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On graded n-primaly ideals

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Abstract. Let G be an abelian group, R be a G-graded commutative ring and I be a graded ideal of (R , G). In this paper, we introduce the concepts of G- n-adjoint, not G -n-primary and uniformly not G -n-primary sets for I, for any positive integer n. We show that G- n-adjoint sets of I are not necessarily graded ideals. Thus we define a graded ideal to be graded n-primaly if G- n-adjoint set of I is graded ideal. We also introduce the concept of n-graded radical of I and study the relation between graded ideal I, G- n-adjoint sets of I, n-graded radical of I and graded radical of I. Also we investigate the relation between graded prime ideal or graded primary ideal on one hand and graded n-primaly ideals on the other hand, and study all the previous concepts in details illustrated by several examples.

1 Introduction

Let G be an abelian group with identity 1 and R be a commutative ring.. Then R is called a G-graded ring (or simply graded ring) and is denote by (R,G) if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \bigoplus_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$.the elements of R_g are called homogenous of degree g. If $a \in R$, then a can be written uniquely as $\sum_{g\in G} a_g$, where a_g is the component of a in R_g. Also the elements of $h(R) = \bigcup_{g\in G} R_g$ are called the homogenous elements of R, (see [3]).
Graded rings have been studied since 1955,(see for instance [1, 2]), then various researchers interested in these rings and made several important studies in them and construct a new branch in ring theory.This interest in studying graded rings was because there is a wide variety of applications of these rings in geometry and physics, see [3],[4],[5] .
We first introduce the known definitions that we will use throughout the research, see [6] and [7] .A graded ideal I of (R,G) is an ideal I of R in which $I = \bigoplus_{g\in G}(I \bigcap R_g)$.
The proper graded ideal I is graded prime ideal of (R,G) and denoted by G-prime , if whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$. The proper graded ideal I is graded primary ideal of (R,G) and denoted by G-primary , if whenever$a, b \in h(R)$ with $ab \in I$, then $a \in I$ or $b \in Gr(I)$ where Gr(I) is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$, Gr(I) is called The graded radical of I.
Let n be a positive integer. Let I be a graded ideal of (R,G), in Section 2, we introduce the concept of Graded n-primary element to the ideal I, which we denoted by G- n-primary element to I . Thus for a positive integer n, we say that an element $a = \sum_{g\in G} a_g \in R$ of a graded ring R is G- n-primary to I if at least one component a_g of a satisfies the condition that if $a_g^n r \in I$ with $r \in h(R)$ implies that $r^m \in R$ for some positive integer m. Based on this concept, we present sets that are not G-n-primary to I and sets that are uniformly not G-n-primary to I. Then we study the relation between these two sets and find the conditions that make these two sets equivalent. We also introduce the concept of n-graded radical of I and illustrate these concepts by several examples.
The set of all elements that are not G-n-primary to I is called the G-n-adjoint set for I and is denoted by G-n-adj(I).
Section 3 deals with G-n-adjoint set for the graded ideal I. We study the relation between G-n-

adjoint set for the graded ideal I, for different n's, we give several examples based on these G-n-adjoint sets. We also study the relation between these sets, the set $G(I) = \{a = \sum_{g\in G} a_g \in R :$ for all $g \in G$, there exists $r \in h(R) - I$ such that $a_g r \in I\}$ and the sets of n-graded radical of I. Finally, in Section 4 we clarify that the G-n-adjoint sets of a graded ideal I are not necessarily graded ideals. However, if they are graded ideals, then we call the graded ideal I to be graded n-primaly ideals. We present some properties of graded n-primaly ideals and study the conditions that make a graded proper ideal a graded n-primaly. We also study the the relation between graded primary ideal and graded n-primaly ideals and prove that every graded prime ideal graded n-primaly.

Throughout this research, all graded rings are assumed to be commutative with non zero identity 1.

2 Graded n-primary elements to an ideal

We start this section by the following definition

Definition 2.1. Let n be a positive integer. Let I be a graded ideal of (R,G). Let

$$s = \sum_{g\in G} s_g \in R.\ Define\ s^{-n}I = \{a = \sum_{k\in G} a_k \in R : s_g^n a_k \in I,\ for\ all\ g, k \in G\}$$

.

