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Eulerian and Hamiltonian complements of zero-divisor graphs of pseudocomplemented posets

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Abstract. In this paper, Eulerian complements of zero-divisor graphs are classified for a special class of finite pseudocomplemented posets. Also, it is proved that the complement of the zero-divisor graph of a finite pseudocomplemented poset P is Hamiltonian if and only if P has at least three atoms. These results are applied to zero-divisor graphs and intersection graphs of ideals of reduced commutative Artinian rings.

1 Introduction

Zero-divisor graphs have become popular as a means by which new descriptions of algebraic structure can be given in the language of graph theory. The idea began in [4] within the context of commutative rings, where questions regarding chromatic numbers were addressed. The graphs were defined by letting every element of a commutative ring R be a vertex, and two distinct vertices a and b were adjacent if and only if $ab = 0$. Since the appearance of [2], the vertices of zero-divisor graphs of commutative rings have usually been restricted to nonzero zero-divisors, as the zero-divisor relations involving 0 and nonzero-divisors are trivial.

More recently, the zero-divisor graph concept was extended to posets in [8]. Analogous to the definition in [4], the graphs were defined for posets P that have the least element 0 by letting every element of P be a vertex, and two distinct vertices a and b were adjacent if and only if 0 was the only lower bound of $\{a, b\}$ in P. As in [2], the vertices of zero-divisor graphs of posets were restricted in [16] to only include nonzero "zero-divisors" of P. This is the definition that has been adopted in most of the subsequent related work, and it will be used in the majority of the present investigation as well.

It seems equally beneficial to study complements of zero-divisor graphs, as the information reflected in such graphs can be considered dual to that which is reflected by zero-divisor graphs. Moreover, in the case of partially ordered sets, the complements of zero-divisor graphs generalize the closely related notion of "intersection graphs of ideals" of rings, which was first introduced in [6]. That is, given a (not necessarily commutative) ring R, the *intersection graph of ideals* $IG(R)$ of R is the graph whose vertices are the proper nonzero left ideals of R such that distinct vertices I and J are adjacent if and only if $I \cap J \neq \{0\}$. Note that $IG(R)$ coincides with the complement of the zero-divisor graph (in the sense of [8], but without the vertex $\{0\}$) of the poset (under inclusion) of proper left ideals of R.

Interest in the classical Eulerian and Hamiltonian properties of zero-divisor graphs is inspired, not only by their potential for illuminating algebraic structure, but also by their historical relevance. For instance, determining whether a graph is Hamiltonian is an NP-complete problem, which cultivates a broad interest in the discovery of significant classes of Hamiltonian graphs. Eulerian and Hamiltonian zero-divisor graphs of rings have been examined in [13] and [1], respectively, and results on Eulerian and Hamiltonian intersection graphs of ideals were given in [6] and [12]. For example, Eulerian intersection graphs of ideals of $\mathbb{Z}_n$ are characterized for every nonprime integer n greater than 1 in [6, Theorem 5.1].

The current paper considers the Eulerian and Hamiltonian properties for complements of zero-divisor graphs of some special posets that are prototypical of several important ring theoretic structures. Specifically, it is proved that the complement $G^c(P)$ of the zero-divisor graph of a finite pseudocomplemented poset P is Hamiltonian if and only if P has at least three atoms (Theorem 4.1). Also, if P is a finite direct product of finite bounded posets that each has exactly one atom then $G^c(P)$ is Eulerian if and only if P has at least three factors, and every factor of P has even cardinality (Theorem 3.4). These results are applied to show that the complement of the zero-divisor graph of a finite reduced commutative ring R with at least three prime ideals is Hamiltonian (Corollary 4.10), and it is Eulerian if and only if R has characteristic 2 (Corollary 3.5). Moreover, the result in [6, Theorem 5.1] on Eulerian intersection graphs of ideals is generalized (and corrected; see the comments that follow Remark 3.7) to commutative Artinian principal ideal rings (Corollaries 3.8 and 3.9).

2 Preliminaries

Let P be a poset. Given any $\varnothing \neq A \subseteq P$, set $A^\vee = \{b \in P \mid b \geq a$ for every $a \in A\}$ and $A^\wedge = \{b \in P \mid b \leq a$ for every $a \in A\}$. If $a \in P$ then the sets $\{a\}^\vee$ and $\{a\}^\wedge$ will be denoted by $a^\vee$ and $a^\wedge$, respectively.

Suppose that P is a poset with 0. If $\varnothing \neq A \subseteq P$ then the *annihilator* of A is denoted by $A^\perp = \{b \in P \mid \{a, b\}^\wedge = \{0\}$ for all $a \in A\}$, and if $A = \{a\}$ then set $a^\perp = A^\perp$. An element $a \in P$ is an *atom* if $a > 0$ and $\{b \in P \mid 0 < b < a\} = \varnothing$, and P is called *atomic* if for every $b \in P \smallsetminus \{0\}$ there exists an atom $a \in P$ such that $a \leq b$.

A poset P is called *bounded* if P has both the least element 0 and the greatest element 1. An element b of a bounded poset P is a *complement* of $a \in P$ if $\{a, b\}^\wedge = \{0\}$ and $\{a, b\}^\vee = \{1\}$. A *pseudocomplement* of $a \in P$ is an element $b \in P$ such that $\{a, b\}^\wedge = \{0\}$, and $x \leq b$ for every $x \in P$ with $\{a, x\}^\wedge = \{0\}$; that is, b is a pseudocomplement of a if and only if $a^\perp = b^\wedge$. It is straightforward to check that any element $a \in P$ has at most one pseudocomplement, and it will be denoted by a^* (if it exists). A bounded poset P is called *complemented* (respectively, *pseudocomplemented*) if every element of P has a complement (respectively, a^* exists for every $a \in P$).

