Palestine Journal of Mathematics

Vol. 8(2)(2019) , 400–412

Some Results on Z_k-Magic Labeling

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Communicated by Ayman Badawi

MSC 2010 Classifications: 05C78.

Keywords and phrases: A-magic labeling, Z_k-magic labeling, k-magic graph, shell, generalised jahangir, double wheel, splitting graph.

Abstract For any non-trivial abelian group A under addition a graph G is said to be A-*magic* if there exists a labeling $f : E(G) \to A - \{0\}$ such that, the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant. An A-*magic* graph G is said to be Z_k-magic graph if the group A is Z_k the group of integers modulo k and these graphs are referred as k-*magic* graphs. In this paper we prove that shell graph, generalised jahangir graph, $(P_n + P_1) \times P_2$ graph, double wheel graph, mongolian tent graph, flower snark, slanting ladder, double step grid graph, double arrow graph and semi jahangir graph are k-magic and also prove that if the graph G is k-magic with magic constant 0 then the splitting graph of G is k-magic.

1 Introduction

Graph labeling is currently an emerging area in the research of graph theory. A graph labeling is an assignment of integers to vertices or edges or both subject to certain conditions. A detailed survey was done by Gallian in [6]. If the labels of edges are distinct positive integers and for each vertex v the sum of the labels of all edges incident with v is the same for every vertex v in the given graph then the labeling is called a magic labeling. Sedláček [8] introduced the concept of A-magic graphs. A graph with real-valued edge labeling such that distinct edges have distinct non-negative labels and the sum of the labels of the edges incident to a particular vertex is same for all vertices. Low and Lee [7] examined the A-magic property of the resulting graph obtained from the product of two A-magic graphs. Shiu, Lam and Sun [9] proved that the product and composition of A-magic graphs were also A-magic.

For any non-trivial Abelian group A under addition a graph G is said to be A-*magic* if there exists a labeling $f : E(G) \to A - \{0\}$ such that, the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant. An A-*magic* graph G is said to be Z_k-magic graph if the group A is Z_k, the group of integers modulo k. These Z_k-magic graphs are referred to as k-*magic graphs*. Shiu and Low [10] determined all positive integers k for which fans and wheels have a Z_k-magic labeling with a magic constant 0. Motivated by the concept of A-magic graph in [8] and the results in [7], [9] and [10] Jeyanthi and Jeya Daisy [1]-[5] proved that the open star of graphs, subdivision graphs, cycle of graphs and some standard graphs admit Z_k-magic labeling. In this paper we prove that shell graph, generalised jahangir graph, $(P_n + P_1) \times P_2$ graph, double wheel graph, mongolian tent graph, flower snark, slanting ladder, double step grid graph, double arrow graph and semi jahangir graph are k-magic and also prove that if the graph G is k-magic with magic constant 0 then the splitting graph of G is k-magic. We use the following definitions in the subsequent section.

Definition 1.1. A shell S_n is the graph obtained by taking $n-3$ concurrent chords in a cycle C_n. The vertex at which all the chords are concurrent is called the apex.

Definition 1.2. A generalised Jahangir graph $J_{k,s}$ is a graph on $ks+1$ vertices consisting of a cycle C_{ks} and one additional vertex that is adjacent to k vertices of C_{ks} at distance s to each other on C_{ks}.

Definition 1.3. The Cartesian product $(P_n + P_1) \times P_2$ is a graph with the vertex set $V((P_n + P_1) \times P_2) = \{u, u_i, v, v_i : 1 \le i \le n\}$ and the edge set $E((P_n + P_1) \times P_2) = \{uu_i, vv_i, u_iv_i : 1 \le i \le n\} \cup \{u_iu_{i+1}, v_iv_{i+1} : 1 \le i \le n-1\} \cup \{uv\}$.

Definition 1.4. A double wheel graph DW_n of size n can be composed of $2C_n + K_1$ that is, it consists of two cycles of size n, where the vertices of the two cycles are all connected to a common hub.
Definition 1.5. For each point v of a graph G take a new vertex v' and join v' to those points of G adjacent to v. The graph thus obtained is called the splitting graph of G and is denoted as $S'(G)$.
Definition 1.6. A Mongolian tent $M(m,n)$ is a graph obtained from $P_m \times P_n$ by adding one extra vertex above the grid and joining every other vertex of the top row of $P_m \times P_n$ to the new vertex.
Definition 1.7. A flower snark J_n is a graph on $4n$ vertices, for $n \geq 5$ and odd whose vertices are labelled $V_i = \{w_i, x_i, y_i, z_i\}$ for $1 \leq i \leq n$ and whose edges can be partitioned into n star graphs and two cycles as we will next describe. Each quadruple V_i of vertices induce a star graph w_i as its centre vertices z_i induce an odd cycle $(z_1, z_2, \dots z_n, z_1)$ vertices x_i and y_i induce an even cycle $(x_1, x_2, \dots x_n, y_1, y_2, \dots y_n, x_1)$.
Definition 1.8. The slanting ladder SL_n is a graph obtained from two paths $u_1, u_2, \dots u_n$ and $v_1, v_2, \dots v_n$ by joining each u_i with v_{i+1}, $1 \leq i \leq n-1$.
Definition 1.9. Take $P_n, P_n, P_{n-2}, P_{n-4}, \dots P_4, P_2$ paths on $n, n, n-2, n-4, \dots 4, 2$ vertices and arrange them centrally horizontal where $n \equiv 0(mod\ 2), n \neq 2$. A graph obtained by joining vertical vertices of given successive paths is known as a double step grid graph of size n. It is denoted by DSt_n.
Definition 1.10. A double arrow graph A_n^t with width t and length n is obtained by joining two vertices v and w with superior vertices of $P_m \times P_n$ by $m + m$ new edges from both the ends.
Definition 1.11. A semi Jahangir graph, denoted by SJ_n is a connected graph with vertex set $V(SJ_n) = \{p, x_i, y_k : 1 \leq i \leq n+1, 1 \leq k \leq n\}$ and the edge set $E(SJ_n) = \{px_i : 1 \leq i \leq n+1\} \cup \{x_i y_i : 1 \leq i \leq n\} \cup \{y_i x_{i+1} : 1 \leq i \leq n\}$.