Remark 2.2. Let n be a positive integer. Let I be a graded ideal of (R,G).

- If $s \in h(R)$, then $s^{-n}I = \{a = \sum_{k\in G} a_k \in R : s^n a_k \in I,\ for\ all\ k \in G\}$.
- If r is a homogenous element in (R,G), then
  $r \in s^{-n}I$ if and only if for all $g \in G, s_g^n r \in I.$

Remark 2.3. Let I be a graded ideal of (R, G). Let $s = \sum_{g\in G} s_g \in R$ Then $I \subseteq s^{-1}I \subseteq s^{-2}I \subseteq s^{-3}I \subseteq$ ... .But the equality does not holds in general.

Proof. The inclusion is clear and can noticed directly from the definition since I is a graded ideal of (R,G). However, the equality does not hold in general because if we take $G = Z_2, R_0 = Z, R_1 = iZ$, then $R = Z[i] = R_0 \oplus R_1$ is a graded ring and $I = 8R$ is a graded ideal of R, with $(2 + 2i)^{-1}I = 4R, (2 + 2i)^{-2}I = 2R$ and $(2 + 2i)^{-3}I = R$.□

Definition 2.4. Let n be a positive integer. Let I be a graded ideal of (R,G).

(i) An element $a \in h(R)$ is called g-n-primary to I if $a^n r \in I$ with $r \in h(R)$ implies that $r^m \in R$ for some positive integer m.

(ii) An element $a = \sum_{g\in G} a_g \in R$ is called G-n-primary to I if at least one component a_g of a is g-n-primary to I.

Remark 2.5. Let I be a graded ideal of (R,G). We can notice directly from the definition that an element $a = \sum_{g\in G} a_g \in R$ is not G-n-primary to I if for all $g \in G$, there exists $b_{g'} \in h(R)$ with $a_g^n b_{g'} \in I$, but $b_{g'}^m \notin I$ for any positive integer m.

Example 2.6. Let $G = \mathbb{Z}_2, R_0 = \mathbb{Z}, R_1 = i\mathbb{Z}$, then $R = \mathbb{Z}[i] = R_0 \oplus R_1$ is a graded ring and $I = 6R$ is a graded ideal of R. Note that for any positive integer n ,
$2^n(3) \in I$ and $(3i)^n \in I,\ 2, 3 \in h(R)$, but neither $3^m \in I$ nor $2^m \in I$ for any positive Integer m. Thus 2 and $3i$ are not g-n-primary to I , where $g = 0, 1$, respectively. Therefore, $2 + 3i$ is not G-n-primary to I for any positive integer n.

Example 2.7. Let $G = \mathbb{Z}_2, R_0 = \mathbb{Z}, R_1 = i\mathbb{Z}$, then $R = \mathbb{Z}[i] = R_0 \oplus R_1$ is a graded ring and $I = 4R$ is a graded ideal of R. Since 2 is 0-1-primary to I, because if $2r \in 4R$ with $r \in h(R)$, then either $r \in R_0 = \mathbb{Z}$ and this implies that $r \in 2\mathbb{Z}$, or $r \in R_1 = i\mathbb{Z}$ and this implies that

$r \in i2\mathbb{Z}$. So $r^2 \in 4R$. Thus $x = 2 + ib$ is G-1-primary to I ,for all $b \in \mathbb{Z}$. But $2 + 2i$ is not G-2-primary to I , since 2 is not 0-2-primary to I, note that $2^2(1) \in 4R$,however $1^m = 1 \notin I$ for any positive integer m.
Also $2i$ is not 1-2-primary to I since $(2i)^2(1)$, however $1^m = 1 \notin I$ for any positive integer m.

The following result is trivial, since it follows immediately from definitions.

Remark 2.8. If $s \in h(R)$, then s is g- n-primary to I if and only if $s^{-n}I \subseteq Gr(I)$, note that if $r \in h(R)$, then $r \in Gr(I)$ if and only if $r^n \in I$ for some positive integer n.

