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On Para-Sasakian manifolds admitting a special type of semi-symmetric non-metric η-connection

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Abstract. A special type of semi-symmetric non-metric η-connection on a Para-Sasakian manifold has been studied. It is shown that if the Ricci tensor on Para-Sasakian manifolds with respect to the Levi-Civita connection is an Einstein manifold, then the manifold is a η-Einstein manifold admitting a special type of semi symmetric non-metric η-connection. Ricci-Semisymmetricness of a Para-Sasakian manifold with respect to the semi-symmetric non-metric η-connection has also been considered and it is seen the Ricci-Semisymmetric with respect to the Levi-Civita connection and the semi-symmetric non-metric η-connection are equivalent. Finally, an illustrative example is given to verify the result.

1 Introduction

Let (M, g) be a n-dimensional Riemannian manifold admitting a 1-form η which satisfies the conditions

$$(\nabla_X \eta)(Y) = (\nabla_Y \eta)(X)$$

and

$$(\nabla_X \nabla_Y \eta)(Z) = -\eta(Y)g(X,Z) - \eta(Z)g(X,Y) + 2\eta(X)\eta(Y)\eta(Z),$$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Such a manifold is called a Para-Sasakian manifold or briefly a P-Sasakian manifold.

In 1977, Adati and Matsumoto [3] defined Para-Sasakian and Special Para-Sasakian manifolds which are considered as special cases of an almost paracontact manifold introduced by Sato [16]. Para-Sasakian manifolds have been studied by De and Pathak [7], Matsumoto, Ianus and Mihai [14], De, Özgür, Arslan, Murathan and Yildiz [8], Yildiz, Turan and Acet [18], Barman ([4], [5]) and many others.

In 1924, Friedmann and Schouten [11] introduced the idea of semi-symmetric connection on a differentiable manifold. A linear connection $\tilde{\nabla}$ on a differentiable manifold M is said to be a semi-symmetric connection if the torsion tensor T of the connection $\tilde{\nabla}$ satisfies $T(X,Y) = u(Y)X - u(X)Y$, where u is a 1-form and ρ is a vector field defined by $u(X) = g(X, \rho)$, for all vector fields $X, Y \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M.

In 1932, Hayden [12] introduced the idea of semi-symmetric metric connections on a Riemannian manifold (M, g). A semi-symmetric connection $\tilde{\nabla}$ is said to be a semi-symmetric metric connection if $\tilde{\nabla} g = 0$. A relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ of (M, g) was given by Yano [17]: $\tilde{\nabla}_X Y = \nabla_X Y + u(Y)X - g(X,Y)\rho$, where $u(X) = g(X, \rho)$.

After a long gap the study of a semi-symmetric connection $\hat{\nabla}$ satisfying $\hat{\nabla} g \neq 0$, was initiated by Prvanović [15] with the name pseudo-metric semi-symmetric connection and was just followed by Andonie [2]. The semi-symmetric connection $\hat{\nabla}$ is said to be a semi-symmetric non-metric connection, if $\hat{\nabla} g \neq 0$.

In 1992, Agashe and Chafle [1] studied a semi-symmetric non-metric connection $\hat{\nabla}$, whose torsion tensor T satisfies $T(X,Y) = u(Y)X - u(X)Y$ and $(\hat{\nabla}_X g)(Y,Z) = -u(Y)g(X,Z) -$

$u(Z)g(X,Y)$. In 1992, Barua and Mukhopadhyay [6] studied a type of semi-symmetric connection $\hat{\nabla}$ which satisfies $(\hat{\nabla}_X g)(Y,Z) = 2u(X)g(Y,Z) - u(Y)g(X,Z) - u(Z)g(X,Y)$. Since $\hat{\nabla} g \neq 0$, this is another type of semi-symmetric non-metric connection. However, the authors preferred the name semi-symmetric semimetric connection.

In 1994, Liang [13] studied another type of semi-symmetric non-metric connection $\hat{\nabla}$ for which we have $(\hat{\nabla}_X g)(Y,Z) = 2u(X)g(Y,Z)$, where u is a non-zero 1-form and he called this a semi-symmetric recurrent metric connection.

