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ON PSEUDO-SLANT SUBMANIFOLDS OF (κ, μ)-CONTACT SPACE FORMS

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Abstract. The geometry of pseudo-slant submanifolds of (κ, μ)-contact space forms has been studied. The necessary and sufficient condition for a pseudo-slant submanifold to be mixed-geodesic has been obtained along with some results on totally umbilical pseudo-slant submanifolds of (κ, μ)-contact space form.

1 Introduction

It is known that slant submanifolds are the generalization of invariant and anti-invariant submanifolds, many geometers have shown interest in this study. Chen ([7], [8]) initiated this study on complex manifolds. Lotta [16] introduced the concept of slant immersions in to an almost contact metric manifold. Carriazo introduced another new class of submanifolds called hemi-slant submanifolds (it is also called as anti-slant or pseudo-slant submanifold) [5]. Later many geometers like ([9], [10], [12], [13], [15]) studied pseudo-slant submanifolds on various manifolds.

The notion of (κ, μ)-contact space form was introduced by Koufogiorgos [14], which contains the well known class of Sasakian space forms for $\kappa = 1$. Thus it is worthwhile to study pseudo-slant submanifolds in a (κ, μ)-contact space form. Tripathi et al., [6] introduced generalized (κ, μ)-space forms and proved that the functions of a contact metric generalized (κ, μ)-contact space form $M(f_1, \cdots, f_6)$ of dimension greater than or equal to 5 are constant and are related to each other. Motivated by these studies we plan to study pseudo-slant submanifolds of (κ, μ)-contact space forms.

This paper is organized as follows: Section 2 contains some basic formulas and definitions of (κ, μ)-contact metric manifold and their submanifolds. In section 3, we review some definitions and proved some basic results on pseudo-slant submanifold of (κ, μ)-contact metric manifold. Last section deals with the study of totally umbilical pseudo-slant submanifold in (κ, μ)-contact metric manifold and (κ, μ)-contact space forms.

2 Preliminaries

A $(2m+1)$-dimensional smooth manifold $\tilde{M}$ is said to be contact manifold if it carries a global 1-form η satisfying $\eta \wedge (d\eta)^m \neq 0$ everywhere on $\tilde{M}$. And a $(2m+1)$ dimensional almost contact manifold with almost contact structure (ϕ, ξ, η) consisting of $(1,1)$ tensor field ϕ, global 1-form η and a characteristic vector field ξ satisfies ([1], [2]):

$$\phi^2 = -I + \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = 1, \ \phi\xi = 0, \ \eta \circ \phi = 0. \tag{2.2}$$

Let g be the compatible Riemannian metric with almost contact structure (ϕ, ξ, η) such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(X, \xi) = \eta(X). \tag{2.4}$$

Then $\tilde{M}$ equipped with almost contact metric structure (ϕ,ξ,η,g) is called almost contact metric manifold. Let Φ be the fundamental 2-form on $\tilde{M}$ defined by $\Phi(X,Y) = g(X,\phi Y) = -\Phi(Y,X)$. Now if $\Phi = d\eta$ then almost contact metric structure becomes contact metric structure.

We know that in a contact metric manifold $(\tilde{M},\phi,\xi,\eta,g)$, the symmetric tensor h, defined by $2h = \mathcal{L}_\xi\phi$, satisfies the following [1]

$$h\xi = 0,\ h\phi + \phi h = 0,\ \tilde{\nabla}_X\xi = -\phi X - \phi hX,\ \text{tr}(h) = \text{tr}(\phi h) = 0, \tag{2.5}$$

where $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{M}$. A contact metric manifold $\tilde{M}$ is said to be (κ,μ)-contact metric manifold if the structural vector field ξ belongs to (κ,μ)-nullity distribution defined by [3]

$$\begin{aligned}\mathcal{N}(\kappa,\mu) : p \to \mathcal{N}_p(\kappa,\mu) = \{Z \in T_p\tilde{M} \mid \tilde{R}(X,Y)Z =&\kappa(g(Y,Z)X - g(X,Z)Y)\\ &+ \mu(g(Y,Z)hX - g(X,Z)hY)\},\end{aligned}$$

where κ, μ are constants. We know that in a (κ,μ)-contact metric manifold $\tilde{M}$, $h^2 = (\kappa-1)\phi^2$ and therefore $\kappa \leq 1$. If $\kappa = 1$ then $\tilde{M}$ becomes Sasakian manifold.