Example 2.9. Let $G = \mathbb{Z}_2, R_0 = \mathbb{Z}, R_1 = i\mathbb{Z}$, then $R = \mathbb{Z}[i] = R_0 \oplus R_1$ is a graded ring and $I = 4R$ is a graded ideal of R. $Gr(I) = 2R.2^{-1}I = 2R$ and $2^{-2}I = R$.
Thus, by Remark 2.8, 2 is 0-1-primary to I , but 2 is not 0-2-primary to I.

Definition 2.10. Let n be a positive integer. Let I be a graded ideal of (R,G).

(i) A subset A of R is not G-n-primary to I if for every element $a \in A$, a is not G-n-primary to I .

(ii) A subset A of h(R) is not G-n-primary to I if for every element $a = a_g \in A$, for some $g \in G,\ a$ is not g-n-primary to I .

Definition 2.11. Let n be a positive integer. Let I be a graded ideal of (R,G). A subset A of R is uniformly not G-n-primary to I if there exists $b \in R - Gr(I)$ such that the set $\{a_g^n b$, for all $a = \sum_{g \in G} a_g \in A\} \subseteq I$.

Example 2.12. Let n be a positive integer. Let I be a graded ideal of (R,G), where R is a ring with identity $1_R = \sum_{t \in G} 1_R$ .Then the set $\{a_g^n 1_R = a_g^n$ for all $a = \sum_{g \in G} a_g \in I\} \subseteq I$.
Thus I is uniformly not G-n-primary to itself.

Proposition 2.13. *Let n be a positive integer. Let I be a graded ideal of (R,G). If A is uniformly not G-n-primary to I, then A is not G-n-primary to I.*

Proof. Let A be uniformly not G-n-primary to I . Then there exists $b = \sum_{t \in G} b_t$ such that $\{a_g^n b$, for all $a = \sum_{g \in G} a_g \in A\} \subseteq I$. Thus for all $a = \sum_{g \in G} a_g \in A$, for all $g \in G, a_g^n b = a_g^n \sum_{t \in G} b_t \in I$,but $b \notin Gr(I)$. Thus there exists an element $b_t \notin Gr(I)$ for some $t \in G$. Since I is graded ideal of (R,G), $a_g^n b_t \in I$ with $b_t \in h(R)$ but $b_t^m \notin I$ for every positive integer m. Hence a_g is not g-n-primary to I. So for all $a \in A,\ a$ is not G-n-primary to I. Thus A is not G-n-primary to I.□

Remark 2.14. The converse of Proposition 2.13 is not true in general. Let $G = \mathbb{Z}_2, R_0 = \mathbb{Z}, R_1 = i\mathbb{Z}$,then $R = \mathbb{Z}[i] = R_0 \oplus R_1$ is a graded ring and $I = 6R$ is a graded ideal of R, Note that $A = 2, 3i$ is not G-n-primary to I for any positive integer n, as shown in Example 2.6. However, A is not uniformly not G-n-primary to I, since if $2^n b, (3i)^n b \in I$,then b must be in $Gr(I)$.

The following theorem treats the case in which the converse of Proposition 2.13 is true.

Theorem 2.15. *Let n be a positive integer. Let $Gr(I)$ be a G-prime ideal of (R,G), where G is finite. If A is finite subset of R, then A is uniformly not G-n-primary to I if and only if A is not G-n-primary to I.*

Proof. The necessity is by Proposition 2.13. To prove the sufficiency, suppose that $A = \{a_1, a_2, ..., a_m\}$ is not G-n-primary to I. Then for all $a_i = \sum_{g \in G} a_{ig}$, for all $g \in G$, there exists $b_{ig'} \in h(R) - Gr(I)$ such that $a_{ig}^n b_{ig'} \in I$, where $g' \in G$. Let $b_i = \prod_{g \in G} b_{ig'}$, for all $i \in \{1, 2, ..., m\}$. Then $b_i \in R - Gr(I)$, since $Gr(I)$ be G-prime ideal. Let $b = \prod_{i=1}^{m} b_i$, then $b \in R - Gr(I)$ with $\{a_{ig}^n b$, for all $a_i = \sum_{g \in G} a_{ig} \in A\} \subseteq I$. Hence A is uniformly not G-n-primary to I .□

Since for every G-primary ideal I of (R,G), $Gr(I)$ is G-prime ideal of (R,G),see [7] then we can conclude the following result.