2 Main Results

Theorem 2.1. *The shell graph S_n is k-magic when (i) n is odd and (ii) n is even and k is even.*

Proof. Let the vertex set and the edge set of S_n be $V(S_n) = \{v_i : 1 \leq i \leq n\}$ and $E(S_n) = \{v_1 v_{i+2} :\ 1 \leq i \leq n-3\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\}$.
We consider the following two cases.
Case(i): n is odd.
For any element $a \in Z_k - \{0\}$.
Define the edge labeling $f : E(S_n) \to Z_k - \{0\}$ as follows:
$f(v_1 v_{i+2}) = a$ for $1 \leq i \leq n-3$,
$$f(v_i v_{i+1}) = \begin{cases} \frac{(n-3)a}{2}, & \text{for } i \text{ is even,} \\ k - \frac{(n-1)a}{2}, & \text{for } i \text{ is odd, } i \neq 1, n \end{cases}$$
$f(v_1 v_2) = f(v_1 v_n) = k - \frac{(n-3)a}{2}$.
Then the induced vertex labeling $f^+ : V(S_n) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(G)$.
Case(ii): n is even and k is even.
Subcase(i): $k \equiv 0(mod\ 4)$.
Define the edge labeling $f : E(S_n) \to Z_k - \{0\}$ as follows:
$f(v_1 v_{i+2}) = \frac{k}{2}$ for $1 \leq i \leq n-3$,
$f(v_i v_{i+1}) = \frac{3k}{4}$ for $2 \leq i \leq n-1$,
$f(v_1 v_2) = f(v_1 v_n) = k - \frac{k}{4}$.
Then the induced vertex labeling $f^+ : V(S_n) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(G)$.
Subcase(ii): $k \equiv 2(mod\ 4)$.
Define the edge labeling $f : E(S_n) \to Z_k - \{0\}$ as follows:
$f(v_1 v_{i+2}) = \frac{k}{2}$ for $1 \leq i \leq n-3$,
$$f(v_i v_{i+1}) = \begin{cases} \frac{(3k+2)}{4}, & \text{for } i \text{ is even,} \\ \frac{3k-2}{4}, & \text{for } i \text{ is odd, } i \neq 1. \end{cases}$$
$f(v_1 v_2) = \frac{k-2}{4}$, $f(v_1 v_n) = \frac{k+2}{4}$.
Then the induced vertex labeling $f^+ : V(S_n) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(S_n)$. Hence f^+ is constant and it is equal to $0(mod\ k)$. Since the shell graph S_n admits Z_k-magic labeling, then it is a k-magic graph. □

The examples of Z_8-magic labeling of S_9 and S_8 are shown in Figure 1.

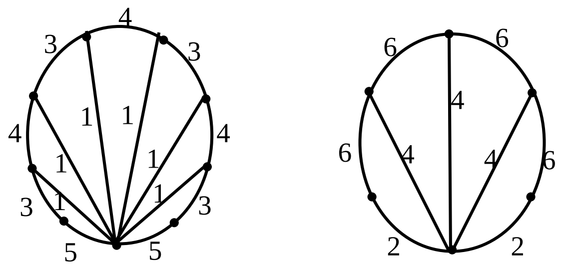

Figure 1: Z_8-magic labeling of S_9 and S_8

Conjecture 2.2. The shell graph S_n is not k-magic when n is even and k is odd.

Theorem 2.3. *The generalised jahangir graph $J_{n,s}$ is k-magic when (i) n is odd and s is odd (ii) n is even and s is even (iii) n is even, s is odd and k is even.*

Proof. Let the vertex set and the edge set of $J_{n,s}$ be $V(J_{n,s}) = \{v, v_i : 1 \le i \le n\} \cup \{v_i^j : 1 \le i \le s-1, 1 \le j \le n\}$ and $E(J_{n,s}) = \{vv_i : 1 \le i \le n\} \cup \{v_j v_1^j :\ 1 \le j \le n\} \cup \{v_i^j v_{i+1}^j : 1 \le i \le s-1, 1 \le j \le n\} \cup \{v_{s-1}^j v_{j+1} : 1 \le j \le n-1\} \cup \{v_{s-1}^n v_1\}$.
We consider the following three cases.
Case(i): n is odd and s is odd.
For any element a such that $k > 2a$.
Define the edge labeling $f : E(J_{n,s}) \to Z_k - \{0\}$ as follows:
$f(vv_i) = a$ for $2 \le i \le n$,
$f(v_1 v_{i+2}) = a$ for $1 \le i \le n-3$,

$$f(v_j v_1^j) = \begin{cases} \frac{(n-1)a}{2}, \text{ for } j \text{ is odd,} \\ k - \frac{(n+1)a}{2}, \text{ for } j \text{ is even,} \end{cases}$$

For j is odd.

$$f(v_i^j v_{i+1}^j) = \begin{cases} k - \frac{(n-1)a}{2}, \text{ for } i \text{ is odd,} \\ \frac{(n-1)a}{2}, \text{ for } i \text{ is even,} \end{cases}$$

For j is even.

$$f(v_i^j v_{i+1}^j) = \begin{cases} k - \frac{(n+1)a}{2}, \text{ for } i \text{ is odd,} \\ \frac{(n+1)a}{2}, \text{ for } i \text{ is even.} \end{cases}$$

Then the induced vertex labeling $f^+ : V(J_{n,s}) \to Z_k$ is $f^+(v) \equiv 0 (mod\ k)$ for all $v \in V(J_{n,s})$.
Case(ii): n is even and s is even .
Define the edge labeling $f : E(J_{n,s}) \to Z_k - \{0\}$ as follows:

$$f(vv_i) = \begin{cases} a, & \text{for } i \text{ is odd,} \\ k-a, & \text{for } i \text{ is even,} \end{cases}$$

$$f(v_j v_1^j) = \begin{cases} 2a, & \text{for } j \text{ is odd,} \\ k-2a, & \text{for } j \text{ is even,} \end{cases}$$