Moreover for a (κ,μ)-contact metric manifold $\tilde{M}$ of dimension $2m+1$ and for all $X,Y \in \Gamma(TM)$, we have [2]

$$(\tilde{\nabla}_X\phi)Y = g(X+hX,Y)\xi - \eta(Y)(X+hX). \tag{2.6}$$

The (κ,μ)-contact metric manifold is said to be (κ,μ) contact space form denoted by $\tilde{M}(c)$ if $\tilde{M}$ has constant ϕ-sectional curvature. Now the curvature tensor of $\tilde{M}(c)$ is given by [14]

$$\tilde{R} =\frac{c+3}{4}R_1 + \frac{c-1}{4}R_2 + \left(\frac{c+3}{4} - \kappa\right)R_3 + \frac{1}{2}R_4 + R_5 + (1-\mu)R_6, \tag{2.7}$$

where,

$$\begin{aligned}
R_1(X,Y)Z =&\{g(Y,Z)X - g(X,Z)Y\}\\
R_2(X,Y)Z =&\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}\\
R_3(X,Y)Z =&\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}\\
R_4(X,Y)Z =&\{g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX\}\\
R_5(X,Y)Z =&g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y,\\
R_6(X,Y)Z =&\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi.
\end{aligned}$$

for any vector fields X,Y,Z.

Let M be a submanifold of a contact metric manifold $\tilde{M}$ with induced metric denoted by the same symbol g. Then the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_XY =\nabla_XY + \sigma(X,Y), \tag{2.8}$$

$$\tilde{\nabla}_XV = -A_VX + \nabla^\perp_XV, \tag{2.9}$$

where ∇ and $\nabla^\perp$ are induced connections on the tangent bundle TM and $T^\perp M$ of M respectively, σ and A_V are the second fundamental form and the shape operator with respect to V respectively. Further σ and A_V are related by

$$g(A_VX,Y) = g(\sigma(X,Y),V), \tag{2.10}$$

for any $X,Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$. The mean curvature vector H of M is given by

$$H = \frac{1}{l}tr(\sigma) = \frac{1}{l}\sum_{i=1}^{l}\sigma(e_i,e_i), \tag{2.11}$$

where l is the dimension of M and $\{e_1,e_2,\cdots,e_l\}$ is the local orthonormal frame of M.

- A submanifold is said to be totally umbilical if

$$\sigma(X,Y) = g(X,Y)H, \tag{2.12}$$

where H is the mean curvature vector.

- A submanifold is said to be totally geodesic if $\sigma(X,Y) = 0$.

- A submanifold is said to be minimal if $H = 0$.

Also, we have

$$\sigma^r_{ij} = g(\sigma(e_i,e_j),e_r) \text{ and } ||\sigma||^2 = \sum_{i,j=1}^{l} g(\sigma(e_i,e_j),\sigma(e_i,e_j)), \tag{2.13}$$

for $1 \leq i,j \leq l, l+1 \leq r \leq 2m+1$. Now for any submanifold M of a Riemannian manifold $\tilde{M}$ and for any $X,Y,Z \in \Gamma(TM)$, the covariant derivative of σ is defined by

$$(\tilde{\nabla}_X\sigma)(Y,Z) = \nabla^{\perp}_X\sigma(Y,Z) - \sigma(\nabla_XY,Z) - \sigma(Y,\nabla_XZ). \tag{2.14}$$