The *direct product* of posets $P_1, \ldots, P_k$ is the poset $\prod_{i=1}^k P_i$ with $\leq$ defined such that $a \leq b$ if and only if $a(i) \leq b(i)$ (where $a(i), b(i) \in P_i$ are the ith components of a and b, respectively) for every $i \in \{1, \ldots, k\}$. For any $\varnothing \neq A \subseteq \prod_{i=1}^k P_i$, note that $A^\vee = \{b \in \prod_{i=1}^k P_i \mid b(i) \geq a(i)$ for every $a \in A$ and $i \in \{1, \ldots, k\}\}$. Similarly, $A^\wedge = \{b \in \prod_{i=1}^k P_i \mid b(i) \leq a(i)$ for every $a \in A$ and $i \in \{1, \ldots, k\}\}$.

Let G be a finite simple graph with vertex-set $V(G)$. A *Hamiltonian path* in G is a path that contains every element of $V(G)$. Similarly, a *Hamiltonian cycle* in G is a cycle that contains every element of $V(G)$, and G is called *Hamiltonian* if it has a Hamiltonian cycle.

An *Eulerian trail* in G is a trail that traverses every edge of G. Similarly, an *Eulerian circuit* in G is a circuit that traverses every edge of G, and G is called *Eulerian* if it has an Eulerian circuit. It is well known that a finite graph G has an Eulerian trail if and only if G is connected, and either G has no vertices of odd degree or G has exactly two vertices of odd degree. Also, G is Eulerian if and only if G is finite, connected, and every vertex of G has even degree ([5, Theorem I.12]).

In this paper, every ring R is commutative with identity. The *zero-divisor graph* of R is the graph $\Gamma(R)$ whose vertices are the nonzero zero-divisors of R such that distinct vertices a and b are adjacent if and only if $ab = 0$. Throughout, the complement of $\Gamma(R)$ will be denoted by $\Gamma^c(R)$.

Let P be a poset with 0. Define a *zero-divisor* of P to be any element of the set $Z(P) = \{a \in P \mid$ there exists $b \in P \smallsetminus \{0\}$ such that $\{a, b\}^\wedge = \{0\}\}$. As in [16], the *zero-divisor graph* of P is the graph $G(P)$ whose vertices are the elements of $Z(P) \smallsetminus \{0\}$ such that two vertices a and b are adjacent if and only if $\{a, b\}^\wedge = \{0\}$. If $Z(P) \neq \{0\}$ then clearly $G(P)$ has at least two vertices, and $G(P)$ is connected with diameter at most three ([16, Proposition 2.1]). Throughout, the complement of $G(P)$ is denoted by $G^c(P)$.

Given a bounded poset P, let $G^*(P)$ be the graph with $V(G^*(P)) = P \smallsetminus \{0, 1\}$ such that distinct vertices a and b are adjacent if and only if $\{a, b\}^\wedge = \{0\}$. The complement of $G^*(P)$, denoted by $(G^*)^c(P)$, provides a prototype for the graphs $IG(R)$ of rings R. Note that $G^c(P)$ is the subgraph of $(G^*)^c(P)$ induced by $Z(P) \setminus \{0\}$.

Recall that if R is a finite *reduced* (i.e., R has a trivial nilradical) commutative ring then $\Gamma(R)$ is isomorphic to the (poset-theoretic) zero-divisor graph of a finite direct product of finite bounded linearly ordered sets (see the discussion prior to Corollary 3.5). Moreover, if R is an Artinian principal ideal ring then its poset of ideals is isomorphic to a finite direct product of finite bounded linearly ordered sets (see the proof of Corollary 3.8). This motivates the hypotheses of Section 3, where every poset is of the form $\prod_{i=1}^k P_i$ ($k \in \mathbb{N}$) such that $P_i \neq \{0\}$ is a finite bounded poset with $Z(P_i) = \{0\}$ for every $i \in \{1, \dots, k\}$. Note that if P is a finite poset with $P \neq \{0\}$ then $Z(P) = \{0\}$ if and only if P contains exactly one atom. The results of Section 4 are based on finite pseudocomplemented posets, which generalize the posets of Section 3 by the following proposition.

Proposition 2.1. *If $P_1, \dots, P_k$ ($k \in \mathbb{N}$) are pseudocomplemented posets then $P = \prod_{i=1}^k P_i$ is a pseudocomplemented poset. In particular, $P = \prod_{i=1}^k P_i$ is pseudocomplemented if P_i is bounded with $Z(P_i) = \{0\}$ for every $i \in \{1, \dots, k\}$.*

Proof. Note that if $a, b \in P$ then $\{a, b\}^\wedge = \{x \in P \mid x(i) \in \{a(i), b(i)\}^\wedge$ for every $i \in \{1, \dots, k\}\}$. Given $a \in P$, we claim that the pseudocomplement of a in P is $x = (a(1)^*, \dots, a(k)^*)$ (where $a(i)^*$ is the pseudocomplement of $a(i)$ in P_i).

It is easy to observe that $\{a, x\}^\wedge = \{(0, \dots, 0)\}$. Now, suppose that $\{a, b\}^\wedge = \{(0, \dots, 0)\}$ for some $b \in P$. Then $\{a(i), b(i)\}^\wedge = \{0\}$ for every $i \in \{1, \dots, k\}$. Hence, $b(i) \leq a(i)^*$ in P_i for every $i \in \{1, \dots, k\}$. Therefore, $b \leq x$, and it follows that $a^* = x$ in P.

The "in particular" statement follows since if P_i is bounded with $Z(P_i) = \{0\}$ then P_i is pseudocomplemented (with $0^* = 1$ and $a^* = 0$ for every $a \in P_i \setminus \{0\}$). □

Throughout, the set of positive integers and the ring of integers modulo n will be denoted by $\mathbb{N}$ and $\mathbb{Z}_n$, respectively. Also, set $\mathcal{D} = P \setminus Z(P)$ (the notation $\mathcal{D}$ will not be ambiguous since the underlying poset P will always be evident). References on commutative rings and graphs are given in [3] and [5], respectively.

