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Analytic Odd Mean Labeling of Some Graphs

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Abstract. Let $G = (V, E)$ be a graph with p vertices and q edges. A graph G is analytic odd mean if there exist an injective function $f : V \to \{0, 1, 3, 5 \ldots, 2q - 1\}$ with an induce edge labeling $f^* : E \to Z$ such that for each edge uv with $f(u) < f(v)$,

$$f^*(uv) = \begin{cases} \left\lceil \frac{f(v)^2-(f(u)+1)^2}{2} \right\rceil, & if \quad f(u) \neq 0 \\ \left\lceil \frac{f(v)^2}{2} \right\rceil, & if \quad f(u) = 0 \end{cases}$$

is injective. We say that f is an analytic odd mean labeling of G. In this paper we prove that fan F_n, double fan $D(F_n)$, double wheel $D(W_n)$, closed helm CH_n, total graph of cycle $T(C_n)$, total graph of path $T(P_n)$, armed crown $C_n \Theta P_m$, generalized peterson graph $GP(n, 2)$ are analytic odd mean graphs.

1 Introduction

Throughout this paper we consider only finite, simple and undirected graph $G = (V, E)$ with p vertices and q edges and notations not defined here are used in the sense of Harary [1]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling. An excellent survey of graph labeling is available in [5]. The concept of mean labeling was introduced in [6]. A graph G is called a mean graph if there is an injective function $f : V \to \{0, 1, 2, 3 \ldots, q\}$ with an induce edge labeling $f^* : E \to Z$ given by $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ is injective. The concept of analytic mean labeling was due to Tharmaraj and Sarasija in [7]. A graph G is analytic mean graph if it admits a bijection $f : V \to \{0, 1, 2, \ldots, p - 1\}$ such that the induced edge labeling $f^* : E \to Z$ given by $f^*(uv) = \left\lceil \frac{f(u)^2-f(v)^2}{2} \right\rceil$ with $f(u) > f(v)$ is injective. Motivated by the results in [7], we introduced a new mean labeling called analytic odd mean labeling in [2]. A graph G is an analytic odd mean if there exist an injective function $f : V \to \{0, 1, 3, 5 \ldots, 2q - 1\}$ with an induce edge labeling $f^* : E \to Z$ such that for each edge uv with $f(u) < f(v)$, $f^*(uv) = \begin{cases} \left\lceil \frac{f(v)^2-(f(u)+1)^2}{2} \right\rceil, & if \quad f(u) \neq 0 \\ \left\lceil \frac{f(v)^2}{2} \right\rceil, & if \quad f(u) = 0 \end{cases}$ is injective. We say that f is an analytic odd mean labeling of G. We proved that path P_n, cycle C_n, complete graph K_n, complete bipartite graph $K_{m,n}$, wheel graph W_n, flower graph Fl_n, ladder graph L_n, comb $P_n \odot K_1$, graph $L_n \odot K_1$ and union of two cycles are analytic odd mean graphs in [2]. Further results on analytic odd mean labeling are also obtained in [3] and [4].
We use the following definitions in the subsequent section to prove the main results.

Definition 1.1. A fan graph F_n is obtained from a path P_n by adding a new vertex and joining it to all the vertices of the path by an edge.

Definition 1.2. A double fan DF_n is obtained by $P_n + 2K_1$.

Definition 1.3. A double wheel graph DW_n of size n can be composed of $2C_n + K_1$ that is, it consists of two cycles of size n, where the vertices of the cycles are all connected to a common hub.

Definition 1.4. The closed helm CH_n is the graph obtained from a helm H_n by joining each pendent vertex to form a cycle.

Definition 1.5. The total graph $T(G)$ of the graph G has the vertex set $V(G) \cup E(G)$ in which two vertices are adjacent whenever they are either adjacent or incident in G.

Definition 1.6. A generalised Peterson graph $P(n, m), n \geq 3, 1 \leq m < n/2$ is a 3-regular graph with 2n vertices $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ and edges $(u_i v_i), (u_i u_{i+1})$, $(v_i v_{i+m})$ for all $1 \leq i \leq n$, where the subscripts are taken modulo n.

Definition 1.7. An armed crown $C_n \Theta P_m$ is a cycle attached with paths of equal length at each vertex of the cycle where P_m is a path of length m-1.