Also for the submanifold M, the Riemannian curvature tensor $\tilde{R}$ of $\tilde{M}$ is given by

$$\tilde{R}(X,Y)Z = R(X,Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X + (\tilde{\nabla}_X\sigma)(Y,Z) - (\tilde{\nabla}_Y\sigma)(X,Z), \tag{2.15}$$

where R is the Riemannian curvature tensor of M. The tangent and normal components of the above equation are, respectively

$$(\tilde{R}(X,Y)Z)^T = R(X,Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X, \tag{2.16}$$

$$(\tilde{R}(X,Y)Z)^{\perp} = (\tilde{\nabla}_X\sigma)(Y,Z) - (\tilde{\nabla}_Y\sigma)(X,Z), \tag{2.17}$$

for any $X,Y,Z \in \Gamma(TM)$. Note that M is said to be curvature invariant submanifold of $\tilde{M}$ if $(\tilde{R}(X,Y)Z)^{\perp} = 0$.

The Ricci equation is given by

$$g(\tilde{R}(X,Y)U,V) = g(R^{\perp}(X,Y)U,V) + g([A_U,A_V]X,Y), \tag{2.18}$$

for any $X,Y \in \Gamma(TM)$ and $U,V \in \Gamma(T^{\perp}M)$. Here $R^{\perp}$ denotes the Riemannian curvature tensor tensor of $T^{\perp}M$ and if it is zero then the normal connection of M is flat.

Definition 2.1. A (κ,μ)-contact metric manifold $\tilde{M}$ is said to be η-Einstein manifold if its Ricci tensor S is of the form $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$, where a and b are smooth functions on $\tilde{M}$ and $X,Y \in \Gamma(TM)$.

Before going to main results we first recall a lemma of [18],

Lemma 2.2. *If (M,ϕ,ξ,η,g) is a contact Riemannian manifold and ξ belongs to the (κ,μ)-nullity distribution, then $\kappa \leq 1$. If $\kappa < 1$, then M admits three mutually orthogonal and integrable distributions $D(0), D(\lambda)$ and $D(-\lambda)$ defined by the eigenspaces of h, where $\lambda = \sqrt{1-\kappa}$. Further, if $X \in D(\lambda)$, then $hX = \lambda X$ and if $X \in D(-\lambda)$ then $hX = -\lambda X$.*

3 Pseudo-slant submanifolds of (κ,μ)-contact metric manifold

Let M be a submanifold of (κ,μ) contact metric manifold $\tilde{M}$. Then for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$ we can write

$$\phi X = TX + NX, \tag{3.1}$$

$$\phi V = tV + nV, \tag{3.2}$$

where TX and NX (respectively tV and nV) are the tangential and normal component of ϕX (respectively ϕV). Using (2.1) in the above equations one can get

$$T^2 = -tN - I + \eta\circ\xi, \quad NT + nN = 0, \tag{3.3}$$

$$n^2 = -I - Nt, \qquad Tt + tn = 0. \tag{3.4}$$

Furthermore, from (2.4), (3.1) and (3.2) we can say T and n are skew-symmetric tensor fields. Also for $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, we can obtain relation between N and t as

$$g(NX, V) = -g(X, tV). \tag{3.5}$$

In view of (2.7) and (2.18) we get

$$\begin{aligned} g(\tilde{R}^\perp(X,Y)V,U) =& \frac{c-1}{4}\{g(X,\phi V)g(\phi Y,U) - g(Y,\phi V)g(\phi X,U) + 2g(X,\phi Y)g(\phi V,U)\} \\ &+ \frac{1}{2}\{g(hY,V)g(hX,U) - g(hX,V)g(hY,U) + g(\phi hX,V)g(\phi hY,U) \\ &- g(\phi hY,V)g(\phi hX,U)\} - g([A_U,A_V]X,Y), \end{aligned} \tag{3.6}$$

for $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^\perp M)$. And in view of (2.7), (2.16) and (2.17), we obtain

$$R(X,Y)Z = \frac{c+3}{4}R_1 + \frac{c-1}{4}R_2^T + \left(\frac{c+3}{4} - \kappa\right)R_3 + \frac{1}{2}R_4^T + R_5 + (1-\mu)R_6, \tag{3.7}$$