For j is odd.

$$f(v_i^j v_{i+1}^j) = \begin{cases} k - \frac{(n-1)a}{2}, \text{ for } i \text{ is odd,} \\ \frac{(n-1)a}{2}, \text{ for } i \text{ is even,} \end{cases}$$

For j is even.

$$f(v_i^j v_{i+1}^j) = \begin{cases} a, \text{ for } i \text{ is odd,} \\ k-a, \text{ for } i \text{ is even.} \end{cases}$$

Then the induced vertex labeling $f^+ : V(J_{n,s}) \to Z_k$ is $f^+(v) \equiv 0 (mod\ k)$ for all $v \in V(J_{n,s})$.
Case(iii): n is even, s is odd and k is even.
Subcase(i): $k \equiv 0 (mod\ 4)$.
Define the edge labeling $f : E(J_{n,s}) \to Z_k - \{0\}$ as follows:
$f(vv_i) = \frac{k}{2}$ for $1 \le i \le n$,
$f(v_j v_1^j) = \frac{k}{4}$ for $1 \le i \le n$,

$f(v_i^j v_{i+1}^j) = \begin{cases} \frac{3k}{4}, \text{ for } i \text{ is odd,} \\ \frac{k}{4}, \text{ for } i \text{ is even.} \end{cases}$

Then the induced vertex labeling $f^+ : V(J_{n,s}) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(J_{n,s})$.

Subcase(ii): $k \equiv 2(mod\ 4)$.

Define the edge labeling $f : E(J_{n,s}) \to Z_k - \{0\}$ as follows:

$f(vv_i) = \frac{k}{2}$ for $1 \le i \le n$,

$f(v_j v_1^j) = \begin{cases} \frac{(k-2)}{4}, & \text{for } j \text{ is odd,} \\ \frac{(k+2)}{4}, & \text{for } j \text{ is even,} \end{cases}$

For j is odd.

$f(v_i^j v_{i+1}^j) = \begin{cases} \frac{(3k+2)}{4}, \text{ for } i \text{ is odd,} \\ \frac{(k-2)}{4}, \text{ for } i \text{ is even,} \end{cases}$

For j is even.

$f(v_i^j v_{i+1}^j) = \begin{cases} \frac{(3k-2)}{4}, \text{ for } i \text{ is odd,} \\ \frac{(k+2)}{4} \text{ for } i \text{ is even.} \end{cases}$

Then the induced vertex labeling $f^+ : V(J_{n,s}) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(J_{n,s})$. Since the generalised jahangir graph $J_{n,s}$ admits Z_k-magic labeling, then it is a k-magic graph. □

The examples of Z_{10} and Z_{11}-magic labeling of $J_{5,3}$ and $J_{4,6}$ are shown in Figure 2.

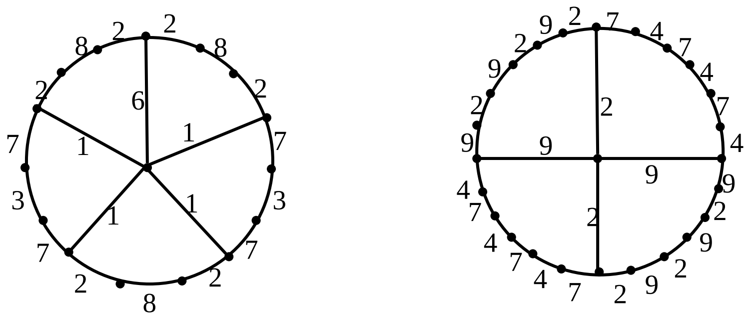

Figure 2: Z_{10} and Z_{11}-magic labeling of $J_{5,3}$ and $J_{4,6}$

Conjecture 2.4. The generalised jahangir graph $J_{n,s}$ is not k-magic when (i) n is odd and s is even (ii) n is even, s is odd and k is odd.

Theorem 2.5. *The graph $(P_n + P_1) \times P_2$ is k-magic when n is odd.*

Proof. Let the vertex set and the edge set of $(P_n + P_1) \times P_2$ be $V((P_n + P_1) \times P_2) = \{u, u_i, v, v_i : 1 \le i \le n\}$ and $E((P_n + P_1) \times P_2) = \{uu_i, vv_i, u_iv_i : 1 \le i \le n\} \cup \{u_iu_{i+1}, v_iv_{i+1} : 1 \le i \le n-1\} \cup \{uv\}$.

We consider the following two cases.

Case(i): $n = 3$.

For any element a such that $k > 2a$.

Define the edge labeling $f : E((P_n + P_1) \times P_2) \to Z_k - \{0\}$ as follows:

$f(uu_1) = f(uu_3) = k - a$,

$f(u_1u_2) = f(u_2u_3) = k - a$,

$f(uu_2) = f(u_2v_2) = a$,

$f(u_1v_1) = f(u_3v_3) = 2a$.

Then the induced vertex labeling $f^+ : V((P_n + P_1) \times P_2) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V((P_n + P_1) \times P_2)$.

Case(ii): $n > 3$.

For any element a such that $k > \frac{(n-1)a}{2}$.

Define the edge labeling $f : E((P_n + P_1) \times P_2) \to Z_k - \{0\}$ as follows:

$f(uu_1) = k - \frac{(n-3)a}{2}$,

$f(uu_n) = k - \frac{(n-3)a}{2}$,

$f(uu_i) = a$ for $2 \le i \le n-1$,

$f(vv_1) = k - \frac{(n-3)a}{2}$,
$f(vv_n) = k - \frac{(n-3)a}{2}$,
$f(vv_i) = a$ for $2 \leq i \leq n-1$,
$f(u_1v_1) = f(u_nv_n) = \frac{(n-1)a}{2}$,
$f(u_iv_i) = a$ for $2 \leq i \leq n-1$,
$f(u_iu_{i+1}) = k - a$ for $1 \leq i \leq n-1$,
$f(v_iv_{i+1}) = k - a$ for $1 \leq i \leq n-1$,
$f(uv) = k - a.$
Then the induced vertex labeling $f^+ : V((P_n + P_1) \times P_2) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V((P_n + P_1) \times P_2)$. Since $(P_n + P_1) \times P_2$ admits Z_k-magic labeling, then it a k-magic graph. □

The examples of Z_6 and Z_7-magic labeling of $(P_7 + P_1) \times P_2$ and $(P_3 + P_1) \times P_2$ are shown in Figure 3.