2 Main Results

In this section we prove that fan F_n, double fan $D(F_n)$, double wheel $D(W_n)$, closed helm CH_n, total graph of cycle $T(C_n)$, total graph of path $T(P_n)$, armed crown $C_n \Theta P_m$, generalized peterson graph $GP(n, 2)$ are analytic odd mean graphs.

Theorem 2.1. *The fan graph F_n is an analytic odd mean graph.*

Proof. Let the vertex set and edge set of fan graph be $V(F_n) = \{v, v_i : 1 \leq i \leq n\}$ and $E(F_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v v_i : 1 \leq i \leq n\}$
Now $|V(F_n)| = n + 1$ and$|E(F_n)| = 2n - 1$.
We define an injective map $f : V(F_n) \to \{0, 1, 3, 5, \ldots, 4n - 3\}$ by
$f(v) = 0$ and $f(v_i) = 4i - 3$ for $1 \leq i \leq n$.
The induced edge labeling f^* is defined as follows:
$f^*(v_i v_{i+1}) = 12i - 1$ for $1 \leq i \leq n - 1$
and $f^*(v v_i) = 8i^2 - 12i + 5$ for $1 \leq i \leq n$.
We observe that the edge labels of $v_i v_{i+1}$ are increased by 12 as i increases and the difference of edge labels $v v_i$ are increased by 16 as i increases. Hence the edge labels are distinct and odd . Hence F_n admits an analytic odd mean labeling.
An analytic odd mean labeling of F_6 is shown in Figure 1.

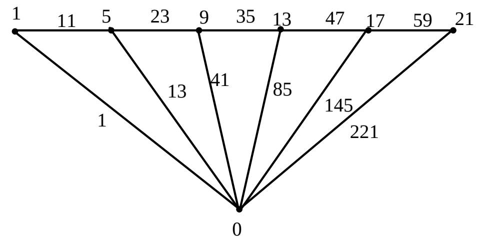

Figure 1

Theorem 2.2. *The double fan graph DF_n is an analytic odd mean graph.*

Proof. Let the vertex set and edge set of fan graph be $V(DF_n) = \{v, u, v_i : 1 \leq i \leq n\}$ and $E(DF_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v v_i : 1 \leq i \leq n\} \cup \{u v_i : 1 \leq i \leq n\}$. Now $|V(DF_n)| = n + 2$ and$|E(DF_n)| = 3n - 1$.
We define an injective map $f : V(DF_n) \to \{0, 1, 3, 5, \ldots, 6n - 3\}$ by
$f(v) = 0$; $f(u) = 3$
and $f(v_i) = 4i - 3$ for $1 \leq i \leq n$.
The induced edge labeling f^* is defined as follows:
$f^*(v_i v_{i+1}) = 12i - 1$ for $1 \leq i \leq n - 1$
$f^*(v v_i) = 8i^2 - 12i + 5$ for $1 \leq i \leq n$
and $f^*(u v_i) = 8i^2 - 12i - 3$ for $1 \leq i \leq n$.
We observe that the edge labels of $v_i v_{i+1}$ are increased by 12 as i increases and the difference of edge labels $v v_i$ and $u v_i$ are increased by 16 as i increases 2 to n. Hence the edge labels are odd and distinct. Therefore DF_n admits an analytic odd mean labeling.
An analytic odd mean labeling of DF_6 is shown in Figure 2.

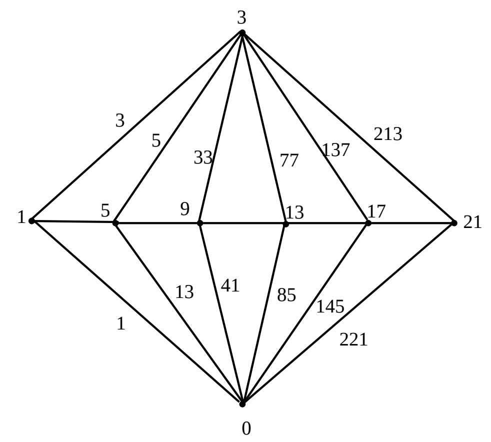

Figure-2

Theorem 2.3. *The double wheel graph $DW_n, n \geq 3$ is an analytic odd mean graph.*