Corollary 2.16. *Let n be a positive integer. Let I be a G-primary ideal of (R,G), $Gr(I)$ is G-prime ideal of (R,G) where G is finite. If A is finite subset of R, then A is uniformly not G-n-primary to I if and only if A is not G-n-primary to I.*

Now, consider the following definition.

Definition 2.17. Let n be a positive integer. Let I be a graded ideal of (R,G). The n-graded radical of I, denoted by $Gr_n(I)$, where

$$Gr_n(I) = \{x = \sum_{g \in G} x_g \in R : \ for \ all \ g \in G, x_g^n \in I\}.$$

The following result is trivial, since it follows immediately from definitions.

Remark 2.18. Let n be a positive integer. Let I be a graded ideal of (R,G). Then

- $Gr_1(I) = I$.
- $I \subseteq Gr_n(I) \subseteq Gr(I)$.

Example 2.19. Let $G = \mathbb{Z}_2, R_0 = \mathbb{Z}, R_1 = i\mathbb{Z}$, then $R = \mathbb{Z}[i] = R_0 \oplus R_1$ is a graded ring with $Gr_2(4R) = Gr(4R) = 2R, Gr_2(8R) = 4R$, and $Gr_3(8R) = Gr(8R) = 2R$.

Proposition 2.20. *Let n be a positive integer. Let I be a proper graded ideal of (R,G). Then $Gr_n(I)$ is uniformly not G-n-primary to I.*

Proof. Let $A = Gr_n(I)$ and $b = 1_R$, then $b \in R - Gr_n(I)$ with $\{a_{ig}^n b$, for all $a = \sum_{g \in G} a_g \in A\} \subseteq I$. Hence A is uniformly not G-n-primary to I.□

3 G-n-adjoint sets for an ideal

Definition 3.1. Let n be a positive integer. Let I be a graded ideal of (R,G). The set of all elements that are not G-n-primary to I is called the G-n-adjoint set for I and is denoted by G-n-adj(I). That is $G - n - adj(I) = \{a = \sum_{g \in G} a_g : \ for \ all \ g \in G, \exists b_{g'} \in h(R) - Gr(I)$ with $a_g^n b_{g'} \in I\}$.

Remark 3.2. If I is a graded ideal of (R,G), then $G - n - adj(I) \neq R$, for every positive integer n.

Proof. If $G - n - adj(I) = R$, then $1 \in G - n - adj(I)$. Thus there exists $b \in h(R) - Gr(I)$, such that $b = 1(b) \in I$, which is a **contradiction**.□

Example 3.3. Let $G = \mathbb{Z}_2, R_0 = \mathbb{Z}, R_1 = i\mathbb{Z}$, and $R = \mathbb{Z}[i] = R_0 \oplus R_1$, then R is a graded ring with $G - 1 - adj(4R) = 4R$
$G - n - adj(4R) = 2R$, for every positive integer $n \geq 2$.
$G - n - adj(6R) = 2R \cup 3R$ for every positive integer n.
$G - 1 - adj(8R) = 8R$.
$G - 2 - adj(8R) = 4R$.
$G - n - adj(8R) = 2R$, for every positive integer $n \geq 3$.
$G - 1 - adj(9R) = 9R$.
$G - n - adj(9R) = 3R$, for every positive integer $n \geq 2$.
$G - 1 - adj(12R) = 4R \cup 3R$.
$G - n - adj(12R) = 2R \cup 3R$, for every positive integer $n \geq 2$.

Remark 3.4. Let n be a positive integer. Let I be a graded ideal of (R,G), then

(i) $G - n - adj(I)$ is not necessarily a graded ideal of (R,G), since in Example 3.3 $G - n - adj(6R) = 2R \cup 3R$ is a subset of R, which is not an ideal of (R,G).