After introduction in section 2, we give a brief account of P-Sasakian manifolds. In section 3, we define a special type of semi-symmetric non-metric η-connection on P-Sasakian manifolds. Section 4 is devoted to establish the relation between the curvature tensors with respect to the special type of the semi-symmetric non-metric η-connection and the Levi-Civita connection on P-Sasakian manifolds and prove that if the Ricci tensor on Para-Sasakian manifolds with respect to the Levi-Civita connection is an Einstein manifold, then the manifold is a η-Einstein manifold admitting a special type of semi symmetric non-metric η-connection. In section 5, Ricci-Semisymmetricness of a Para-Sasakian manifold with respect to the semi-symmetric non-metric η-connection has also been considered and it is seen the Ricci-Semisymmetric with respect to the Levi-Civita connection and the semi-symmetric non-metric η-connection are equivalent. Finally, we construct an example of a 5-dimensional Para-Sasakian manifold admitting a special type of semi-symmetric non-metric η-connection whose curvature tensor satisfies the skew-symmetric property, the first Bianchi identity and also to verify the result of Section 5.

2 P-Sasakian manifolds

A n-dimensional differentiable manifold M is said to be an almost para-contact structure (ϕ, ξ, η, g), if there exists ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is the Riemannian metric on M which satisfy the conditions

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \quad g(X,\xi) = \eta(X), \tag{2.1}$$

$$\phi^2(X) = X - \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$(\nabla_X \eta)Y = g(X, \phi Y) = (\nabla_Y \eta)X, \tag{2.4}$$

for any vector fields X, Y on M.

If moreover, (ϕ, ξ, η, g) satisfy the conditions

$$d\eta = 0, \quad \nabla_X \xi = \phi X, \tag{2.5}$$

$$(\nabla_X \phi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.6}$$

then M is called a para-Sasakian manifold or briefly a P-Sasakian manifold.

In a P-Sasakian manifold the following relations hold ([3], [16]) :

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.7}$$

$$R(\xi, X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.8}$$

$$R(\xi, X)\xi = X - \eta(X)\xi, \tag{2.9}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.10}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{2.11}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.12}$$

where R and S are the curvature tensor and the Ricci tensor of the Levi-Civita connection respectively.

3 semi-symmetric non-metric η-connection on P-Sasakian manifolds

Theorem 3.1: The necessary and sufficient conditions that a linear connection $\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X + g(\phi X, Y)\xi - \eta(X)\eta(Y)\xi$ is a special type of semi-symmetric non-metric η-connection on P-Sasakian manifolds.

Proof. A special type of semi-symmetric non-metric η -connection on a P-Sasakian manifold. Let (M, g) be a P-Sasakian Manifold with the Levi-Civita connection ∇ and we define a linear connection $\bar{\nabla}$ on M by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X + g(\phi X, Y)\xi - \eta(X)\eta(Y)\xi. \tag{3.1}$$

Using (2.4) and (3.1), the torsion tensor T of M with respect to the connection $\bar{\nabla}$ is given by

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] = \eta(Y)X - \eta(X)Y. \tag{3.2}$$

The linear connection $\bar{\nabla}$ satisfying (3.2) is a semi-symmetric connection.

So the equation (3.1) with the help of (2.1) turns into

$$\begin{aligned}(\bar{\nabla}_X g)(Y,Z) = \bar{\nabla}_X g(Y,Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z) = -\eta(Y)g(X + \phi X, Z)\\ -\eta(Z)g(X + \phi X, Y) + 2\eta(X)\eta(Y)\eta(Z) \neq 0.\end{aligned} \tag{3.3}$$

The linear connection $\bar{\nabla}$ satisfying (3.2) and (3.3) is called a semi-symmetric non-metric connection.

By making use of (2.1), (2.4) and (3.1), it is obvious that

$$(\bar{\nabla}_X \eta)(Y) = \bar{\nabla}_X \eta(Y) - \eta(\bar{\nabla}_X Y) = 0. \tag{3.4}$$

The linear connection $\bar{\nabla}$ define by (3.1) satisfying (3.2), (3.3) and (3.4) is a special type of semi-symmetric non-metric η-connection on P-Sasakian manifolds.

