - x_{n-1}\| &= \|x_{n-1} - [\alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})\{T^{k-1}x_{n-1} \\
&\quad -\lambda_n\mu F(T^{k-1}x_{n-1})\}]\| \\
&\leq \|x_{n-1} - T^{k-1}x_{n-1}\| + \lambda_n\mu\|F(T^{k-1}x_{n-1})\|
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
\|x_n &- Tx_n\| \leq \|x_{n-1} - T^k x_n\| + L(L+2)\|x_{n-1} - T^{k-1}x_{n-1}\| \\
&+\lambda_n\mu(L+1)L_1^2L_2\|x_{n-1} - p\| + \lambda_n\mu(L+1)\|F(p)\|.
\end{aligned} \tag{2.24}$$

Hence $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Thus completing the proof of (b). Since $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ which converges strongly to p and $\lim_{n\to\infty}\|x_n - p\| = 0$ exists, by Lemma 1.3. Thus completing the proof. □

3 CONCLUDING REMARKS

Remark 3.1. If T is in addition completely continuous or demicompact, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T. Furthermore, if T satisfies condition (A), then $\liminf_{n\to+\infty} d(x_n, F(T)) = 0$, so under the conditions of Theorem 2.1, if T satisfies condition (A), then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.

Remark 3.2. The strong monotonicity condition imposed on F in [8] is not required in our results.

Remark 3.3. Our results extend, generalize and complement the results of Wang [8], Osilike, Isiogugu and Nwokoro [14] and others in literature.

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