Proof. Let the vertex set and edge set of double wheel graph be $V(DW_n) = \{v_i : 0 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$ and $E(DW_n) = \{v_0v_i, v_0u_i : 1 \leq i \leq n\} \cup \{v_iv_{i+1}, u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{v_nv_1\} \cup \{u_nu_1\}$. Now $|V(DW_n)| = 2n + 1$ and $|E(DW_n)| = 4n$.
If $n = 4$, we label the vertices v_i as $f(v_1) = 1$, $f(v_2) = 5$, $f(v_3) = 13$ and $f(v_4) = 9$. Clearly the double wheel graph DW_4 is an analytic odd mean graph.
For $n \geq 3$ and $n \neq 4$, we define an injective map $f : V(DW_n) \rightarrow \{0, 1, 3, 5, \ldots, 8n-1\}$ by $f(v) = 0$
$f(v_i) = 4i - 3$ for $1 \leq i \leq n$
and $f(u_i) = 4n + 4i - 3$ for $1 \leq i \leq n$.
The induced edge labeling f^* is defined as follows:
$f^*(v_0v_i) = 8i^2 - 12i + 5$ for $1 \leq i \leq n$
$f^*(v_iv_{i+1}) = 12i - 1$ for $1 \leq i \leq n - 1$
$f^*(v_nv_1) = 8n^2 - 12n + 3$
$f^*(v_0u_i) = 4n(2n - 3) + 4i(2i - 3) + 16ni + 5$ for $1 \leq i \leq n$
$f^*(u_iu_{i+1}) = 12n + 12i - 1$ for $1 \leq i \leq n - 1$
and $f^*(u_nu_1) = 24n^2 - 32n + 3$.
Clearly all the edge labels are odd. We observe that the edge labels v_iv_{i+1} and u_iu_{i+1} are increased by 12 as i increases from 1 to n-1. Also the edge labels of v_0v_i are increased by 16i-4 and that of $v_0u_i$are increased by 16n+16i-4 as i increases from 1 to n-1. Therefore all the edge labels are distinct. Hence the double wheel graph DW_n admits an analytic odd mean labeling.
An analytic odd mean labeling of DW_6 is shown in Figure 3.

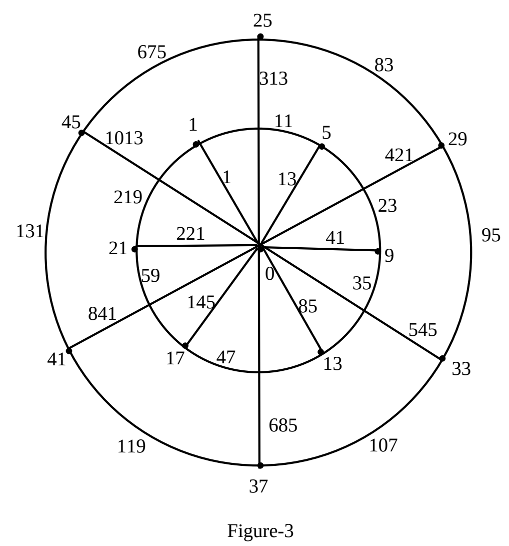

Figure-3

Theorem 2.4. *The closed helm graph $CH_n, n \geq 3$ is an analytic odd mean graph.*