(ii) We can show, directly from the definition, that for a proper graded ideal I of (R,G), $I \subseteq G-1-adj(I) \subseteq G-2-adj(I) \subseteq G-3-adj(I) \subseteq ...$.

(iii) The equality in (ii) does not hold in general, since as in Example 3.3, $G-1-adj(8R) = 8R, G-2-adj(8R) = 4R$ and $G-3-adj(8R) = 2R$.

The following result follows directly from the previous remark.

Proposition 3.5. *If I is a graded proper ideal of (R,G), where R is a Noetherian ring and $G-n-adj(I)$ are ideals of R for every positive integer n, then there exists a positive integer m such that $\bigcup_{n=1}^{\infty} G-n-adj(I) = G-m-adj(I)$.*

Example 3.6. Let $G = \mathbb{Z}_2, R_0 = \mathbb{Z}, R_1 = i\mathbb{Z}$, and $R = \mathbb{Z}[i] = R_0 \oplus R_1$. Since R is a Noetherian graded ring, then

$$\bigcup_{n=1}^{\infty} G-n-adj(4R) = G-2-adj(4R).$$

$$\bigcup_{n=1}^{\infty} G-n-adj(8R) = G-3-adj(8R).$$

$$\bigcup_{n=1}^{\infty} G-n-adj(9R) = G-2-adj(9R).$$

Now, remember the following definition , see [8]

Definition 3.7. Let I be a graded ideal of (R,G). Define the following two sets $g(I) = \{a \in R :$ a is not g-prime to $I\} = \{a \in h(R) : ar \in I$ for some $r \in h(R) - I\}$, and
$G(I) = \{a = \sum_{g\in G} a_g \in R : \; for \; all \; g \in G, a_g$ is not g-prime to $I\}$
$= \{a = \sum_{g\in G} a_g \in R : \; for \; all \; g \in G, \; there \; exists \; r \in h(R) - I$ such that $a_g r \in I\}$.

Proposition 3.8. *For any graded ideal I of (R,G), $G-1-adj(I) \subseteq G(I)$.*

Proof. Let $a \in G-1-adj(I)$. Then $a = \sum_{g\in G} a_g \in R$ and for all $g \in G$, there exists$r = b_{g'} \in h(R) - I$ such that $a_g r \in I$. Since $I \subseteq Gr(I)$(see [2]) , then $b_{g'} \in h(R) - I$ with $a_g r \in I$, so $a \in Gr(I)$.□

Remark 3.9. Since in the graded ring $R = \mathbb{Z}[i] = R_0 \oplus R_1$, where $G = \mathbb{Z}_2, R_0 = \mathbb{Z}, R_1 = i\mathbb{Z}$, $G-1-adj(4R) = 4R$ and $G(4R) = 2R$, then the equality in Proposition 3.8 does not hold in general.

Theorem 3.10. *If I is a G-prime ideal of (R,G), then $G-1-adj(I) = G(I)$.*

Proof. Let $a \in G(I)$, then $a = \sum_{g\in G} a_g \in R$ and for all $g \in G$, there exists $r \in h(R) - I$ such that $a_g r \in I$. Since I is G-prime, then $r \in h(R) - Gr(I)$ with $a_g r \in I$. Thus $a \in G-1-adj(I)$. By Proposition 3.8, the equality holds.□

The following result follows immediately from the previous theorem and Remark 3.4

Corollary 3.11. *If I is a G-prime ideal of (R,G), then $G(I) = G-1-adj(I) \subseteq G-2-adj(I) \subseteq G-3-adj(I) \subseteq ...$, that is ,*
$G(I) \subseteq G-n-adj(I)$, for every positive integer n.