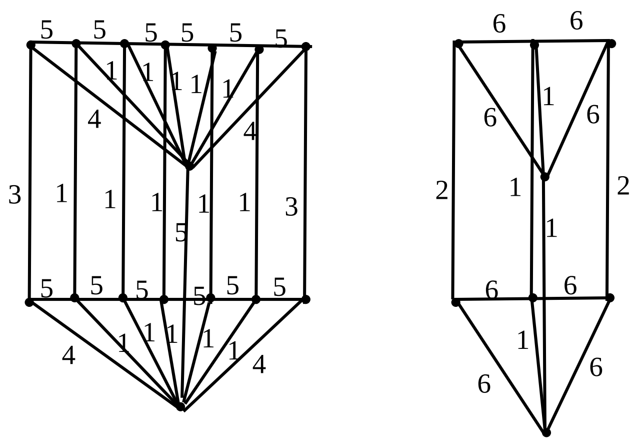

Figure 3: Z_6 and Z_7-magic labeling of $(P_7 + P_1) \times P_2$ and $(P_3 + P_1) \times P_2$

Conjecture 2.6. The graph $(P_n + P_1) \times P_2$ is not k-magic when n is even.

Theorem 2.7. *The double wheel graph DW_n is k-magic.*

Proof. Let the vertex set and the edge set of DW_n be
$V(DW_n) = \{v, v_i, u_i : 1 \leq i \leq n\}$ and $E(DW_n) = \{vv_i, vu_i : 1 \leq i \leq n\} \cup \{v_iv_{i+1}, u_iu_{i+1} : 1 \leq i \leq n-1,\} \cup \{v_nv_1, u_nu_1\}$
For any element a such that $k > 2a$.
Define the edge labeling $f : E(DW_n) \to Z_k - \{0\}$ as follows:
$f(vv_i) = 2a$ for $1 \leq i \leq n,$
$f(vu_i) = k - 2a$ for $1 \leq i \leq n,$
$f(v_iv_{i+1}) = k - a$ for $1 \leq i \leq n-1,$
$f(v_nv_1) = k - a,$
$f(u_iu_{i+1}) = a$ for $1 \leq i \leq n-1,$
$f(u_nu_1) = a.$
Then the induced vertex labeling $f^+ : V(DW_n) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(DW_n)$. Since the double wheel graph DW_n admits Z_k-magic labeling, then it is a k-magic graph. □

An example of Z_9-magic labeling of DW_5 is shown in Figure 4.

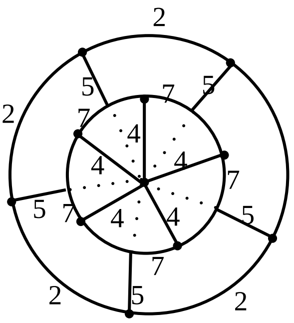

Figure 4: Z_9-magic labeling of DW_5

Theorem 2.8. *The mongolian tent graph $M(m,n)$ is k-magic when m is even.*

Proof. Let the vertex set and the edge set of $M(m,n)$ be
$V(M(m,n)) = \{v, u_{i,j} : 1 \le i \le n, 1 \le j \le m\}$ and $E(M(m,n)) = \{u_{i,j}u_{i,j+1} : 1 \le i \le n, 1 \le j \le m-1\} \cup \{u_{i,j}u_{i+1,j} : 1 \le i \le n-1, 1 \le j \le m\} \cup \{vu_{1,j} : 1 \le j \le m\}$.
For any element a such that $k > \frac{ma}{2}$.
Define the edge labeling $f : E(M(m,n)) \to Z_k - \{0\}$ as follows:
$f(u_{i,j}u_{i,j+1}) = k - a$ for $1 \le i \le n-1, 1 \le j \le m-1,$
$f(u_{i,j}u_{i+1,j}) = a$ for $1 \le i \le n, 2 \le j \le m-1,$
$f(u_{n,j}u_{n,j+1}) = \begin{cases} k - \frac{ma}{2}, & \text{for } j \text{ is odd,} \\ \frac{(m-2)a}{2}, & \text{for } j \text{ is even,} \end{cases}$
$f(u_{i,1}u_{i+1,1}) = \begin{cases} \frac{ma}{2}, & \text{for } i \text{ is odd,} \\ k - \frac{(m-2)a}{2}, & \text{for } i \text{ is even,} \end{cases}$
$f(vu_{1,1}) = f(vu_{1,m}) = k - \frac{(m-2)a}{2},$
$f(vu_{1,j}) = a$ for $2 \le j \le m-1$.
Then the induced vertex labeling $f^+ : V(M(m,n)) \to Z_k$ is $f^+(v) \equiv 0 (mod\ k)$ for all $v \in V(M(m,n))$. Since the mongolian tent graph $M(m,n)$ admits Z_k-magic labeling, then it is a k-magic graph. □

An example of Z_{11}-magic labeling of $M(6,3)$ is shown in Figure 5.