Conversely, we show that a linear connection $\bar{\nabla}$ defined on M satisfying (3.2), (3.3) and (3.4) is given by (3.1). Let H be a tensor field of type $(1,2)$ and

$$\bar{\nabla}_X Y = \nabla_X Y + H(X,Y). \tag{3.5}$$

Then we conclude that

$$T(X,Y) = H(X,Y) - H(Y,X). \tag{3.6}$$

Further using (3.5), it follows that

$$\begin{aligned}(\bar{\nabla}_X g)(Y,Z) = \bar{\nabla}_X g(Y,Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z) = -g(H(X,Y),Z)\\ -g(Y, H(X,Z)).\end{aligned} \tag{3.7}$$

In view of (3.3) and (3.7) yields,

$$\begin{aligned}g(H(X,Y),Z) + g(Y, H(X,Z)) = \eta(Y)g(X + \phi X, Z) + \eta(Z)g(X + \phi X, Y)\\ -2\eta(X)\eta(Y)\eta(Z).\end{aligned} \tag{3.8}$$

Also using (3.8) and (3.6), we derive that

$$g(T(X,Y),Z) + g(T(Z,X),Y) + g(T(Z,Y),X) = 2g(H(X,Y),Z) - 2\eta(Z)g(X + \phi X, Y)$$

$$+2\eta(X)\eta(Y)\eta(Z).$$

From the above equation yields,

$$g(H(X,Y),Z) = \frac{1}{2}[g(T(X,Y),Z) + g(T(Z,X),Y) + g(T(Z,Y),X)] \\ +\eta(Z)g(X+\phi X,Y) - \eta(X)\eta(Y)\eta(Z). \tag{3.9}$$

Let T' be a tensor field of type $(1,2)$ given by

$$g(T'(X,Y),Z) = g(T(Z,X),Y). \tag{3.10}$$

Adding (2.1), (3.2) and (3.10), we obtain

$$T'(X,Y) = \eta(X)Y - g(X,Y)\xi. \tag{3.11}$$

From (3.9) we have by using (3.10) and (3.11)

$$g(H(X,Y),Z) = \frac{1}{2}[g(T(X,Y),Z) + g(T'(X,Y),Z) + g(T'(Y,X),Z)] \\ \eta(Z)g(X+\phi X,Y) - \eta(X)\eta(Y)\eta(Z) = \eta(Y)g(X,Z) \\ +\eta(Z)g(\phi X,Y) - \eta(X)\eta(Y)\eta(Z). \tag{3.12}$$

Now contracting Z in (3.12) and using (2.1), implies that

$$H(X,Y) = \eta(Y)X + g(\phi X,Y)\xi - \eta(X)\eta(Y)\xi. \tag{3.13}$$

Combining (3.5) and (3.13), it follows that

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X + g(\phi X,Y)\xi - \eta(X)\eta(Y)\xi.$$

This completes the proof of the theorem.

4 Curvature tensor of a P-Sasakian manifold with respect to the semi-symmetric non-metric η -connection

In this section we obtain the expressions of the curvature tensor and the Ricci tensor of M with respect to the semi-symmetric non-metric η-connection defined by (3.1).

Analogous to the definitions of the curvature tensor of M with respect to the Levi-Civita connection ∇, we define the curvature tensor $\bar{R}$ of M with respect to the semi-symmetric non-metric η -connection $\bar{\nabla}$ by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z, \tag{4.1}$$

where $X, Y, Z \in \chi(M)$, the set of all differentiable vector fields on M.