Proof. Let $V(CH_n) = \{v, v_i, u_i : 1 \leq i \leq n\}$ and $E(CH_n) = \{vv_i, v_iu_i : 1 \leq i \leq n\} \cup \{v_iv_{i+1}, u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{v_nv_1, u_nu_1\}$. Now $|V(CH_n)| = 2n+1$ and $|E(CH_n)| = 4n$.
If $n = 4$, we label the vertices v_i as $f(v_1) = 1$, $f(v_2) = 5$, $f(v_3) = 13$ and $f(v_4) = 9$. Clearly the closed helm graph CH_4 is an analytic odd mean graph.
For $n \geq 3$ and $n \neq 4$, we define an injective map $f : V(CH_n) \to \{0, 1, 3, 5, \ldots, 8n-1\}$ by $f(v) = 0$
$f(v_i) = 4i - 3$ for $1 \leq i \leq n$
$f(u_i) = 4n + 4i - 3$ for $1 \leq i \leq n$.
The induced edge labeling f^* is defined as follows:
$f^*(vv_i) = 8i^2 - 12i + 5$ for $1 \leq i \leq n$
$f^*(v_iv_{i+1}) = 12i - 1$ for $1 \leq i \leq n-1$
$f^*(v_nv_1) = 8n^2 - 12n + 3$
$f^*(u_iu_{i+1}) = 12n + 12i - 1$ for $1 \leq i \leq n-1$
$f^*(u_nu_1) = 24n^2 - 32n + 3$
and $f^*(v_iu_i) = 4n(2n-3) + 4i(4n-1) + 3$ for $1 \leq i \leq n$.
Clearly all the edge labels are odd and distinct. Hence CH_n admits an analytic odd mean labeling.
An analytic odd mean labeling of CH_8 is shown in Figure 4.

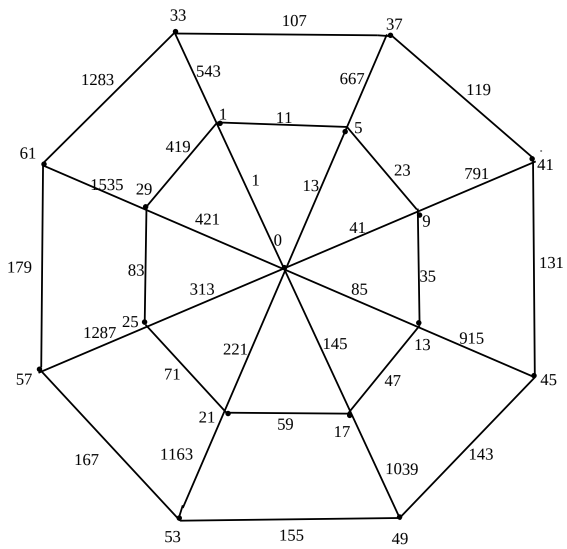

Figure-4

Theorem 2.5. *The total graph of cycle $T(C_n), n \geq 3$ of length n is an analytic odd mean graph.*

Proof. Let the vertex set and edge set of total graph of cycle be $V(T(C_n)) = \{v_i, u_i : 1 \leq i \leq n\}$ and $E(T(C_n)) = \{v_iv_{i+1}, u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 1 \leq i \leq n\} \cup \{v_1v_n, v_1u_n, u_1u_n\}$ respectively. Now $|V(T(C_n))| = 2n$ and $|E(T(C_n))| = 4n$.
If $n = 4$, we label the vertices v_i as $f(v_1) = 1$, $f(v_2) = 5$, $f(v_3) = 13$ and $f(v_4) = 9$. Clearly the total graph of cycle $T(C_4)$ is an analytic odd mean graph.
For $n \geq 3$ and $n \neq 4$, we define an injective map $f : V(T(C_n)) \to \{0, 1, 3, 5, \ldots, 8n-1\}$ by $f(v_i) = 4i - 3$ and $f(u_i) = 4n + 4i - 3$ for $1 \leq i \leq n$.
The induced edge labeling f^* is defined as follows:
$f^*(u_1u_n) = 24n^2 - 32n + 3$
$f^*(v_1v_n) = 8n^2 - 12n + 3$
$f^*(v_iv_{i+1}) = 12i - 1$ for $1 \leq i \leq n-1$
$f^*(u_iu_{i+1}) = 12n + 12i - 1$ for $1 \leq i \leq n-1$
$f^*(v_1u_n) = 32n^2 - 24n + 3$
$f^*(v_iu_i) = 4n(2n + 4i - 7) - 12i + 17$ for $2 \leq i \leq n$
and $f^*(v_{i+1}u_i) = 4n(2n-3) + 4i(4n-5) + 3$ for $1 \leq i \leq n-1$.
Clearly all edge labels are odd. We observed that the edge labels of u_iv_i are increased by $16n-4$ as i increases from 1 to n and that of u_iv_{i+1} are increased by $16n-20$ as i increases from 1 to n-1. Also the edge labels of u_iu_{i+1} and v_iv_{i+1} are increased by 12 as $i$increases from 1 to n-1. So all the edge labels are distinct. Hence the total graph of cycle $T(C_n)$ admits an analytic odd mean labeling. An analytic odd mean labeling of $T(C_7)$ is shown in Figure 5.