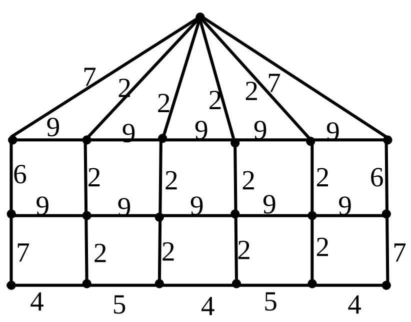

Figure 5: Z_{11}-magic labeling of $M(6,3)$

Theorem 2.9. *The flower snark graph J_n is k-magic.*

Proof. Let the vertex set and the edge set of J_n be
$V(J_n) = \{w_i, x_i, y_i, z_i : 1 \le i \le n\}$ and $E(J_n) = \{z_iz_{i+1}, x_ix_{i+1}, y_iy_{i+1} : 1 \le i \le n-1\} \cup \{z_iw_i, w_ix_i, w_iy_i : 1 \le i \le n\} \cup \{z_nz_1, x_nx_1, y_ny_1, x_1y_n\}$.
For any element a such that $k > 3a$.
Define the edge labeling $f : E(J_n) \to Z_k - \{0\}$ as follows:
$f(z_iz_{i+1}) = k - a$ for $1 \le i \le n-1,$
$f(z_nz_1) = k - a,$
$f(z_iw_i) = 2a$ for $1 \le i \le n,$
$f(w_ix_i) = f(w_iy_i) = k - a$ for $1 \le i \le n,$

$f(x_ix_{i+1}) = \begin{cases} 2a, \text{ for } i \text{ is odd,} \\ k-a, \text{ for } i \text{ is even,} \end{cases}$
$f(y_iy_{i+1}) = \begin{cases} k-a, \text{ for } i \text{ is odd}, i \neq n, \\ 2a, \text{ for } i \text{ is even,} \end{cases}$
$f(x_nx_1) = 2a,$
$f(y_ny_1) = 2a,$
$f(x_1y_n) = k-3a.$
Then the induced vertex labeling $f^+ : V(J_n)) \to Z_k$ is $f^+(v) \equiv 0(mod\, k)$ for all $v \in V(J_n)$.
Since the flower snark graph J_n admits Z_k-magic labeling, then it is a k-magic graph. □

An example of Z_{13}-magic labeling of J_5 is shown in Figure 6.

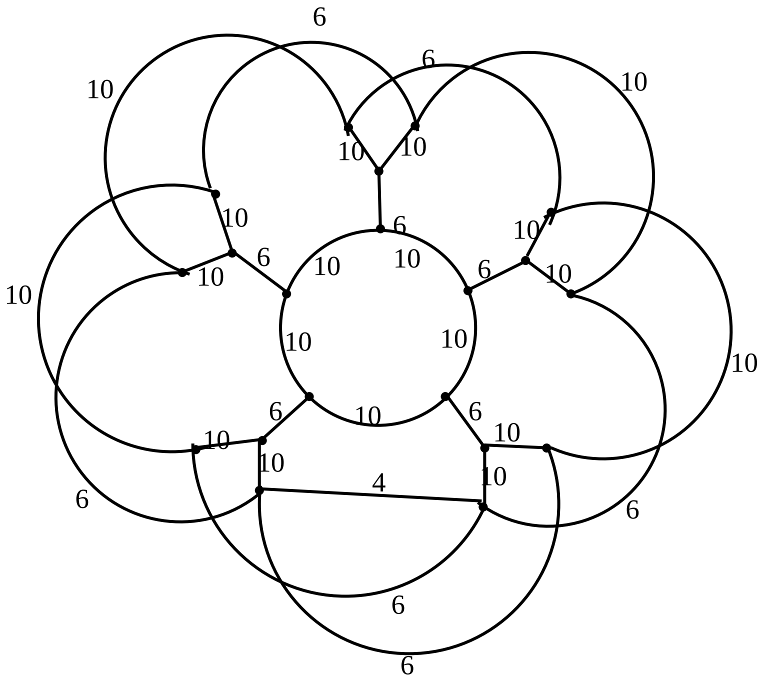

Figure 6: Z_{13}-magic labeling of J_5

Theorem 2.10. *The slanting ladder graph SL_n is k-magic when n is even.*

Proof. Let the vertex set and the edge set of SL_n be
$V(SL_n) = \{v_i, u_i : 1 \leq i \leq n\}$ and $E(SL_n) = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{v_iu_{i+1} : 1 \leq i \leq n-1\}$.
For any element a such that $k > 2a$.
Define the edge labeling $f : E(SL_n) \to Z_k - \{0\}$ as follows:
$f(u_1u_2) = f(v_{n-1}v_n) = a$,
$f(v_iv_{i+1}) = \begin{cases} k-a, \text{ for } i \text{ is odd}, i \neq n-1, \\ a, \text{ for } i \text{ is even,} \end{cases}$
$f(u_iu_{i+1}) = \begin{cases} k-2a, \text{ for } i \text{ is even,} \\ 2a, \text{ for } i \text{ is odd}, i \neq 1, \end{cases}$
$f(v_1u_2) = 2a$, $f(v_{n-1}u_n) = k-a,$
$f(v_iu_{i+1}) = a$ for $2 \leq i \leq n-2$.
Then the induced vertex labeling $f^+ : V(SL_n)) \to Z_k$ is $f^+(v) \equiv a(mod\, k)$ for all $v \in V(SL_n)$.
Since the slanting ladder graph SL_n admits Z_k-magic labeling, then it is a k-magic graph. □

An example of Z_9-magic labeling of SL_6 is shown in Figure 7.