3 Eulerian graphs

In this section, the Eulerian condition is studied in the graphs $G^c(P)$ and $(G^*)^c(P)$ for posets $P = \prod_{i=1}^k P_i$ ($k \in \mathbb{N}$) such that $P_i \neq \{0\}$ is a finite bounded poset with $Z(P_i) = \{0\}$ for every $i \in \{1, \dots, k\}$. Applications to zero-divisor graphs of finite reduced rings and intersection graphs of ideals of Artinian principal ideal rings are also provided. The investigation begins with the following two lemmas, which determine connectivity, and count the degrees of the vertices in the graphs $G^c(P)$ and $(G^*)^c(P)$. We adopt the convention that a graph is *connected* if it has exactly one component, and hence the null graph is not connected.

Lemma 3.1. *Let $P = \prod_{i=1}^k P_i$ ($k \in \mathbb{N}$) such that $P_i \neq \{0\}$ is a finite bounded poset with $Z(P_i) = \{0\}$ for every $i \in \{1, \dots, k\}$. Then $G^c(P)$ is connected if and only if $k \geq 3$. In this case,* $\mathrm{diam}(G^c(P)) = 2$. *Moreover, if $|P_i| \geq 3$ for some $i \in \{1, \dots, k\}$ then $(G^*)^c(P)$ is connected.*

Proof. The "moreover" statement is clear since if $|P_i| \geq 3$ (i.e., $P_i \setminus \{0, 1\} \neq \varnothing$) for some $i \in \{1, \dots, k\}$ then $V((G^*)^c(P)) \cap (P \setminus Z(P)) \neq \varnothing$. To prove the first assertion, observe that if $k = 1$ then $G^c(P)$ is null, and if $k = 2$ then $G^c(P)$ is the disjoint union of the two complete graphs induced by $(P_1 \times \{0\}) \setminus \{(0,0)\}$ and $(\{0\} \times P_2) \setminus \{(0,0)\}$. Hence, if $G^c(P)$ is connected then $k \geq 3$.

Conversely, suppose that $k \geq 3$, and let $x, y \in V(G^c(P))$. If $i, j \in \{1, \dots, k\}$ such that $x(i), y(j) \neq 0$, then consider any vertex $z \in V(G^c(P))$ such that $z(i) = z(j) = 1$ (such a vertex z exists since $k \geq 3$). Then the vertices x, z, y belong to a path in $G^c(P)$. This completes the proof of the first statement, and shows that if $G^c(P)$ is connected then $\mathrm{diam}(G^c(P)) \leq 2$. Hence, the second assertion follows since the zero-divisor graph $G(P)$ is connected with at least two vertices (so that $G^c(P)$ is not complete). □

In the next two lemmas, note that the assumption "$x \in V(G^c(\prod_{i=1}^k P_i))$" implies $x(i) = 0$ and $x(j) \neq 0$ for some $i, j \in \{1, \dots, k\}$ (in particular, it implies $k \geq 2$).

Lemma 3.2. *Let $P = \prod_{i=1}^k P_i$ ($k \in \mathbb{N}$) such that $P_i \neq \{0\}$ is a finite bounded poset with $Z(P_i) = \{0\}$ for every $i \in \{1, \dots, k\}$. The following statements hold.*

(1) If $x \in V(G^c(P))$ then

$$\deg(x) = \textstyle\prod_{i=1}^k |P_i| - \prod_{i=1}^k \big(|P_i| - 1\big) - \prod_{x(i)=0} |P_i| - 1.$$

(2) If $x \in V((G^*)^c(P))$ then

$$\deg(x) = \textstyle\prod_{i=1}^k |P_i| - \prod_{x(i)=0} |P_i| - 2$$

(where the empty product is defined as 1).

Proof. Let $G \in \{G^c(P), (G^*)^c(P)\}$. If $x, y \in P$ then $y \in x^\perp$ if and only if $y(i) = 0$ for every $i \in \{1, \dots, k\}$ such that $x(i) \neq 0$. Thus, $|x^\perp| = \prod_{x(i)=0} |P_i|$. Therefore, if $x \in V(G)$ then

$$\begin{aligned} \deg(x) &= |V(G) \setminus (x^\perp \cup \{x\})| \\ &= |V(G)| - |x^\perp \setminus \{0\}| - |\{x\}| \\ &= |V(G)| - \left[\prod_{x(i)=0} |P_i| - 1 \right] - 1 \\ &= |V(G)| - \prod_{x(i)=0} |P_i|. \end{aligned}$$

Observe that (2) is now clear since $|V((G^*)^c(P))| = |P \setminus \{0,1\}| = \prod_{i=1}^k |P_i| - 2$. Also, $|\mathcal{D}| = \prod_{i=1}^k (|P_i| - 1)$ since $x \in \mathcal{D}$ if and only if $x(i) \neq 0$ for every $i \in \{1, \dots, k\}$, and hence $|V(G^c(P))| = |P| - |\mathcal{D} \cup \{0\}| = \prod_{i=1}^k |P_i| - \big(\prod_{i=1}^k (|P_i| - 1) + 1\big)$. Therefore, (1) follows immediately. ◻

Now, the focus is put on the graphs $G^c(P)$. The next result gives criteria to determine the parities of the degrees of vertices in $G^c(P)$.

Lemma 3.3. *Let $P = \prod_{i=1}^k P_i$ ($k \in \mathbb{N}$) such that $P_i \neq \{0\}$ is a finite bounded poset with $Z(P_i) = \{0\}$ for every $i \in \{1, \dots, k\}$, and set $J = \{i \in \{1, \dots, k\} \mid |P_i|$ is even$\}$. If $x \in V(G^c(P))$ then $\deg(x)$ is even if and only if either $J = \{1, \dots, k\}$, or $\varnothing \neq J \subsetneq \{1, \dots, k\}$ and $x(j) \neq 0$ for every $j \in J$.*

Proof. The "if" portion easily follows by Lemma 3.2(1). Conversely, suppose that $J \neq \{1, \dots, k\}$. By Lemma 3.2(1), $\deg(x)$ is odd if $J = \varnothing$, so assume $J \neq \varnothing$. Then $\prod_{i=1}^k |P_i|$ and $\prod_{i=1}^k (|P_i| - 1)$ are even since $J \neq \varnothing$ and $J \neq \{1, \dots, k\}$, respectively. Moreover, if $x(j) = 0$ for some $j \in J$ then $\prod_{x(i)=0} |P_i| + 1$ is odd, and therefore $\deg(x)$ is odd by Lemma 3.2(1). ◻

The preparation to establish the first main theorem of this section is now in place.