$$\begin{aligned} \text{where, } R_2^T =& g(X,\phi Z)TY - g(Y,\phi Z)TX + 2g(X,\phi Y)TZ \\ \text{and } R_4^T =& g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)ThY - g(\phi hY,Z)ThX \end{aligned}$$

and

$$\begin{aligned} (\tilde{\nabla}_X\sigma)(Y,Z) - (\tilde{\nabla}_Y\sigma)(X,Z) = & \left(\frac{c-1}{4}\right)\{g(X,\phi Z)NY - g(Y,\phi Z)NX + 2g(X,\phi Y)NZ\} \\ &+ \frac{1}{2}\{g(\phi hX,Z)NhY - g(\phi hY,Z)NhX\}, \end{aligned} \tag{3.8}$$

for all $X, Y, Z \in \Gamma(TM)$. Again from (2.6), we obtain the following;

$$(\nabla_X T)Y = A_{NY}X + t\sigma(X,Y) + g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{3.9}$$

$$(\nabla_X N)Y = n\sigma(X,Y) - \sigma(X,TY), \tag{3.10}$$

$$(\nabla_X t)V = A_{nV}X - TA_VX, \tag{3.11}$$

$$(\nabla_X n)V = -\sigma(X,tV) - NA_VX, \tag{3.12}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.

Now let us recall some definitions of of classes of submanifolds. Let M be a submanifold, then M is said to be

(i) Invariant submanifold if N is identically zero in (3.1), i.e., $\phi X \in TM,\ \forall X \in TM$.

(ii) Anti-invariant submanifold if T is identically zero in (3.1), i.e., $\phi X \in T^\perp M,\ \forall X \in TM$.

(iii) Slant submanifold if there exists an angle $\theta(x) \in [0, \pi/2]$ between ϕX and TM for all non-zero vector X tangent to M at x called slant angle which is constant.

(iv) Pseudo-slant submanifold if there exists distributions D_θ and $D^\perp$ such that (1) TM admits orthogonal direct composition $TM = D_\theta \oplus D^\perp, \xi \in D_\theta$, (2) D_θ is a slant distribution with slant angle $\theta \neq \pi/2$ and (3) $D^\perp$ is an anti-invariant distribution [12].

From the above definitions we can note that slant submanifold is the generalization of invariant (if $\theta = 0$) and anti-invariant (if $\theta = \pi/2$) submanifolds. A proper slant submanifold is neither invariant nor anti-invariant submanifold i.e., $\theta \in (0, \pi/2)$. Hence in general we have the following theorem which characterize slant submanifolds of almost contact metric manifolds;

Theorem 3.1. *[4] Let M be a slant submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in \Gamma(TM)$. Then, M is slant submanifold if and only if there exist a constant $\gamma \in [0,1]$ such that*

$$T^2 = -\gamma(I - \eta \otimes \xi), \tag{3.13}$$

furthermore, in this case, if θ is the slant angle of M, then $\gamma = \cos^2\theta$.

Corollary 3.2. *[4] Let M be a slant submanifold of an almost contact metric manifold $\tilde{M}$ with slant angle θ. Then for any $X, Y \in \Gamma(TM)$, we have*

$$g(TX, TY) = \cos^2\theta\{g(X, Y) - \eta(X)\eta(Y)\}, \tag{3.14}$$

$$\text{and } g(NX, NY) = \sin^2\theta\{g(X, Y) - \eta(X)\eta(Y)\}. \tag{3.15}$$

If we denote the orthogonal complementary of ϕTM in $T^{\perp}M$ by ν, then the normal bundle $T^{\perp}M$ can be decomposed as follows

$$T^{\perp}M = N(D_\theta) \oplus N(D^{\perp}) \oplus \nu.$$

From the above decomposition one can see that $g(X, Z) = 0$, for each $X \in \Gamma(D_\theta)$ and $Z \in \Gamma(D^{\perp})$. And hence $g(NX, NZ) = g(\phi X, \phi Z) = g(X, Z) = 0$. Also we know that $t\sigma = 0$, so (3.9) reduces to

$$(\nabla_X T)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX),$$

for all $X, Y \in D_\theta$. Hence we infer that the induced structure T is a (κ, μ)-contact metric structure on M if the ambient manifold $\tilde{M}$ is a (κ, μ)-contact metric manifold.