Proposition 3.12. *Let n be a positive integer. Let I be a graded proper ideal of (R,G). $I \subseteq Gr_n(I) \subseteq G-n-adj(I)$.*

Proof. It is clear as noted in Remark 2.18 that $I \subseteq Gr_n(I)$. Now, let $x = \sum_{g\in G} x_g \in Gr_n(I)$, then $x \in R$ and for all $g \in G, x_g^n \in I$. Thus $x_g^n(1) \in I$. Since $1 \in R - Gr(I)$, then $x \in G-n-adj(I)$.□

Proposition 3.13. *Let n be a positive integer. Let I be a G-prime ideal of (R,G). Then*

$$I \subseteq Gr_n(I) \subseteq Gr(I) \subseteq G - n - adj(I).$$

Proof. By Remark 2.18, $I \subseteq Gr_n(I) \subseteq Gr(I)$. Now, let $a = \sum_{g\in G} a_g \in Gr(I)$, then for all $g \in G$, there exists a positive integer n_g such that $a_g^{n_g} \in I$. for all $g \in G$, let m_g be the smallest such positive integer, then

(i) If $m_g = n$, then $a_g^{m_g}(1) = a_g^n(1) \in I, 1 \in R - Gr(I)$.

(ii) If $m_g < n$, then $a_g^{m_g} \in I$ implies $a_g^n(1) = a_g^{n-m_g} a_g^{m_g} \in I, 1 \in R - Gr(I)$.

(iii) If $m_g > n$, then $a_g^{m_g} \in I a_g^n a_g^{m_g-n} \in I$ with $a_g^{m_g-n} \in R - I$ implies $a_g^{m_g-n} \in R - Gr(I)$ because I is G-prime ideal of (R,G). Let $b = a_g^{m_g-n}$, then $b \in R - Gr(I)$ with $a_g^n b \in I$.

Therefore, $a \in G - n - adj(I)$.□

4 Graded n-primaly ideals

We noticed in the previous section that the G-n-adjoint sets of a graded ideal I of (R,G) are not necessarily graded ideals of (R,G). However, in some cases they will be graded ideals of (R,G). In this section, we will study the graded ideal in which G-n-adjoint sets of it are graded ideals. We call these kinds of graded ideals a graded n-primaly ideals as in the following definition.

Definition 4.1. Let n be a positive integer. A graded ideal I of (R,G) is a graded n-primaly ideal of (R,G) if $G - n - adj(I)$ is a graded ideal of (R,G).

Example 4.2. Let $G = \mathbb{Z}_2, R_0 = \mathbb{Z}, R_1 = i\mathbb{Z}$, and $R = \mathbb{Z}[i] = R_0 \oplus R_1$. By Example 3.3 we notice that $4R, 8R$ and $9R$ are graded n-primaly ideals of (R,G), while $6R$ and $12R$ are not graded n-primaly ideals of (R,G), for every positive integer n.

Proposition 4.3. *Let n be a positive integer. Let I be a graded ideal of (R,G). If $G - n - adj(I)$ is uniformly not G-1-primary to I with the property that for all $r \in R$ and for all $a \in G - n - adj(I), ra \in G - n - adj(I)$, then I is graded n-primaly ideal of (R,G).*

Proof. We have to show that G-n-adj(I) is a graded ideal of (R,G). That is

(i) $G - n - adj(I)$ is an ideal of R, and

(ii) $G - n - adj(I) \subseteq \bigoplus_{g\in G}(G - n - adj(I) \cap R_g)$

Note that (ii) is satisfied by the definition of $G - n - adj(I)$. Now, since $G - n - adj(I)$ satisfies the property that for all $r \in R$ and for all $a \in G - n - adj(I), ra \in G - n - adj(I)$, it is enough to show that $G - n - adj(I)$ is closed under addition. Let $A = G - n - adj(I)$.