Theorem 3.4. *Let $P = \prod_{i=1}^k P_i$ ($k \in \mathbb{N}$) such that $P_i \neq \{0\}$ is a finite bounded poset with $Z(P_i) = \{0\}$ for every $i \in \{1, \dots, k\}$. The following statements are equivalent.*

(1) $G^c(P)$ is Eulerian.

(2) $G^c(P)$ has an Eulerian trail.

(3) $k \geq 3$ and $|P_i|$ is even for every $i \in \{1, \dots, k\}$.

Proof. Note that (1) implies (2) trivially, and (3) implies (1) by Lemmas 3.1 and 3.3. To show (2) implies (3), note that $k \geq 3$ by Lemma 3.1. Suppose that there exists $i \in \{1, \dots, k\}$ such that $|P_i|$ is odd. If $|P_i|$ is odd for every $i \in \{1, \dots, k\}$ then Lemma 3.3 implies $\deg(x)$ is odd for every $x \in V(G^c(P))$. But $|V(G^c(P))| \geq 3$ since $k \geq 3$ (e.g., P has at least three atoms), so $G^c(P)$ has no Eulerian trail. Thus, let $j \in \{1, \dots, k\}$ such that $|P_j|$ is even. Then

$|\{x \in V(G^c(P)) \mid x(j) = 0\}| \geq 3$ (e.g., since $k \geq 3$, at least three vertices x satisfy $x(j) = 0$ with either $x(s) \neq 0$, $x(t) \neq 0$, or $x(s), x(t) \neq 0$ for some distinct $s, t \in \{1, \ldots, k\} \setminus \{j\}$). Hence, by Lemma 3.3, $G^c(P)$ has at least three vertices of odd degree. Therefore, $G^c(P)$ has no Eulerian trail. □

Recall that if R is a reduced Artinian ring with exactly k prime ideals then there exist fields $F_1, \ldots, F_k$ such that $R \cong F_1 \times \cdots \times F_k$ ([3, Theorem 8.7]). By endowing every F_i with a linear order such that 0 and 1 are the least and the greatest elements of F_i, respectively, a poset $\prod_{i=1}^k F_i$ is obtained such that $Z(F_i) = \{0\}$ for every $i \in \{1, \ldots, k\}$. In this case, it is straightforward to check that the ring-theoretic zero-divisor graph $\Gamma(\prod_{i=1}^k F_i)$ equals the poset-theoretic zero-divisor graph $G(\prod_{i=1}^k F_i)$ (cf. [14, Remark 3.4]). Since $|F_i|$ is even if and only if F_i is finite and has characteristic 2, the following application of Theorem 3.4 is immediate.

Corollary 3.5. *If R is a finite reduced commutative ring then $\Gamma^c(R)$ is Eulerian if and only if R is of characteristic 2 and has at least three prime ideals.*

Next, we turn the focus to the graphs $(G^*)^c(P)$.

Theorem 3.6. *Let $P = \prod_{i=1}^k P_i$ ($k \in \mathbb{N}$) such that $P_i \neq \{0\}$ is a finite bounded poset with $Z(P_i) = \{0\}$ for every $i \in \{1, \ldots, k\}$. Then $(G^*)^c(P)$ is Eulerian if and only if one of the following statements holds.*

(1) $k \geq 3$ and $|P_i| = 2$ for every $i \in \{1, \ldots, k\}$.

(2) $|P_i|$ is odd for every $i \in \{1, \ldots, k\}$.

Proof. If (1) holds then $(G^*)^c(P) = G^c(P)$ is Eulerian by Theorem 3.4, and if (2) holds then Lemmas 3.1 and 3.2(2) imply that $(G^*)^c(P)$ is Eulerian. Conversely, assume that $(G^*)^c(P)$ is Eulerian. If there exist $i, j \in \{1, \ldots, k\}$ such that $|P_i|$ is odd and $|P_j|$ is even then consider the element $x \in P$ such that $x(i) = 0$, while $x(t) = 1$ for every $t \in \{1, \ldots, k\} \setminus \{i\}$. It is clear from Lemma 3.2(2) that $\deg(x)$ is odd in $(G^*)^c(P)$, which is a contradiction. Therefore, either $|P_i|$ is even for every $i \in \{1, \ldots, k\}$, or $|P_i|$ is odd for every $i \in \{1, \ldots, k\}$.

To complete the proof, suppose that $|P_i|$ is even for every $i \in \{1, \ldots, k\}$. If $|P_i| > 2$ for some $i \in \{1, \ldots, k\}$, then consider $x \in P$ such that $x(i) \in P_i \setminus \{0, 1\}$, while $x(j) = 1$ for every $j \in \{1, \ldots, k\} \setminus \{i\}$. Then $\deg(x) = \prod_{i=1}^k |P_i| - (1) - 2$ in $(G^*)^c(P)$ by Lemma 3.2(2), which is odd. This is a contradiction, and therefore $|P_i| = 2$ for every $i \in \{1, \ldots, k\}$. In particular, $(G^*)^c(P) = G^c(P)$, and hence $k \geq 3$ by Theorem 3.4. □

Remark 3.7. *In contrast to Theorem 3.4, it can happen that $(G^*)^c(P)$ has an Eulerian trail, but is not Eulerian. For example, this is the case if $P = \{0, 1\} \times \{0, 1, 2\}$.*

Conditions were given in [6, Theorem 5.1] in order to characterize the intersection graphs of the principal ideal rings $\mathbb{Z}_n$. However, the result contains a minor oversight since, for example, the characterization implies the false assertion that $IG(\mathbb{Z}_6)$ is Eulerian (the error is the omission of the condition "$k \geq 3$" in the case when $n = p_1 \cdots p_k$ for distinct primes $p_1, \ldots, p_k$). The following corollaries generalize (and correct) [6, Theorem 5.1].