Using (2.2) and (3.1) in (4.1), we obtain

$$\bar{R}(X,Y)Z = R(X,Y)Z - (\nabla_Y\eta)(Z)X + (\nabla_X\eta)(Z)Y + \eta(X)(\nabla_Y\eta)(Z)\xi \\ -\eta(Y)(\nabla_X\eta)(Z)\xi + g(\phi Y,Z)\phi X + g(\phi Y,Z)X - \eta(X)g(\phi Y,Z)\xi \\ -g(\phi X,Z)\phi Y - g(\phi X,Z)Y + \eta(Y)g(\phi X,Z)\xi \\ +\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X. \tag{4.2}$$

By making use of (2.4) and (2.5) in (4.2), we have

$$\bar{R}(X,Y)Z = R(X,Y)Z + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - \eta(Y)\eta(Z)\phi X \\ +\eta(X)\eta(Z)\phi Y. \tag{4.3}$$

So the equation (4.3) turns into

$$\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z,$$

and

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0. \tag{4.4}$$

We call (4.4) the first Bianchi identity with respect to a special type semi-symmetric non-metric η-connection on P-Sasakian manifolds.

Taking the contrations of (4.3) with X and using (2.1), it follows that

$$\bar{S}(Y,Z) = S(Y,Z) + \alpha g(\phi Y,Z) - g(Y,Z) + (1-\alpha)\eta(Y)\eta(Z), \tag{4.5}$$

where $\bar{S}$ and S denote the Ricci tensors of M with respect to $\bar{\nabla}$ and ∇ respectively and $\alpha = g(e_i, \phi e_i)$.

From (4.5), implies that

$$\bar{S}(Y,Z) = \bar{S}(Z,Y).$$

Summing up all of above equations we can state the following proposition:

Proposition 4.1: For a P-Sasakian manifold M with respect to a special type of semi-symmetric non-metric η -connection $\bar{\nabla}$

(i) The curvature tensor $\bar{R}$ is given by $\bar{R}(X,Y)Z = R(X,Y)Z + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - \eta(Y)\eta(Z)\phi X + \eta(X)\eta(Z)\phi Y,$

(ii) The Ricci tensor $\bar{S}$ is given by $\bar{S}(Y,Z) = S(Y,Z) + \alpha g(\phi Y,Z) - g(Y,Z) + (1-\alpha)\eta(Y)\eta(Z),$

(iii)$\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z,$

(iv)$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0,$

(v) The Ricci tensor $\bar{S}$ is symmetric.

Definition 4.1: A P-Sasakian manifold with respect to the special type of semi-symmetric non-metric η -connections is said be a η-Einstein if its Ricci tensor $\bar{S}$ is of the form $\bar{S}(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z)$, where a and b are smooth functions on the manifold.

Definition 4.2: A P-Sasakian manifold with respect to the Levi-Civita connections is said be an Einstein if its Ricci tensor S is of the form

$$S(Y,Z) = \lambda g(Y,Z), \tag{4.6}$$

where λ is smooth function on the manifold.

Theorem 4.1: If a P-Sasakian manifold is an Einstein manifold admitting the Levi-Civita connection and trace of ϕ vanishes, then the manifold is is a η-Einstein manifold with respect to a special type of semi-symmetric non-metric η -connection.

Proof. Combining (4.5) and (4.6), we get

$$\bar{S}(Y,Z) = \lambda g(Y,Z) + \alpha g(\phi Y,Z) - g(Y,Z) + (1-\alpha)\eta(Y)\eta(Z). \tag{4.7}$$

If $\alpha = 0$, in equation (4.7)turns into

$$\bar{S}(Y,Z) = \lambda g(Y,Z) + \eta(Y)\eta(Z)$$

Therefore, $\bar{S}(Y,Z) = (\lambda-1)g(Y,Z) + \eta(Y)\eta(Z)$, where $a = (\lambda-1)$ and $b = 1$.
From which it follows that the P-Sasakian manifolds is a η-Einstein manifold with respect to the special type of semi-symmetric non-metric η -connections. This proves Theorem 4.1.

5 Ricci-semisymmetric on P-Sasakian manifolds with respect to a special type semi-symmetric non-metric η-connection $\bar{\nabla}$

Theorem 5.1: A P-Sasakian manifold is Ricci-semisymmetric with respect to a special type of semi-symmetric non-metric η-connection iff the manifold is also Ricci-semisymmetric with respect to the Levi-Civita connection.