Using (2.9) and (2.6) we can state the following result.

Lemma 3.3. *In a pseudo-slant submanifold of (κ, μ)-contact metric manifold, $A_{NZ}W = A_{NW}Z$ for any $Z, W \in \Gamma(D^{\perp})$.*

Theorem 3.4. *Let M be a proper pseudo-slant submanifold of (κ, μ)-contact metric manifold $\tilde{M}$. Then (1) N is parallel. (2) t is parallel. (3) $A_{nV}Y + A_V TY = 0$, for any $Y \in \Gamma(TM), V \in \Gamma(T^{\perp}M)$ are equivalent.*

Proof. Let N be parallel. For any $X, Y \in \Gamma(TM), V \in \Gamma(T^{\perp}M)$, we have from (3.10) that

$$g((\nabla_X N)Y, V) = g(n\sigma(X, Y), V) - g(\sigma(X, TY), V) = 0. \tag{3.16}$$

In view of (2.10) and (3.11), the above equation become

$$-g(A_{nV}X, Y) + g(TA_V X, Y) = -g(A_{nV}X - TA_V X, Y) = -g((\nabla_X t)V, Y) = 0$$

Hence t is parallel. Converse part is obvious. Again if N is parallel, then from (3.16), we have

$$-g(A_{nV}Y, X) - g(A_V X, TY) = -g(A_{nV}Y + A_V TY, X) = 0.$$

Converse part is trivial. This proves our assertion. □

Theorem 3.5. *Let M be a proper pseudo-slant submanifold of a (κ, μ)-contact metric manifold. Then the covariant derivative of T is skew-symmetric.*

Proof. For any $X, Y, Z \in \Gamma(TM)$, we have from (3.9) that

$$g((\nabla_X T)Y, Z) = g(A_{NY}X + t\sigma(X, Y) + g(X + hX, Y)\xi - \eta(Y)(X + hX), Z).$$

Using (2.10) in the above equation we get

$$\begin{aligned} g((\nabla_X T)Y, Z) =& g(\sigma(X, Z), NY) + g(t\sigma(X, Y), Z) + g(X + hX, Y)\eta(Z) \\ &- \eta(Y)g(X + hX, Z) \\ =& -\{g(t\sigma(X, Z), Y) + g(\sigma(X, Y), NZ) - g(X + hX, Y)\eta(Z) \\ &+ \eta(Y)g(X + hX, Z)\} = -g((\nabla_X T)Z, Y). \end{aligned}$$

This completes the proof. □

Theorem 3.6. *Let M be a proper pseudo-slant submanifold of a (κ,μ)-contact metric manifold. Then the covariant derivative of n is skew-symmetric.*

Proof. For any $X \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$, in view of (3.12) and (2.10) and following same procedure of the above theorem we obtain $g((\nabla_X n)V, U) = -g(V, (\nabla_X n)U)$. □

Lemma 3.7. *Let M be a proper pseudo-slant submanifold of a (κ,μ)-contact metric manifold $\tilde{M}$. Then n is parallel if and only if the shape operator A_V of M satisfies the condition $A_V tU = A_U tV$ for all $U, V \in \Gamma(T^{\perp}M)$.*

Proof. Let n be parallel. Then from (3.12), (2.10) and (3.5) we have

$$g((\nabla_X n)V, U) = -g(\sigma(X, tV), U) - g(NA_V X, U) = 0$$
$$-g(A_U tV, X) + g(A_V X, tU) = 0$$
$$-g(A_U tV - A_V tU, X) = 0.$$

Hence we get $A_U tV = A_V tU$ for $X \in \Gamma(TM), U, V \in \Gamma(T^{\perp}M)$. □

Definition 3.8. A pseudo-slant submanifold M of a (κ,μ)-contact metric manifold $\tilde{M}$ is said to be mixed-geodesic submanifold if $\sigma(X, Z) = 0$ for any $X \in \Gamma(D_\theta)$ and $Z \in \Gamma(D^{\perp})$.