Let R be a commutative ring with identity. Recall that R is a *special principal ideal ring* (or, *SPIR* for brevity) if R is a local Artinian principal ideal ring (cf. [9]). If R is an SPIR with maximal ideal M then there exists $n \in \mathbb{N}$ such that $M^n = \{0\}$, $M^{n-1} \neq \{0\}$, and if I is an ideal of R then $I = M^i$ for some $i \in \{0, 1, \ldots, n\}$ ([9, Proposition 4]). In this case, M is nilpotent with the index of nilpotency equal to n, and the lattice of ideals of R is isomorphic to the poset $\{0, 1, \ldots, n\}$.

By [9, Lemma 10], R is an Artinian principal ideal ring if and only if there exist SPIRs $R_1, \ldots, R_k$ such that $R \cong R_1 \times \cdots \times R_k$ (it is also a straightforward consequence of the structure theorem of Artinian rings in [3, Theorem 8.7]). The next corollary characterizes Eulerian graphs $IG(R)$ for such rings R.

Corollary 3.8. *Suppose that R is a commutative Artinian principal ideal ring, and let $R_1, \ldots, R_k$ ($k \in \mathbb{N}$) be SPIRs such that $R \cong R_1 \times \cdots \times R_k$. If M_i is the maximal ideal of R_i ($i \in \{1, \ldots, k\}$), then $IG(R)$ is Eulerian if and only if one of the following statements holds.*

(1) $k \geq 3$ and R_i is a field for every $i \in \{1, \dots, k\}$.

(2) The index of nilpotency of M_i is even for every $i \in \{1, \dots, k\}$.

In particular, if R is a reduced commutative Artinian ring with at least three prime ideals then $IG(R)$ is Eulerian.

Proof. For every $i \in \{1, \dots, k\}$, let n_i be the index of nilpotency of M_i. Hence, the lattice of ideals of R_i is isomorphic to the poset $\{0, 1, \dots, n_i\}$, and thus $IG(R) \cong (G^*)^c(\prod_{i=1}^k \{0, 1, \dots, n_i\})$. Also, R_i is a field if and only if $|\{0, 1, \dots, n_i\}| = 2$. Therefore, the first assertion follows by Theorem 3.6, and the "in particular" statement holds by (1) and [3, Theorem 8.7]. □

The next corollary provides the correction to [6, Theorem 5.1], and generalizes the result by relaxing the "distinct primes" condition.

Corollary 3.9. *If $p_1, \dots, p_k \in \mathbb{N}$ are (not necessarily distinct) prime numbers then the graph $IG(\mathbb{Z}_{p_1^{n_1}} \times \cdots \times \mathbb{Z}_{p_k^{n_k}})$ is Eulerian if and only if either $k \geq 3$ and $n_1 = \cdots = n_k = 1$, or $n_i \in \mathbb{N}$ is even for every $i \in \{1, \dots, k\}$.*

Proof. The result is an immediate consequence of Corollary 3.8 since $\mathbb{Z}_{p_i^{n_i}}$ is an SPIR whose maximal ideal has the index of nilpotency equal to n_i for every $i \in \{1, \dots, k\}$. □

Remark 3.10. *Corollary 3.8 can fail if R is not a principal ideal ring. In fact, if $R = \mathbb{Z}_2[X, Y]/(X, Y)^2$ then R is a finite local ring whose maximal ideal $M = (\overline{X}, \overline{Y})$ has even index of nilpotency, but $IG(R) \cong K_{1,3}$ (with vertices $(\overline{X})$, $(\overline{Y})$, $(\overline{X+Y})$, and M) is not Eulerian.*

4 Hamiltonian graphs

The goal in this section is to prove the following theorem, which characterizes Hamiltonian $G^c(P)$ for finite pseudocomplemented posets P. It is observed in the discussion that follows the proof of Lemma 4.8 that $G^c(P)$ can be Hamiltonian without P being pseudocomplemented, but also that the pseudocomplemented condition is necessary to prove the assertion. As a consequence, it is shown in Corollary 4.10 that complements of zero-divisor graphs of finite reduced commutative rings with at least three prime ideals are Hamiltonian.

Theorem 4.1. *If P is a finite pseudocomplemented poset then $G^c(P)$ is a Hamiltonian graph if and only if P has at least three atoms.*

The "only if" statement is clear since $G^c(P)$ is null if P has only one atom, and if P has exactly two atoms a and b then $G^c(P)$ is the disjoint union of the complete graphs induced by $a^\vee \cap V(G^c(P))$ and $b^\vee \cap V(G^c(P))$ (cf. the illustration discussed prior to Figure 2). The next seven lemmas will be sufficient to prove the "if" statement. For Lemmas 4.2, 4.3, and 4.4, let $p_1, p_2, \dots, p_k$ ($2 \leq k \in \mathbb{N}$) be the atoms of P. Observe that if $d \in \mathcal{D}$ then $p \leq d$ for every atom p of P. Thus, set $A_{p_1} = p_1^\vee \setminus \mathcal{D}$, and define $A_{p_j} = p_j^\vee \setminus \left(\mathcal{D} \cup \left(\bigcup_{i=1}^{j-1} A_{p_i}\right)\right)$ for every $j \in \{2, \dots, k\}$.