Proof. we characterize Ricci-semisymmetric $\bar{R} \cdot \bar{S}$ on a P-Sasakian manifold admitting a special type of semi-symmetric non-metric η-connection $\bar{\nabla}$.
Definition 5.1: A Riemannian manifold is Ricci-semisymmetric with respect to the Levi-Civita connection ∇, that is, $(R(X,Y) \cdot S)(U,V) = 0$.

Then from the above equation, we can write

$$\bar{R} \cdot \bar{S} = \bar{S}(\bar{R}(X,Y)U,V) + \bar{S}(U,\bar{R}(X,Y)V) \tag{5.1}$$

Putting $V = \xi$ in (5.1) and using (2.1), (4.3) and (4.5), it follows that

$$\begin{aligned}\bar{R} \cdot \bar{S} = R \cdot S + \alpha g(\phi U, R(X,Y)\xi) - g(R(X,Y)\xi,U) + 2(1-\alpha)\eta(R(X,Y)\xi)\eta(U) \\ -g(R(X,Y)U,\xi) - \eta(Y)[S(U,\phi X) + \alpha g(\phi U,\phi X) - g(U,\phi X)] \\ +\eta(X)[S(U,\phi Y) + \alpha g(\phi U,\phi Y) - g(U,\phi Y)],\end{aligned} \tag{5.2}$$

where trace of $\phi = \alpha$.

We take $U = \xi$ in (5.2) and using (2.1), we obtain

$$\bar{R} \cdot \bar{S} = R \cdot S - 2\alpha g(R(X,Y)\xi,\xi). \tag{5.3}$$

Combining (2.1), (2.10) and (5.3), we get

$$\bar{R} \cdot \bar{S} = R \cdot S.$$

Hence the proof of Theorem is completed.
Lemma 5.1: [9]A n-dimensional $(n > 2)$ P-Sasakian manifold is Ricci- semisymmetric if and only if it is an Einstein manifold.

Therefore, from Theorem 5.1 and Lemma 5.1 we can state the following theorem:
Theorem 5.2: A n-dimensional $(n > 2)$ P-Sasakian manifold is Ricci- semisymmetric with respect to a special type of semi-symmetric non-metric η-connection if and only if it is an Einstein manifold.
Lemma 5.2: [10] Let M be a P-Sasakian manifold. Then the following conditions are equivalent:
i) M is an Einstein manifold.
ii) The Ricci is parallel, $\nabla S = 0$.
iii) $R(X,Y) \cdot S = 0$ for any X and Y.

Hence, from Theorem 5.1 and Lemma 5.2 we can state the following theorem:
Theorem 5.3: Let M be a P-Sasakian manifold with respect to a special type of semi-symmetric non-metric η-connection. Then the following conditions are equivalent:
i) M is an Einstein manifold with respect to the Levi-Civita connection.
ii) The Ricci is parallel admitting the Levi-Civita connection, $\nabla S = 0$.
iii) $\bar{R}(X,Y) \cdot \bar{S} = 0$ for any X and Y.

6 Example

Now, we give an example of a 5-dimensional P-Sasakian manifold admitting a special type of semi-symmetric non-metric η-connection $\bar{\nabla}$, which verify the skew-symmetric property and the first Bianchi identity of the curvature tensors $\bar{R}$ of $\bar{\nabla}$.

We consider the 5-dimensional manifold $\{(x,y,z,u,v) \in R^5\}$, where (x,y,z,u,v) are the standard coordinates in R^5.
We choose the vector fields

$$e_1 = \frac{\partial}{\partial x},\ e_2 = e^{-x}\frac{\partial}{\partial y},\ e_3 = e^{-x}\frac{\partial}{\partial z},\ e_4 = e^{-x}\frac{\partial}{\partial u},\ e_5 = e^{-x}\frac{\partial}{\partial v},$$

which are linearly independent at each point of M.
Let g be the Riemannian metric defined by

$$g(e_i,e_j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j;\ i,j = 1,2,3,4,5. \end{cases}$$