Theorem 3.9. *Let M be a totally umbilical proper pseudo-slant submanifold of a (κ,μ)-contact metric manifold $\tilde{M}$. If t is parallel then M is either mixed-geodesic or anti-invariant submanifold.*

Proof. Let $X \in D_\theta, Y \in D^{\perp}$ and t is parallel. By Theorem 3.4, we have N is parallel and hence $(\nabla_X N)Y = 0$. This implies

$$n\sigma(X, Y) - \sigma(X, TY) = 0.$$

Replacing X by TX in the above equation we get

$$n\sigma(TX, Y) - \sigma(TX, TY) = 0.$$

Since M is totally umbilical, from (2.12) and (3.13) the above equation reduces to

$$-\sigma(T^2X, Y) = \cos^2\theta\sigma(X, Y) = 0.$$

Hence we get either $\theta = \pi/2$ (M is anti-invariant) or $\sigma(X, Y) = 0$ (M is mixed-geodesic). This completes our proof. □

4 Pseudo-slant submanifolds of (κ,μ)-contact space forms

Let us define $\{e_1, \cdots, e_p, e_{p+1} = \sec\theta Te_1, \cdots, e_{2p} = \sec\theta Te_p, e_{2p+1} = \xi, e_{2p+2}, \cdots, e_{2p+q+1}\}$ as an orthonormal basis of $\Gamma(TM)$ such that $\{e_1, \cdots, e_{2p+1}\} \in \Gamma(D_\theta)$ and $\{e_{2p+2}, \cdots, e_{2p+q+1}\} \in \Gamma(D^{\perp})$.

Theorem 4.1. *Let M be a proper pseudo-slant submanifold of a (κ,μ)-contact space form $\tilde{M}(c)$. Then the Ricci tensor S of M is given by*

$$\begin{aligned} S(X, W) =&(K_1 + K_3 + K_5 + K_6)g(X, W) + (2K_2 + K_4 - K_5)\eta(X)\eta(W) \\ &+ (2p + q - 1)g(\sigma(X, W), H) - \sum_{l=1}^{2p+q} g(\sigma(e_l, W), \sigma(X, e_l)), \end{aligned} \tag{4.1}$$

where

$$K_1 = (p-2)\left(\frac{c+3}{4}\right) + (2p-3-\mu)(\pm\lambda) + \frac{p-2}{2}(\pm\lambda^2) + \frac{3(c-1)}{4} + \kappa,$$
$$K_2 = (1-\mu)(\pm\lambda) - \frac{c-3}{2} - \kappa + \frac{(\pm\lambda^2)}{2},$$
$$K_3 = (p-2)\left(\frac{c+3}{4} \pm 2\lambda + \frac{(\pm\lambda^2)}{2}\right) + \left(\frac{c-1}{4} - \frac{(\pm\lambda^2)}{2}\right)\cos^2\theta + \frac{c-1}{2},$$
$$K_4 = \left(\frac{c+3}{4} \pm 2\lambda + \frac{(\pm\lambda^2)}{2}\right) - \left(\frac{c-1}{4} - \frac{(\pm\lambda^2)}{2}\right)\cos^2\theta - \frac{c-1}{2} - (p-1)\left(\frac{c+3}{4} - \kappa + (1-\mu)(\pm\lambda)\right),$$
$$K_5 = \pm\lambda + \frac{(\pm\lambda^2)}{2} + \kappa - (1-\mu)(\pm\lambda) - \sin^2\theta,$$
$$K_6 = (q-2)\left(\frac{c+3}{4}\right) + (2q-3-\mu)(\pm\lambda) + \frac{q-2}{2}(\pm\lambda^2) + \frac{3(c-1)}{4} + \kappa,$$

for any $X, W \in \Gamma(TM)$.