Lemma 4.2. *Let P be finite poset with 0 that has at least two atoms, and let p be an atom of P. The elements of $p^\vee \setminus \mathcal{D}$ induce a maximal complete subgraph of $G^c(P)$. Moreover, if $p_1, p_2, \dots, p_k$ ($2 \leq k \in \mathbb{N}$) are the atoms of P then A_{p_i} induces a complete subgraph of $G^c(P)$ for every $i \in \{1, \dots, k\}$, and $V(G^c(P)) = \dot\bigcup_{i=1}^k A_{p_i}$.*

Proof. It is clear that $V(G^c(P)) = \dot\bigcup_{i=1}^k A_{p_i}$, and the sets $p_i^\vee \setminus \mathcal{D}$ and A_{p_i} ($i \in \{1, \dots, k\}$) induce complete subgraphs of $G^c(P)$. If p is an atom then the complete subgraph of $G^c(P)$ induced by $p^\vee \setminus \mathcal{D}$ is maximal since a vertex $x \in V(G^c(P))$ is adjacent to p if and only if $x \in p^\vee \setminus \mathcal{D}$. □

As in [15], a bounded poset P is called *distributive* if, for all $a, b, c \in P$, the equality $\{\{a\} \cup \{b, c\}^\vee\}^\wedge = \{\{a, b\}^\wedge \cup \{a, c\}^\wedge\}^{\vee\wedge}$ holds. By [15, Theorem 1], this definition generalizes the usual notion of a distributive lattice (i.e., a bounded lattice is distributive in the usual sense if and only if it is a distributive poset). Moreover, as in [7], P is called *Boolean* if P is distributive and complemented. Clearly, every Boolean algebra is a Boolean poset but the converse can fail (in fact, by [7, Theorem 3], there exists a Boolean poset of order $2n$ for every $n \in \mathbb{N}$).

It is well-known that a complement of an element of a Boolean poset P is the pseudocomplement (cf. [10, Lemma 2.4]). In particular, if P is Boolean then P is pseudocomplemented, and every element $x \in P$ has the unique complement x'. This observation is used in the following result, which shows that A_{p_i} does not contain any dual atoms for $i \geq 3$.

Lemma 4.3. *Let P be a finite Boolean poset, and let $p_1, \ldots, p_k$ ($3 \leq k \in \mathbb{N}$) be the atoms of P. Then $A_{p_1} \cap \{p'_1, \ldots, p'_k\} = \{p'_2, \ldots, p'_k\}$, $A_{p_2} \cap \{p'_1, \ldots, p'_k\} = \{p'_1\}$, and if $i \in \{3, \ldots k\}$ then $A_{p_i} \cap \{p'_1, \ldots, p'_k\} = \varnothing$.*

Proof. Clearly, $\{p_1, p_j\}^\wedge = \{0\}$ for every $j \in \{2, \ldots, k\}$. Since complementation in a Boolean poset is pseudocomplementation, $p_1 \leq p'_j$ for every $j \in \{2, \ldots, k\}$. Now, from the construction of A_{p_1}, it is clear that $A_{p_1} \cap \{p'_1, \ldots, p'_k\} = \{p'_2, \ldots, p'_k\}$. Similarly, $p_2 \leq p'_1$, so $A_{p_2} \cap \{p'_1, \ldots, p'_k\} = \{p'_1\}$. Thus, $\{p'_1, \ldots, p'_k\} \subseteq A_{p_1} \cup A_{p_2}$, and therefore $A_{p_i} \cap \{p'_1, \ldots, p'_k\} = \varnothing$ for every $i \in \{3, \ldots, k\}$. □

Next, the main result of this section is proved for the special case when P is Boolean. For this, note that $V(G^c(P)) = P \setminus \{0, 1\}$ (e.g., since P is complemented).

Lemma 4.4. *If P is a finite Boolean poset with at least three atoms then $G^c(P)$ is Hamiltonian.*

Proof. Let $p_1, \ldots, p_k$ ($3 \leq k \in \mathbb{N}$) be the atoms of P. Define $B_{p_1} = A_{p_1} \setminus \{p'_2, \ldots, p'_k\}$, $B_{p_2} = A_{p_2} \setminus \{p'_1\}$, and set $B_{p_i} = A_{p_i}$ for every $i \in \{3, \ldots, k\}$. By Lemmas 4.2 and 4.3, it follows that $V(G^c(P)) = \left(\biguplus_{i=1}^k B_{p_i}\right) \uplus \{p'_1, \ldots, p'_k\}$. Also, if $i \in \{1, \ldots, k\}$ then Lemma 4.2 implies B_{p_i} induces a complete subgraph of $G^c(P)$, which therefore has a Hamiltonian path that begins at p_i and ends at, say, $x_i \in B_{p_i}$.

If $i, j \in \{1, \ldots, k\}$ with $i \neq j$ then $\{x_i, p'_j\}^\wedge \neq \{0\}$; otherwise, $0 < x_i \leq p''_j = p_j$, i.e., $x_i = p_j$ (since p_j is an atom), which contradicts the containment $x_i \in B_{p_i}$. Moreover, $\{p_i, p'_j\}^\wedge \neq \{0\}$ (as $\{p_i, p_j\}^\wedge = \{0\}$ implies $p_i \leq p'_j$) . Hence, the subgraph of $G^c(P)$ induced by $\{p'_k, p'_1\} \cup B_{p_{k-1}}$ has a Hamiltonian path that begins at p'_k (which is adjacent to $p_{k-1} \in B_{p_{k-1}}$ by setting $i = k-1$ and $j = k$) and ends at p'_1 (which is adjacent to $x_{k-1} \in B_{p_{k-1}}$ by setting $i = k-1$ and $j = 1$ in the first statement of this paragraph). Similarly, the subgraph induced by $\{p'_1, p'_2\} \cup B_{p_k}$ has a Hamiltonian path that begins at p'_1 and ends at p'_2, and if $i \in \{1, \ldots, k-2\}$ then the subgraph induced by $\{p'_{i+1}, p'_{i+2}\} \cup B_{p_i}$ has a Hamiltonian path that begins at p'_{i+1} and ends at p'_{i+2}. Therefore, since $V(G^c(P)) = \left(\biguplus_{i=1}^k B_{p_i}\right) \uplus \{p'_1, \ldots, p'_k\}$, a Hamiltonian cycle in $G^c(P)$ is given by the union of these Hamiltonian paths (see Figure 1). □