Let η be the 1-form defined by
$$\eta(Z) = g(Z, e_1),$$
for any $Z \in \chi(M)$.
Let ϕ be the $(1,1)$-tensor field defined by
$$\phi(e_1) = 0,\ \phi(e_2) = e_2,\ \phi(e_3) = e_3,\ \phi(e_4) = e_4,\ \phi(e_5) = e_5.$$
Using the linearity of ϕ and g, we have
$$\eta(e_1) = 1,\ \phi^2 Z = Z - \eta(Z)e_1$$
and
$$g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U),$$
for any vector fields $Z, U \in \chi(M)$. Thus for $e_1 = \xi$, the structure (ϕ, ξ, η, g) defines an almost paracontact metric structure on M.
Then we have
$$\begin{aligned}
&[e_1, e_2] = -e_2, [e_1, e_3] = -e_3, [e_1, e_4] = -e_4, [e_1, e_5] = -e_5,\\
&[e_2, e_3] = [e_2, e_4] = 0, [e_2, e_5] = [e_3, e_4] = [e_3, e_5] = [e_4, e_5] = 0.
\end{aligned}$$
The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by
$$\begin{aligned}
2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)\\
&\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).
\end{aligned}$$
Using Koszul's formula, we get the following:
$$\begin{aligned}
&\nabla_{e_1} e_1 = 0,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = 0,\ \nabla_{e_1} e_4 = 0,\ \nabla_{e_1} e_5 = 0,\\
&\nabla_{e_2} e_1 = e_2,\ \nabla_{e_2} e_2 = -e_1,\ \nabla_{e_2} e_3 = 0,\ \nabla_{e_2} e_4 = 0,\ \nabla_{e_2} e_5 = 0,\\
&\nabla_{e_3} e_1 = e_3,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = -e_1,\ \nabla_{e_3} e_4 = 0,\ \nabla_{e_3} e_5 = 0,\\
&\nabla_{e_4} e_1 = e_4,\ \nabla_{e_4} e_2 = 0,\ \nabla_{e_4} e_3 = 0,\ \nabla_{e_4} e_4 = -e_1,\ \nabla_{e_4} e_5 = 0,\\
&\nabla_{e_5} e_1 = e_5,\ \nabla_{e_5} e_2 = 0,\ \nabla_{e_5} e_3 = 0,\ \nabla_{e_5} e_4 = 0,\ \nabla_{e_5} e_5 = -e_1.
\end{aligned}$$
In view of the above relations, we see that
$$\nabla_X \xi = \phi X,\ (\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \text{ for all } e_1 = \xi.$$
Therefore the manifold is a P-Sasakian manifold with the structure (ϕ, ξ, η, g).
Using (3.1) in above equations, we obtain
$$\begin{aligned}
&\bar{\nabla}_{e_1} e_1 = 0,\ \bar{\nabla}_{e_1} e_2 = 0,\ \bar{\nabla}_{e_1} e_3 = 0,\ \bar{\nabla}_{e_1} e_4 = 0,\ \bar{\nabla}_{e_1} e_5 = 0,\\
&\bar{\nabla}_{e_2} e_1 = 2e_2,\ \bar{\nabla}_{e_2} e_2 = 0,\ \bar{\nabla}_{e_2} e_3 = 0,\ \bar{\nabla}_{e_2} e_4 = 0,\ \bar{\nabla}_{e_2} e_5 = 0,\\
&\bar{\nabla}_{e_3} e_1 = 2e_3,\ \bar{\nabla}_{e_3} e_2 = 0,\ \bar{\nabla}_{e_3} e_3 = 0,\ \bar{\nabla}_{e_3} e_4 = 0,\ \bar{\nabla}_{e_3} e_5 = 0,\\
&\bar{\nabla}_{e_4} e_1 = 2e_4,\ \bar{\nabla}_{e_4} e_2 = 0,\ \bar{\nabla}_{e_4} e_3 = 0,\ \bar{\nabla}_{e_4} e_4 = 0,\ \bar{\nabla}_{e_4} e_5 = 0,\\
&\bar{\nabla}_{e_5} e_1 = 2e_5,\ \bar{\nabla}_{e_5} e_2 = 0,\ \bar{\nabla}_{e_5} e_3 = 0,\ \bar{\nabla}_{e_5} e_4 = 0,\ \bar{\nabla}_{e_5} e_5 = 0.
\end{aligned}$$
Now, we can easily obtain the non-zero components of the curvature tensors as follows:
$$\begin{aligned}
&R(e_1, e_2)e_1 = e_2,\ R(e_1, e_2)e_2 = -e_1,\ R(e_1, e_3)e_1 = e_3,\ R(e_1, e_3)e_3 = -e_1,\\
&R(e_1, e_4)e_1 = e_4,\ R(e_1, e_4)e_4 = -e_1,\ R(e_1, e_5)e_1 = e_5,\ R(e_1, e_5)e_5 = -e_1,\\
&R(e_2, e_3)e_2 = e_3,\ R(e_2, e_3)e_3 = -e_2,\ R(e_2, e_4)e_2 = e_4,\ R(e_2, e_4)e_4 = -e_2,\\
&R(e_2, e_5)e_2 = e_5,\ R(e_2, e_5)e_5 = -e_2,\ R(e_3, e_4)e_3 = e_4,\ R(e_3, e_4)e_4 = -e_3,\\
&R(e_3, e_5)e_3 = e_5,\ R(e_3, e_5)e_5 = -e_3,\ R(e_4, e_5)e_4 = e_5,\ R(e_4, e_5)e_5 = -e_4.
\end{aligned}$$