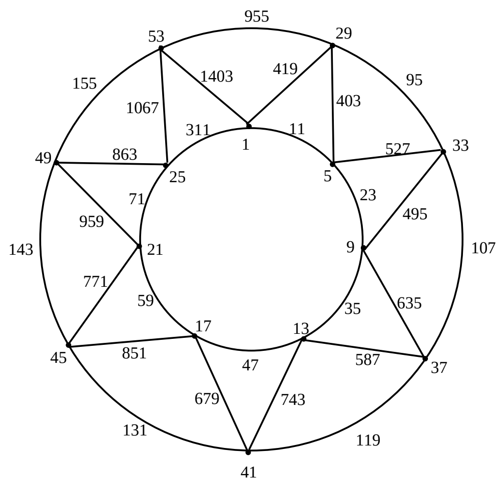

Figure-5

Theorem 2.6. *The total graph of path of n vertices $T(P_n)$ is an analytic odd mean graph.*

Proof. Let the vertex set and edge set of total graph of path be $V(T(P_n)) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n-1\}$ and $E(T(P_n)) = \{v_iv_{i+1}, v_iu_i, v_{i+1}u_i : 1 \leq i \leq n-1\} \cup \{u_iu_{i+1} : 1 \leq i \leq n-2\}$. Now $|V(T(P_n))| = 2n-1$ and $|E(T(P_n))| = 4n-5$.
We define an injective map $f : V(T(P_n)) \to \{0, 1, 3, 5, \ldots, 8n-11\}$ by
$f(v_1) = 0$; $f(v_i) = 4i-5$ for $2 \leq i \leq n$
and $f(u_i) = 4n+4i-7$ for $1 \leq i \leq n-1$. The induced edge labeling f^* is defined as follows:
$f^*(v_1v_2) = 5$
$f^*(v_1u_1) = 8n^2 - 12n + 5$
$f^*(v_iv_{i+1}) = 12i - 7$ for $2 \leq i \leq n-1$
$f^*(u_iu_{i+1}) = 12n + 12i - 13$ for $1 \leq i \leq n-2$
$f^*(v_iu_i) = 4n(2n+4i-7) - 12i + 17$ for $2 \leq i \leq n$
and $f^*(v_{i+1}u_i) = 8n(n+2i) - 28(n+i) + 25$ for $1 \leq i \leq n-1$.
We observe that the edge labels are odd and distinct. Hence $T(P_n)$ admits an analytic odd mean labeling.

An analytic odd mean labeling of $T(P_6)$ is shown in Figure 6.

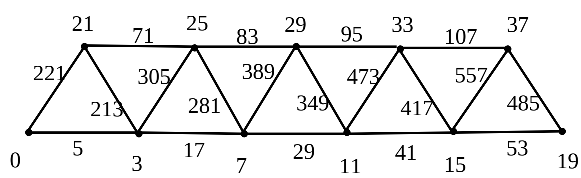

Figure-6

Theorem 2.7. *An armed crown $C_n \Theta P_m, n \geq 3$ and $m \geq 2$ is an analytic odd mean graph.*

Proof. Let $G = C_n \Theta P_m$. Let $u_1, u_2, u_3, \ldots, u_n$ be the vertices of the cycle. Let $v_i^1, v_i^2, v_i^3, \ldots, v_i^m$ be the vertices of the path of length $m-1$ attached with $u_i (1 \leq i \leq n)$ with identification of u_i and v_i^m. Now $|V(G)| = mn$ and $|E(G)| = mn$.
We define an injective map $f : V(G) \to \{0, 1, 3, 5, \ldots, 2mn-1\}$ by
$f(u_i) = f(v_i^m)$
and $f(v_i^j) = 2m(i-1) + 2j - 1$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.
The induced edge labeling f^* is defined as follows:
$f^*(v_i^j v_i^{j+1}) = 2m(i-1) + 2j + 1$ for $1 \leq i \leq n$ and $1 \leq j \leq m-1$
$f^*(v_i^m v_{i+1}^m) = 2m^2(2i+1) - 2m(i+1) + 1$ for $1 \leq i \leq n-1$
and $f^*(v_n^m v_1^m) = 2mn(mn-1) - 2m^2 + 1$.
We observe that the edge labels of $v_i^j v_i^{j+1}$ are increased by 2 as j increases from 1 to $m-1$ and that of $v_i^m v_{i+1}^m$ are increased by $4m^2 - 2m$ as i increases from 1 to $n-1$. Since $f^*(u_1u_2) = f(v_{3m-1}^1)$,