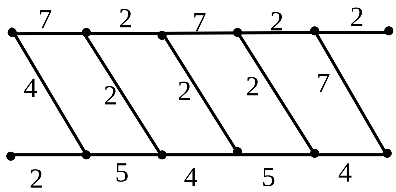

Figure 7: Z_9-magic labeling of SL_6

Theorem 2.11. *The double step grid graph DSt_n is k-magic.*

Proof. Let the vertex set and the edge set of DSt_n be
$V(DSt_n) = \{u_i : 1 \leq i \leq n\} \cup \{v_i^j : 1 \leq i \leq r, 1 \leq j \leq \frac{n}{2}\}$ where $r = n, n-2, n-4 \ldots 4, 2$ and $E(DSt_n) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i u_i^1 : 1 \leq i \leq n\} \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq r-1, 1 \leq j \leq \frac{n}{2} - 1\}$.
For any element a such that $k > 2a$.
Define the edge labeling $f : E(DSt_n) \rightarrow Z_k - \{0\}$ as follows:
$f(u_i u_{i+1}) = a$ for $1 \leq i \leq n-1$,
$f(u_1 u_1^1) = f(u_n u_n^1) = k - a$,
$f(u_i u_i^1) = k - 2a$ for $2 \leq i \leq n-1$,
$f(u_1^1 u_2^1) = f(u_{n-1}^1 u_n^1) = a$,

$$f(u_i^1 u_{i+1}^1) = \begin{cases} k-a, & \text{for } i \text{ is even}, i \neq n, \\ a, & \text{for } i \text{ is odd}, i \neq 1, \end{cases}$$

For $1 \leq i \leq r-1$.

$$f(u_{i+1}^j u_i^{j+1}) = \begin{cases} 2a, & \text{for } i \text{ is odd}, \\ k-2a, & \text{for } i \text{ is even}, \end{cases}$$

For j is even.

$$f(u_i^j u_{i+1}^j) = \begin{cases} k-2a, & \text{for } i \text{ is odd}, \\ 2a, & \text{for } i \text{ is even}, \end{cases}$$

For j is odd, $j \neq 1$,

$$f(u_i^j u_{i+1}^j) = \begin{cases} 2a, & \text{for } i \text{ is odd}, \\ k-2a, & \text{for } i \text{ is even}. \end{cases}$$

Then the induced vertex labeling $f^+ : V(DSt_n) \rightarrow Z_k$ is $f^+(v) \equiv 0 (mod\ k)$ for all $v \in V(DSt_n)$. Since the double step grid graph DSt_n admits Z_k-magic labeling, then it is a k-magic graph. □

An example of Z_5-magic labeling of DSt_{10} is shown in Figure 8.

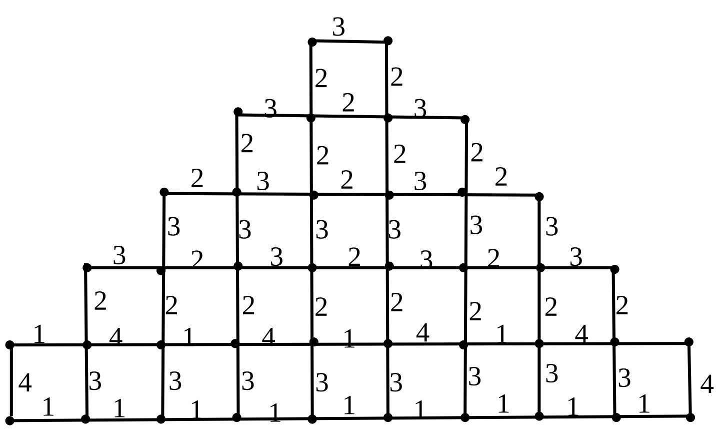

Figure 8: Z_5-magic labeling of DSt_{10}

Theorem 2.12. *The double arrow graph DA_n^t is k-magic when t is even.*

Proof. Let the vertex set and the edge set of DA_n^t be
$V(DA_n^t) = \{x, y, v_{i,j} : 1 \le i \le t,\ 1 \le j \le n\}$ and $E(DA_n^t) = \{xv_{i,1} : 1 \le i \le t\} \cup \{yv_{i,n} : 1 \le i \le t\} \cup \{v_{i,j}v_{i+1,j} : 1 \le i \le t-1, 1 \le j \le n\} \cup \{v_{i,j}v_{i,j+1} : 1 \le i \le t, 1 \le j \le n-1\}$.
We consider the following two cases.
Case(i): n is even.
For any element a such that $k > \frac{ta}{2}$.
Define the edge labeling $f : E(DA_n^t) \to Z_k - \{0\}$ as follows:
$f(xv_{1,1}) = f(yv_{t,1}) = f(yv_{1,n}) = f(yv_{t,n}) = k - \frac{(t-2)a}{2}$,
$f(xv_{i,1}) = f(yv_{i,n}) = a$ for $2 \le i \le t-1$,

$$f(v_{i,j}v_{i+1,j}) = \begin{cases} 2a, \text{ for } i \text{ is odd}, \ 2 \le j \le n-1, \\ k-2a, \text{ for } i \text{ is even}, \ 2 \le j \le n-1, \end{cases}$$

$$f(v_{i,j}v_{i,j+1}) = \begin{cases} k-a, \text{ for } i \text{ is even}, i \ne n, \\ a, \text{ for } i \text{ is odd}, i \ne 1, \end{cases}$$

$$f(v_{i,1}v_{i+1,1}) = f(v_{i,n}v_{i+1,n}) = \begin{cases} \frac{ta}{2}, \text{ for } i \text{ is odd}, \\ k - \frac{ta}{2}, \text{ for } i \text{ is even}, \end{cases}$$

$f(v_{1,j}v_{1,j+1}) = f(v_{t,j}v_{t,j+1}) = k - a$ for $1 \le j \le n-1$,
Then the induced vertex labeling $f^+ : V(DA_n^t) \to Z_k$ is $f^+(v) \equiv 0 (mod\, k)$ for all $v \in V(DA_n^t)$.
Case(ii): n is odd, $t > 4$.
For any element a such that $k > \frac{ta}{2}$.
Define the edge labeling $f : E(DA_n^t) \to Z_k - \{0\}$ as follows:
$f(xv_{1,1}) = f(yv_{t,1}) == k - \frac{(t-2)a}{2}$,
$f(yv_{1,n}) = f(yv_{t,n}) = \frac{(t-2)a}{2}$,
$f(xv_{i,1}) = a$ for $2 \le i \le t-1$,
$f(yv_{i,n}) = k - a$ for $2 \le i \le t-1$,

$$f(v_{i,j}v_{i+1,j}) = \begin{cases} 2a, \text{ for } i \text{ is odd}, \ 2 \le j \le n-1, \\ k-2a, \text{ for } i \text{ is even}, \ 2 \le j \le n-1, \end{cases}$$

$$f(v_{i,j}v_{i,j+1}) = \begin{cases} k-a, \text{ for } i \text{ is even}, \\ a, \text{ for } i \text{ is odd}, i \ne 1, n, \end{cases}$$

$$f(v_{i,1}v_{i+1,1}) = \begin{cases} \frac{ta}{2}, \text{ for } i \text{ is odd}, \\ k - \frac{ta}{2}, \text{ for } i \text{ is even}, \end{cases}$$

$$f(v_{i,n}v_{i+1,n}) = \begin{cases} k - \frac{(t-4)a}{2}, \text{ for } i \text{ is odd}, \\ \frac{(t-4)a}{2}, \text{ for } i \text{ is even}, \end{cases}$$