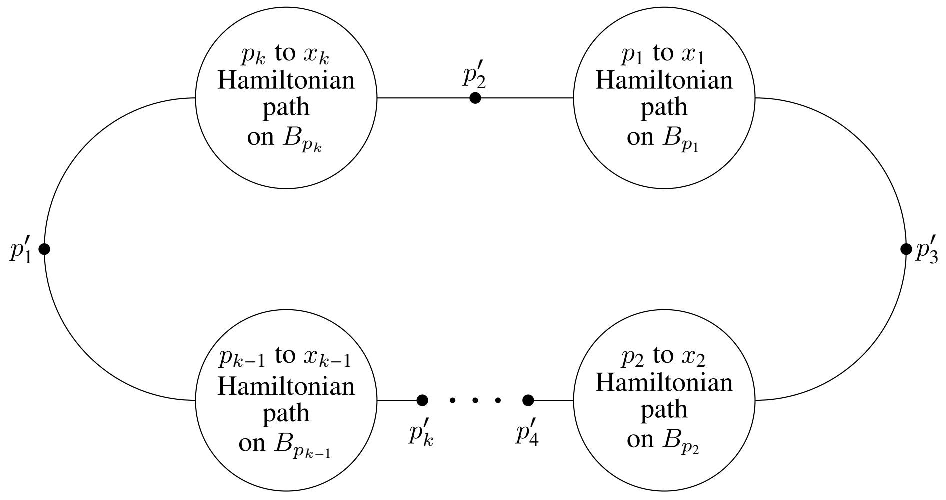

Figure 1. A Hamiltonian cycle of $G^c(P)$

The condition of having at least three atoms in Lemma 4.4 is necessary. For example, if P is the Boolean poset that is depicted in Figure 2 then $G^c(P)$ is not Hamiltonian (in fact, it is not connected).

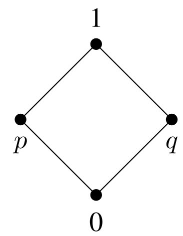

Figure 2. Boolean poset P

To afford an application of Lemma 4.4, it will be important to define and establish properties of a certain poset of equivalence classes of P. For a poset P with 0, an equivalence relation $\sim$ is given on P by $a \sim b$ if and only if $a^\perp = b^\perp$. The set of equivalence classes of P will be denoted by $[P] = \{[a] \mid a \in P\}$, where $[a] = \{x \in P \mid x \sim a\}$. Clearly $[0] = \{0\}$, and if $d \in \mathcal{D}$ then $[d] = \mathcal{D}$.

Note that $[P]$ is a poset under the partial order given by $[a] \le [b]$ if and only if $b^\perp \subseteq a^\perp$. From the observation that $b^\perp \subseteq a^\perp$ whenever $a, b \in P$ with $a \le b$, it follows that the canonical mapping $P \to [P]$ defined by $a \mapsto [a]$ is an order-preserving surjection. Furthermore, if a is an atom of the poset P then, for every $b \in P \setminus \{0\}$, either $a \le b$, or $b \in a^\perp \setminus b^\perp$ (so that $a^\perp \nsubseteq b^\perp$). It follows that if a is an atom of P then $[a]$ is an atom of $[P]$ (the converse is not true; e.g., consider the case where $[a]$ contains an atom p of P with $a \neq p$). Moreover, it is clear that if a and b are distinct atoms of P then $[a] \neq [b]$.

Let P be a pseudocomplemented poset. If $a, b \in P$ then $a^* \le b^*$ if and only if $a^* \in b^\perp$, if and only if $a^\perp = (a^*)^\wedge \subseteq b^\perp$. That is, $a^* \le b^*$ if and only if $[b] \le [a]$ and, in particular, $a^* = b^*$ if and only if $[a] = [b]$. These observations are recorded in (1), (2), and (3) of the following lemma.

Lemma 4.5. *Let P be a poset with 0. If $a, b \in P$ then the following statements hold.*

(1) If $a, b \in P$ are distinct atoms then $[a]$ and $[b]$ are distinct atoms of $[P]$.

(2) If $a \le b$ then $[a] \le [b]$.

(3) If P is pseudomomplemented then $a^* \le b^*$ if and only if $[b] \le [a]$.

(4) $\{a, b\}^\wedge = \{0\}$ if and only if $\{[a], [b]\}^\wedge = \{[0]\}$.

Proof. The statements in (1), (2), and (3) follow by the above discussion. Moreover, the "if" statement of (4) is clear by (2). Conversely, assume that $\{a, b\}^\wedge = \{0\}$ and $[t] \in \{[a], [b]\}^\wedge$. Then $a \in b^\perp \subseteq t^\perp$. Hence, $t \in a^\perp \subseteq t^\perp$. Thus, $t = 0$, i.e., $[t] = [0]$. □

While a pseudocomplemented poset need not be Boolean (e.g., consider the lattice $N_5 = \{\varnothing, \{1\}, \{2\}, \{1,3\}, \{1,2,3\}\}$ under inclusion), [11, Lemma 2.5] and [17, Corollary 6] show that if $[P]$ is pseudocomplemented then $[P]$ is Boolean. This result is specialized in the next lemma.

Lemma 4.6. *If P is a pseudocomplemented poset then $[P]$ is Boolean.*

Proof. By [11, Lemma 2.5] and [17, Corollary 6], it suffices to prove $[P]$ is pseudocomplemented. Let $[a] \in [P]$. The equality $\{[a], [a^*]\}^\wedge = \{[0]\}$ holds by Lemma 4.5(4). Let $[x] \in [P]$ such that $\{[x], [a]\}^\wedge = \{[0]\}$. By Lemma 4.5(4), $\{x, a\}^\wedge = \{0\}$ in P. As P is pseudocomplemented, we have $x \le a^*$. Thus, $[x] \le [a^*]$ by Lemma 4.5(2). Hence, $[a^*] = [a]^*$, and therefore $[P]$ pseudocomplemented. □

Lemma 4.8 is the final result prior to the proof of Theorem 4.1. First, we observe the next lemma, which establishes subgraphs of $G^c(P)$ with Hamiltonian paths.