the edge labels of $f^*(u_1u_2)$ is distinct. Similarly $f^*(u_iu_{i+1})$ are distinct as i increases from 1 to $n-1$. So all the edge labels are distinct. Hence an armed crown $C_n\Theta P_m$ admits an analytic odd mean labeling.
An analytic odd mean labeling of $C_{10}\Theta P_5$ is shown in Figure 7.

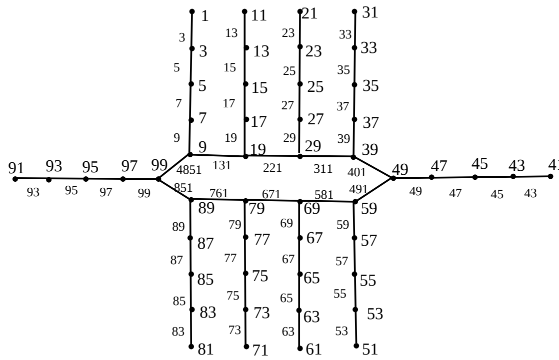

Figure-7

Theorem 2.8. *The generalized Peterson graph $GP(n,2)$ with $n \geq 5$ is an analytic odd mean graph.*

Proof. Let $G = GP(n,2)$ and let the vertex set and the edge set of generalized Peterson graph be $V(G) = \{v_iu_i : 1 \leq i \leq n\}$ and $E(G) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_1u_n\} \cup \{v_iv_{i+2} : 1 \leq i \leq n\} \cup \{v_iu_i : 1 \leq i \leq n\}$. Now $|V(G)| = 2n$ and $|E(G)| = 3n$.
Case(i) $5 \leq n \leq 9$
We define an injective map $f : V(G) \to \{0,1,3,5,\ldots,6n-1\}$ by
$f(u_i) = 2i-1$ for $1 \leq i \leq n-2$
$f(u_{n-1}) = 2n-1$
$f(u_n) = 2n-3$
and $f(v_i) = 6n-2i+1$ for $1 \leq i \leq n$.
The induced edge labeling f^* is defined as follows:
$f^*(u_iu_{i+1}) = 2i+1$ for $1 \leq i \leq n-3$
$f^*(u_{n-2}u_{n-1}) = 6n-7$
$f^*(u_{n-1}u_n) = 2n-1$
$f^*(u_nu_1) = 2n^2-6n+3$
$f^*(v_iv_{i+2}) = 18n-6i-1$ for $1 \leq i \leq n-2$
$f^*(v_{n-1}v_1) = 10n^2-22n-7$
$f^*(v_nv_2) = 10n^2-26n+3$
$f^*(v_{n-1}u_{n-1}) = 6n^2+12n+5$
$f^*(v_nu_n) = 6n^2+8n-1$
and $f^*(v_iu_i) = 6n(3n-2i)+2(3n-i)+1$ for $1 \leq i \leq n-2$.
We observed that $f^*(v_{n-1}v_1), f^*(v_nv_2) > f^*(v_iv_{i+2}), f^*(u_nu_1)$ and $f^*(u_iu_{i+1})$. Also $f^*(u_nu_1) < f^*(v_iv_{i+2})$ as i increases from 1 to n. Clearly all the edge labels are odd and distinct.
Case(ii) $n \geq 10$
We define an injective map $f : V(G) \to \{0,1,3,5,\ldots,6n-1\}$ by
$f(u_i) = 2i-1$ for $1 \leq i \leq n$
and $f(v_i) = 6n-2i+1$ for $1 \leq i \leq n$.
The induced edge labeling f^* is defined as follows:
$f^*(u_iu_{i+1}) = 2i+1$ for $1 \leq i \leq n-1$
$f^*(u_nu_1) = 2n^2-2n-1$
$f^*(v_iu_i) = 6n(3n-2i)+2(3n-i)+1$ for $1 \leq i \leq n$
and $f^*(v_iv_{i+2})$, $f^*(v_{n-1}v_1), f^*(v_nv_2)$ are in Case (i).
We observed that the edge labels of u_iu_{i+1} are increased by 2 as i increases from 1 to $n-1$ and that of v_iv_{i+2} is decreased by 6 as i increases from 1 to $n-2$. Also the edge labels of v_iu_i are decreased by $12n+2$ as i increases from 1 to n. Here $f^*(v_{n-1}v_1)$, $f^*(v_nv_2) > f^*(v_iv_{i+2})$, $f^*(u_nu_1)$ and $f^*(u_iu_{i+1})$. Also $f^*(u_nu_1) > f^*(v_iv_{i+2})$ as i increases from 1 to $n-2$. Therefore all the edge labels are odd and distinct. Hence the generalized Peterson graph $GP(n,2)$ admits an analytic