$f(v_{1,j}v_{1,j+1}) = f(v_{t,j}v_{t,j+1}) = k - a$ for $1 \le j \le n-1$.
Then the induced vertex labeling $f^+ : V(DA_n^t) \to Z_k$ is $f^+(v) \equiv 0 (mod\, k)$ for all $v \in V(DA_n^t)$.
Since the double arrow graph DA_n^t admits Z_k-magic labeling, then it is a k-magic graph. □

An example of Z_{10}-magic labeling of DA_8^6 is shown in Figure 9.

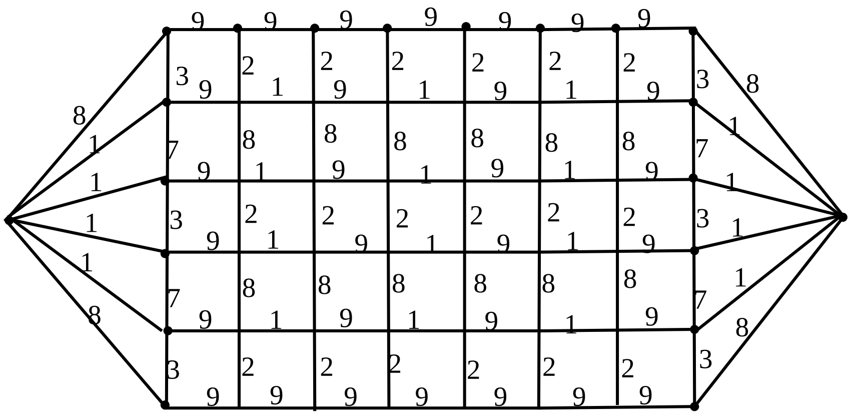

Figure 9: Z_{10}-magic labeling of DA_8^6

Theorem 2.13. *The semi jahangir graph SJ_n is k-magic.*

Proof. Let the vertex set and the edge set of SJ_n be
$V(SJ_n) = \{p, x_i, y_j : 1 \leq i \leq n+1, 1 \leq j \leq n\}$ and $E(SJ_n) = \{px_i : 1 \leq i \leq n+1\} \cup \{x_i y_i : 1 \leq i \leq n, \} \cup \{y_i x_{i+1} : 1 \leq i \leq n\}$
For any element a such that $k > 2a$.
Define the edge labeling $f : E(SJ_n) \to Z_k - \{0\}$ as follows:
$f(px_1) = k - a$,

$$f(px_{n+1}) = \begin{cases} a, & \text{for } n \text{ is odd}, \\ k-a, & \text{for } n \text{ is even}, \end{cases}$$

$$f(px_i) = \begin{cases} 2a, & \text{for } i \text{ is even}, i \neq n+1, \\ k-2a, & \text{for } i \text{ is odd}, i \neq 1, \end{cases}$$

$$f(x_i y_i) = \begin{cases} a, & \text{for } i \text{ is odd}, \\ k-a, & \text{for } i \text{ is even}, \end{cases}$$

$$f(y_i x_{i+1}) = \begin{cases} k-a, & \text{for } i \text{ is odd}, \\ a, & \text{for } i \text{ is even}. \end{cases}$$

Then the induced vertex labeling $f^+ : V(SJ_n) \to Z_k$ is $f^+(v) \equiv 0 (mod\ k)$ for all $v \in V(SJ_n)$.
Since the semi jahangir graph SJ_n admits Z_k-magic labeling, then it is a k-magic graph. □

An example of Z_7-magic labeling of SJ_5 is shown in Figure 10.

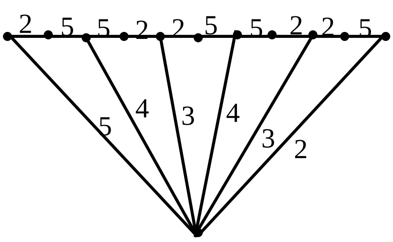

Figure 10: Z_7-magic labeling of SJ_5

Theorem 2.14. *The double wheel graphs DW_{n_1}, DW_{n_2} are connected by a path P_m is k-magic when n_1, n_2 are odd.*

Proof. Let G be the graph obtained by two double wheel graphs connected by a path P_m. Let the vertex set and the edge set of G be $V(G) = \{x_1, x_2\} \cup \{u_i^1, v_i^1 : 1 \leq i \leq n_1\} \cup \{u_i^2, v_i^2 : 1 \leq i \leq n_2\} \cup \{w_i : 1 \leq i \leq m\}$ where $w_1 = u_1^1$ and $w_n = u_1^2$ and
$E(G) = \{x_1 v_i^1 : 1 \leq i \leq n_1\} \cup \{x_2 v_i^2 : 1 \leq i \leq n_2\} \cup \{x_1 u_i^1 : 1 \leq i \leq n_1\} \cup \{x_2 u_i^2 : 1 \leq i \leq n_2\} \cup \{u_i^1 u_{i+1}^1 : 1 \leq i \leq n_1 - 1\} \cup \{u_{n_1}^1 u_1^1\} \cup \{v_i^1 v_{i+1}^1 : 1 \leq i \leq n_1 - 1\} \cup \{v_{n_1}^1 v_1^1\} \cup \{u_i^2 u_{i+1}^2 : 1 \leq i \leq n_2 - 1\} \cup \{u_{n_2}^2 u_1^2\} \cup \{v_i^2 v_{i+1}^2 : 1 \leq i \leq n_2 - 1\} \cup \{v_{n_2}^2 v_1^2\} \cup \{w_i w_{i+1} : 1 \leq i \leq m-1\}$