Lemma 4.7. *Let P be a poset with 0, and let $x \in V(G^c(P))$. The elements of $[x]$ form a complete subgraph of $G^c(P)$. In particular, if P is finite and $y \in [x] \setminus \{x\}$ then the subgraph of $G^c(P)$ induced by $[x]$ contains a Hamiltonian path that begins with x and ends with y.*

Proof. Let $a, b \in [x]$ with $a \neq b$. On the contrary, suppose that a and b are not adjacent in $G^c(P)$, that is, $\{a, b\}^\wedge = \{0\}$. As $a, b \in [x]$, we have $a^\perp = b^\perp = x^\perp$. But from $\{a, b\}^\wedge = \{0\}$, we have $a \in b^\perp = a^\perp$, a contradiction. Hence, the elements of $[x]$ form a complete subgraph of $G^c(P)$. The "in particular" statement is clear. □

Lemma 4.8. *If P is a finite poset with 0 such that $G^c([P])$ is Hamiltonian then $G^c(P)$ is Hamiltonian.*

Proof. Suppose that $x_1, \ldots, x_n \in V(G^c(P))$ such that $[x_1] - [x_2] - \cdots - [x_n] - [x_1]$ is a Hamiltonian cycle in $G^c([P])$. For every $i \in \{1, \ldots, n\}$, choose $y_i \in [x_i] \setminus \{x_i\}$ if $|[x_i]| > 1$, and otherwise let $y_i = x_i$. By Lemma 4.5(4), y_n is adjacent to x_1, and y_i is adjacent to x_{i+1} for every $i \in \{1, \ldots, n-1\}$. Hence, Lemma 4.7 implies that the subgraph of $G^c(P)$ induced by $\{y_{n-1}, x_1\} \cup [x_n]$ has a Hamiltonian path containing the edges $y_{n-1}x_n$ and $y_n x_1$ that begins at y_{n-1} and ends at x_1. Similarly, the subgraph induced by $\{y_n, x_2\} \cup [x_1]$ has a Hamiltonian path containing the edges $y_n x_1$ and $y_1 x_2$ that begins at y_n and ends at x_2, and the subgraph induced by $\{y_i, x_{i+2}\} \cup [x_{i+1}]$ has a Hamiltonian path containing the edges $y_i x_{i+1}$ and $y_{i+1} x_{i+2}$ that begins at y_i and ends at x_{i+2} for every $i \in \{1, \ldots, n-2\}$. Therefore, since $V(G^c(P)) = \dot\bigcup_{i=1}^{n} [x_i]$, a Hamiltonian cycle in $G^c(P)$ is given by the union of these Hamiltonian paths. □

The converse of Lemma 4.8 can fail. For example, let P be the poset depicted in Figure 3(A). The poset $[P]$ is given in Figure 3(B), and it is clear that $G^c(P)$ is Hamiltonian whereas $G^c([P])$ is not Hamiltonian (see Figure 4). Also, it is clear that P is neither pseudocomplemented nor complemented (e.g., consider the element $b \in P$). In particular, $G^c(P)$ is Hamiltonian without P being pseudocomplemented, and $G^c([P])$ shows that the "pseudocomplemented" condition of Theorem 4.1 is necessary. However, if P is pseudocomplemented then we have the pleasant situation mentioned in Theorem 4.1, which can now be readily proved.

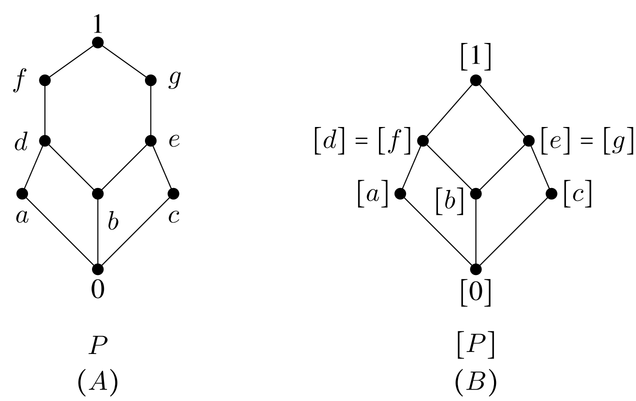

Figure 3. The posets P and $[P]$

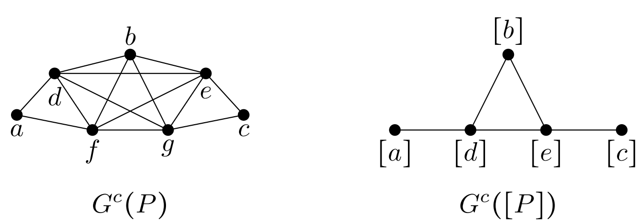

Figure 4. The graphs $G^c(P)$ and $G^c([P])$

Proof of Theorem 4.1. By Lemmas 4.5(1) and 4.6, $[P]$ is Boolean with at least three atoms. Therefore, $G^c([P])$ is Hamiltonian by Lemma 4.4. Hence, the result follows from Lemma 4.8. □

To close this section, we provide the analogues of Theorem 3.4 and Corollary 3.5 for the Hamiltonian condition.

Corollary 4.9. *If* $P = \prod_{i=1}^{k} P_i$ *(*$3 \leq k \in \mathbb{N}$*) such that* $P_i \neq \{0\}$ *is a finite bounded poset with* $Z(P_i) = \{0\}$ *for every* $i \in \{1, \ldots, k\}$ *then* $G^c(P)$ *is Hamiltonian.*

Proof. By Proposition 2.1, $P = \prod_{i=1}^{k} P_i$ is pseudocomplemented, and hence the result follows from Theorem 4.1. □

Corollary 4.10. *If* R *is a finite reduced commutative ring with at least three prime ideals then* $\Gamma^c(R)$ *is Hamiltonian.*

Proof. The result follows immediately by Corollary 4.9 and the discussion prior to Corollary 3.5. □

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