Proof. For any $X, Y, Z, W \in \Gamma(TM)$, by (2.7), (2.15) and Lemma 2.2 we have

$$\begin{aligned} g(R(X,Y)Z,W) = &\left(\frac{c+3}{4} \pm 2\lambda + \frac{(\pm\lambda^2)}{2}\right)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ &+ \left(\frac{c-1}{4} - (\pm\lambda^2)\right)\{g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W)\} \\ &+ \frac{c-1}{2} g(X,\phi Y)g(\phi Z,W) \\ &+ \left(\frac{c+3}{4} - \kappa + (1-\mu)(\pm\lambda)\right)\{\eta(X)\eta(Z)g(Y,W) \\ &- \eta(Y)\eta(Z)g(X,W) + \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z)\} \\ &+ g(\sigma(X,W),\sigma(Y,Z)) - g(\sigma(Y,W),\sigma(X,Z)). \end{aligned} \tag{4.2}$$

Now consider the orthonormal basis as defined in the beginning of this section. By taking $Y = Z = e_i, e_j, \xi, e_k$ where $1 \le i \le p, p+1 \le j \le 2p, 2p+2 \le k \le 2p+q+1$ repeatedly in (4.2) and adding the resultant equations we obtain

$$\begin{aligned} S(X,W) = &\sum_{i=1}^{p} g(R(X,e_i)e_i,W) + \sum_{i=p+1}^{2p} g(R(X,\sec\theta Te_j)\sec\theta Te_j,W) \\ &+ g(R(X,\xi)\xi,W) + \sum_{k=2p+2}^{2p+q+1} g(R(X,e_k)e_k,W). \end{aligned} \tag{4.3}$$

On simplifying, we obtain

$$\begin{aligned} g(R(X,e_i)e_i,W) = &K_1 g(X,W) + K_2\eta(X)\eta(W) \\ &+ g(\sigma(X,W),\sigma(e_i,e_i)) - g(\sigma(e_i,W),\sigma(X,e_i)), \end{aligned} \tag{4.4}$$

where $K_1 = (p-2)\left(\frac{c+3}{4}\right) + (2p-3-\mu)(\pm\lambda) + \frac{p-2}{2}(\pm\lambda^2) + \frac{3(c-1)}{4} + \kappa,$

and $K_2 = (1-\mu)(\pm\lambda) - \frac{c-3}{2} - \kappa + \frac{(\pm\lambda^2)}{2},$

$$g(R(X,\sec\theta Te_j)\sec\theta Te_j,W) = K_3 g(X,W) + K_4\eta(X)\eta(W) + g(\sigma(X,W),\sigma(\sec\theta Te_j,\sec\theta Te_j)) - g(\sigma(\sec\theta Te_j,W),\sigma(X,\sec\theta Te_j)), \tag{4.5}$$

$$\text{where } K_3 = (p-2)\left(\frac{c+3}{4}\pm 2\lambda+\frac{(\pm\lambda^2)}{2}\right)+\left(\frac{c-1}{4}-\frac{(\pm\lambda^2)}{2}\right)\cos^2\theta+\frac{c-1}{2},$$
$$\text{and } K_4 = \left(\frac{c+3}{4}\pm 2\lambda+\frac{(\pm\lambda^2)}{2}\right)-\left(\frac{c-1}{4}-\frac{(\pm\lambda^2)}{2}\right)\cos^2\theta-\frac{c-1}{2} - (p-1)\left(\frac{c+3}{4}-\kappa+(1-\mu)(\pm\lambda)\right),$$

$$g(R(X,\xi)\xi,W) = K_5\{g(X,W)-\eta(X)\eta(W)\}, \tag{4.6}$$
$$\text{where } K_5 = \pm\lambda+\frac{(\pm\lambda^2)}{2}+\kappa-(1-\mu)(\pm\lambda)-\sin^2\theta,$$

$$g(R(X,e_k)e_k,W) = K_6 g(X,W)+K_2\eta(X)\eta(W) + g(\sigma(X,W),\sigma(e_k,e_k)) - g(\sigma(e_k,W),\sigma(X,e_k)), \tag{4.7}$$

$$\text{where } K_6 = (q-2)\left(\frac{c+3}{4}\right)+(2q-3-\mu)(\pm\lambda)+\frac{q-2}{2}(\pm\lambda^2)+\frac{3(c-1)}{4}+\kappa,$$