and

$$\bar{R}(e_1, e_2)e_1 = 2e_3,\ \bar{R}(e_1, e_4)e_1 = 2e_4,$$
$$\bar{R}(e_1, e_3)e_1 = 2e_3,\ \bar{R}(e_1, e_5)e_1 = 2e_5.$$

With the help of the above curvature tensors with respect to a special type of semi-symmetric non-metric η-connection, we find the Ricci tensors as follows:

$$\bar{S}(e_2, e_2) = \bar{S}(e_3, e_3) = \bar{S}(e_4, e_4) = \bar{S}(e_5, e_5) = -2.$$

Let X, Y, Z and U be any four vector fields given by

$X = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5$, $Y = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5$, $U = c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5$ and $V = d_1e_1 + d_2e_2 + d_3e_3 + d_4e_4 + d_5e_5$, where a_i, b_i, c_i, d_i, for all $i = 1, 2, 3, 4, 5$ are all non-zero real numbers.

Using the above curvature tensors admitting the semi-symmetric non-metric η-connection, we obtain

$$\bar{R}(X, Y)Z = -2(a_1b_2c_1e_2 + a_1b_3c_1e_3 + a_1b_4c_1e_4 + a_1b_5c_1e_5) = -\bar{R}(Y, X)Z.$$

Hence we also conclude that from the equation(4.4), we get

$$\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = 0.$$

Therefore, the curvature tensor of a P-Sasakian manifold admitting a special type of semi-symmetric non-metric η-connection $\bar{\nabla}$ is satisfied the skew-symmetric property and the first Bianchi identity of the curvature tensors $\bar{R}$ of $\bar{\nabla}$. Now, we see that the Ricci-Semisymmetric with respect to the semi-symmetric non-metric η-connections from the above relations as follow:

$$\bar{R} \cdot \bar{S} = [(a_1b_2 - a_2b_1)(c_2d_1 + c_1d_2) + (a_1b_3 - a_3b_1)(c_3d_1 + c_1d_3)$$
$$+(a_1b_4 - a_4b_1)(c_4d_1 + c_1d_4) + (a_1b_5 - a_5b_1)(c_5d_1 + c_1d_5) = 0.$$

Hence P-Sasakian manifolds will be Ricci-Semisymmetric with respect to the semi-symmetric non-metric η-connections if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \frac{a_4}{b_4} = \frac{a_5}{b_5}$.

The above arguments tell us that the 5-dimensional P-Sasakian manifolds with respect to the semi-symmetric non-metric η-connections under consideration agrees with the Section 5.

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