Let $a = \sum_{g\in G} a_g, b = \sum_{g\in G} b_g \in A$. Since A is uniformly not G-1-primary to I, then $\exists d \in R - Gr(I)$ such that $\{c_g d : c = \sum_{g\in G} c_g \in A\} \subseteq I$. Thus for all $g \in G, a_g^m(d) b_g^m(d) \in I$, for every positive integer m. Thus for all $g \in G (a_g + b_g)^n(d) = \sum_{k=0}^{n} a_g^{n-k} b_g^k(d) \in I$ with $d \in R - Gr(I)$. Since $d \in R$, then for all $g \in G(a_g + b_g)^n(d) = (a_g + b_g)^n(d) \sum_{h\in G} d_h \in I$. But $d \notin Gr(I)$. Thus there exists an element $d_h \notin Gr(I)$ for some $h \in G$. Since I is graded ideal of $(R, G), (a_g + b_g)^n h_d$ with $b_h \in h(R) - Gr(h)$. Hence $a + b \in A$.□

Proposition 4.4. *Let I be a graded proper ideal of (R,G). If $G-1-adj(I) \subseteq I$, for every positive integer n, then I is G-primary ideal of (R,G).*

Proof. Note first that by Remark 2.5.8, $G - 1 - adj(I) = I$. Let $a, b \in h(R)$ such that $ab \in I$ and $b \notin Gr(I)$, then $a \in G - 1 - adj(I) = I$. Thus I is G-primary ideal of (R,G).□

Example 4.5. By Proposition 4.4 and Example 3.3, $4R, 8R$ and $9R$ are G-primary ideals of $(\mathbb{Z}[i], \mathbb{Z}_2)$.

Proposition 4.6. *Let n be a positive integer. Let I be a G-primary ideal of (R,G). Then* $Gr_n(I) = G - n - adj(I)$*.*

Proof. By Proposition 3.13, it is enough to show that $Gr_n(I) \supseteq G - n - adj(I)$. Let $a = \sum_{g \in G} a_g \in G - n - adj(I)$, then, forall $g \in G$, there exists $r = b_{g'} \in h(R) - Gr(I)$ such that $a_g^n r \in I$. Since I is G-primary ideal of (R,G) with $r \notin Gr(I)$, then $a_g^n \in I$. Therefore, $a \in Gr_n(I)$.□

Proposition 4.7. *Let n be a positive integer. Let I be a G-primary ideal of (R,G) with the property that* $Gr(I) \subseteq Gr_n(I)$*, then I is a graded n-primaly ideal of (R,G) .*

Proof. First note that by Remark 2.18, $Gr(I) \supseteq Gr_n(I)$. Now, we have $Gr(I) = Gr_n(I)$. Since I is G-primary ideal of (R,G), then by Proposition 4.6, $Gr(I) = Gr_n(I) = G - n - adj(I)$. Since $Gr(I)$ is a graded ideal of (R,G) (see [2]), then $G - n - adj(I)$ is a graded ideal of (R,G). Therefore, I is a graded n-primaly ideal of (R,G).□

Theorem 4.8. *Let I be a graded proper ideal of (R,G). If I is a G-prime ideal of (R,G), then* $G - n - adj(I) = Gr_n(I) = Gr(I) = I$*, for every positive integer n.*

Proof. Let I be a G-prime ideal of (R,G). By Proposition 3.13, it is enough to show that $G - n - adj(I) \subseteq I$. Let $a = \sum_{g \in G} a_g \in G - n - adj(I)$, then, for all $g \in G$, there exists $r = b_{g'} \in h(R) - Gr(I)$ such that $a_g^n r \in I$. Since I is G-primary ideal of (R,G) with $r \notin Gr(I)$. Since $I \subseteq Gr(I)$ (see [6]), then $r \notin I$. But I is a G-prime ideal of (R,G), $so a_g^n \in I$. Therefore, $a_g \in I$. Since g is an arbitrary element in G then $a = \sum_{g \in G} a_g \in I$. □
Now, the following two corollaries follow immediately from Theorem 4.8.

Corollary 4.9. *If I is a G-prime ideal of (R,G), then* $G - n - adj(I)$ *is a G-prime ideal of (R,G), for every positive integer n.*

Corollary 4.10. *If I is a G-prime ideal of (R,G), then I is a graded n-primaly ideal of (R,G) for every positive integer n*

Remark 4.11. The converse of Corollary 4.10 is not satisfied in general , since as noted in Example 4.2, $4R$ is a graded n-primaly ideal of (R,G) for every positive integer n, however it is not G-prime ideal of (R,G).

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