For $1\le l\le 2p+q$, put

$$\sum_{l=1}^{2p+q} g(\sigma(e_l,W),\sigma(X,e_l)) = \sum_{i=1}^{p} g(\sigma(W,e_i),\sigma(X,e_i)) + \sum_{j=p+1}^{2p} g(\sigma(\sec\theta Te_j,W),\sigma(X,\sec\theta Te_j)) + \sum_{i=2p+2}^{2p+q+1} g(\sigma(e_k,W),\sigma(X,e_k)) \tag{4.8}$$

Thus we get (4.1) by using (2.11) in (4.4) to (4.8) and then substituting in (4.3). □

Theorem 4.2. *Let M be a pseudo-slant submanifold of a (κ,μ)-contact space form $\tilde M(c)$. Then the scalar curvature ρ of M is given by*

$$\rho = (K_1+K3+K_5+K_6)(2p+q+1) + (2K_2+K_4-K_5)((2p+q)^2-1)||H||^2-||\sigma||^2. \tag{4.9}$$

Proof. Using the fact that

$$\rho = \sum_{i=1}^{2p+q+1} S(e_i,e_i),$$

and (4.1) we get (4.9). □

Corollary 4.3. *Every totally umbilical pseudo-slant submanifold M of a (κ,μ)-contact space form $\tilde M(c)$ is an η-Einstein submanifold.*

Proof. Using (4.1) and (2.12) we have

$$S(X,W) =(K_1+K_3+K_5+K_6)g(X,W)+(2K_2+K_4-K_5)\eta(X)\eta(W) + (2p+q-1)g(g(X,W)H,H) - \sum_{l=1}^{2p+q+1} g(g(e_l,W)H, g(X,e_l)H),$$

This completes our proof. □

Theorem 4.4. *Let M be a pseudo-slant submanifold of a (κ,μ)-contact space form $\tilde{M}(c)$. If M is curvature-invariant pseudo-slant submanifold, then M is either semi-invariant or anti-invariant submanifold, provided* $3(1-c)\neq(\pm 2\lambda^2)$.

Proof. Let M be a curvature-invariant pseudo-slant submanifold of a (κ,μ)-contact space form $\tilde{M}(c)$. Then from (2.17) and (3.8) we have

$$(\tilde{\nabla}_X\sigma)(Y,Z)-(\tilde{\nabla}_Y\sigma)(X,Z) = \left(\frac{c-1}{4}\right)\{g(X,TZ)NY - g(Y,TZ)NX + 2g(X,TY)NZ\} + \frac{1}{2}\{g(ThX,Z)NhY - g(ThY,Z)NhX\} = 0,$$

for $X,Y,Z\in\Gamma(TM)$. By taking in to account of Lemma 2.2 and putting $X=Z$ we obtain

$$\left[\frac{3(c-1)}{4}+\frac{1}{2}(\pm\lambda^2)\right] g(TZ,Y)NZ = 0.$$

Now putting $Y=TZ$ in the above equation and taking inner product with NZ we get

$$\left[\frac{3(c-1)}{4}+\frac{1}{2}(\pm\lambda^2)\right] g(TZ,TZ)g(NZ,NZ) = 0.$$

Simplifying the above equation by considering (3.14) and (3.15), we obtain

$$\sqrt{\frac{3(c-1)}{4}+\frac{1}{2}(\pm\lambda^2)}\ \sin 2\theta\,\{g(Z,Z)-\eta^2(Z)\} = 0.$$

Thus we infer that either M is semi-invariant ($\theta=0$) or anti-invariant ($\theta=\pi/2$), provided $3(1-c)\neq(\pm 2\lambda^2)$. □

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