odd mean labeling. An analytic odd mean labeling of $GP(7,2)$ and $GP(10,2)$ are shown in Figures 8 and 9 respectively.

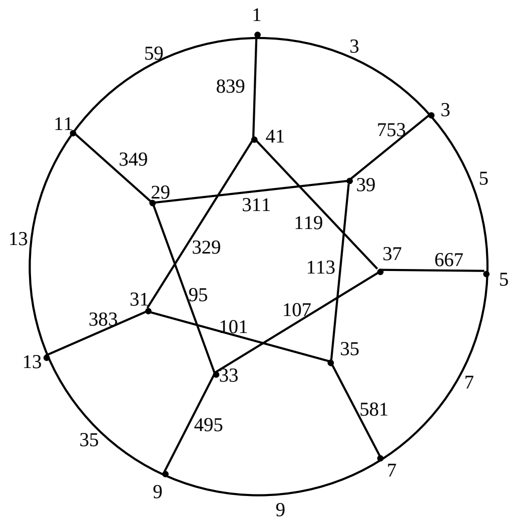

Figure-8

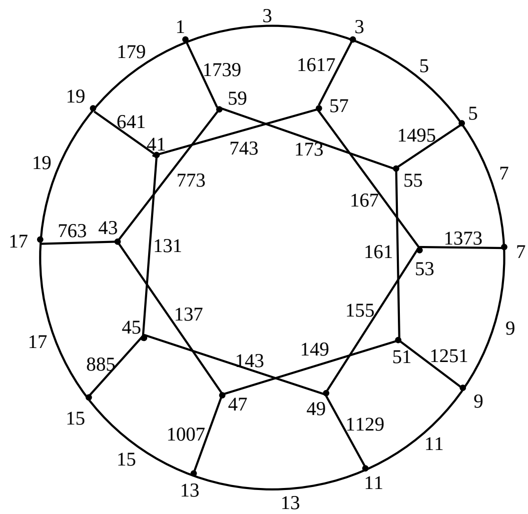

Figure-9

References

[1] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Massachusetts (1972).

[2] P. Jeyanthi, R. Gomathi and Gee-choon Lau, *Analytic odd mean labeling of some standard graphs*, Proceedings of the ICADM,(2018) 62 - 68 .

[3] P.Jeyanthi,R.Gomathi and Gee-Choon Lau, *Analytic odd mean labeling of square and H-graphs*, International Journal of Mathematical Combinatorics, (2018) 61-67.

[4] P.Jeyanthi,R.Gomathi and Gee-Choon Lau, *Some results on analytic odd mean labeling of graph*, Bulletin of International Mathematical Virtual Institute, Vol.9 (3) (2019), 493 - 500.

[5] Joseph A. Gallian, *A Dynamic Survey of Graph Labeling*, The Electronic Journal of Combinatorics, (2017), # DS6.

[6] S. Somasundaram and R. Ponraj, *Mean labelings of graphs*, National Academy Science Letter, 26 (2003),210-213.

[7] T. Tharmaraj and P. B. Sarasija, *Analytic mean labelled graphs*, International Journal of Mathematical Archive, 5(6)(2014), 136 -146.

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