For any element a such that $k > 4a$.
Define the edge labeling $f : E(G) \to Z_k - \{0\}$ as follows:
$f(x_1v_i^1) = 2a$ for $1 \le i \le n_1$,
$f(x_1u_i^1) = k - 2a$ for $1 \le i \le n_1$,
$f(v_i^1v_{i+1}^1) = k - a$ for $1 \le i \le n_1 - 1$,
$f(v_{n_1}^1v_1^1) = k - a$,

$$f(u_i^1u_{i+1}^1) = \begin{cases} 3a, & \text{for } i \text{ is odd,} \\ k-a, & \text{for } i \text{ is even,} \end{cases}$$

$$f(w_iw_{i+1}) = \begin{cases} k-4a, & \text{for } i \text{ is odd,} \\ 4a, & \text{for } i \text{ is even,} \end{cases}$$

$$f(x_2v_i^2) = \begin{cases} k-2a, & \text{for } m \text{ is odd, } 1 \le i \le n_2, \\ 2a, & \text{for } m \text{ is even, } 1 \le i \le n_2 \end{cases}$$

$$f(v_i^2v_{i+1}^2) = \begin{cases} a, & \text{for } m \text{ is odd, } 1 \le i \le n_2, \\ k-a, & \text{for } m \text{ is even, } 1 \le i \le n_2, \end{cases}$$

$$f(v_{n_2}^2v_1^2) = \begin{cases} a, & \text{for } m \text{ is odd, } 1 \le i \le n_2 - 1 \\ k-a, & \text{for } m \text{ is even, } 1 \le i \le n_2 - 1 \end{cases}$$

For m is odd.

$$f(u_i^2u_{i+1}^2) = \begin{cases} k-3a, & \text{for } i \text{ is odd,} \\ a, & \text{for } i \text{ is even,} \end{cases}$$

For m is even.

$$f(u_i^2u_{i+1}^2) = \begin{cases} 3a, & \text{for } i \text{ is odd,} \\ k-a, & \text{for } i \text{ is even.} \end{cases}$$

Then the induced vertex labeling $f^+ : V(G) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(G)$. Since G admits Z_k-magic labeling, then it is a k-magic graph. □

An example of Z_5-magic labeling of G is shown in Figure 11.

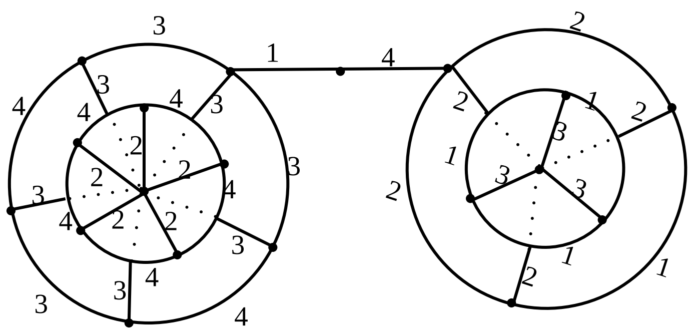

Figure 11: Z_5-magic labeling of G

Theorem 2.15. *The graph G is obtained by joining two copies of shell graphs by a path P_m is k-magic.*

Proof. Let G be the graph obtained by two shell graphs connected by a path P_m. Let the vertex set and the edge set of G be $V(G) = \{v_i^j : 1 \le i \le n, j = 1,2\} \cup \{w_i : 1 \le i \le m\}$ where $w_1 = v_1^1$ and $W_n = v_1^2$ and
$E(G) = \{v_1^jv_{i+2}^j : 1 \le i \le n-3, j = 1,2\} \cup \{v_i^jv_{i+1}^j : 1 \le i \le n-1, j = 1,2\} \cup \{v_n^jv_1^j : j = 1,2\} \cup \{w_iw_{i+1} : 1 \le i \le m-1.\}$
Case(i): m is even.
For any element a such that $k > 2a(n-2)$.
Define the edge labeling $f : E(G) \to Z_k - \{0\}$ as follows:
$f(v_1^jv_{i+2}^j) = 2a$ for $1 \le i \le n-3, j = 1,2$,
$f(v_1^jv_2^j) = f(v_n^jv_1^j) = a$ for $j = 1,2$,
$f(v_i^jv_{i+1}^j) = k - a$ for $2 \le i \le n-1, j = 1,2$,

$f(w_i w_{i+1}) = \begin{cases} k - 2a(n-2), & \text{for } i \text{ is odd,} \\ 2a(n-2), & \text{for } i \text{ is even.} \end{cases}$

Then the induced vertex labeling $f^+ : V(G) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(G)$.

Case(ii): m is odd.

For any element a such that $k > 2a(n-2)$.

Define the edge labeling $f : E(G) \to Z_k - \{0\}$ as follows:

$f(v_1^1 v_{i+2}^1) = 2a$ for $1 \le i \le n-3$,

$f(v_1^1 v_2^1) = f(v_n^1 v_1^1) = a$,

$f(v_i^1 v_{i+1}^1) = k - a$ for $2 \le i \le n-1$,

$f(v_1^2 v_{i+2}^2) = k - 2a$ for $1 \le i \le n-3$,

$f(v_1^2 v_2^2) = f(v_n^2 v_1^2) = k - a$,

$f(v_i^2 v_{i+1}^2) = a$ for $2 \le i \le n-1$,

$f(w_i w_{i+1}) = \begin{cases} k - 2a(n-2), & \text{for } i \text{ is odd,} \\ 2a(n-2), & \text{for } i \text{ is even.} \end{cases}$

Then the induced vertex labeling $f^+ : V(G) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(G)$. Since G admits Z_k-magic labeling, then it is a k-magic graph. □

An example of Z_{15}-magic labeling of G is shown in Figure 12.

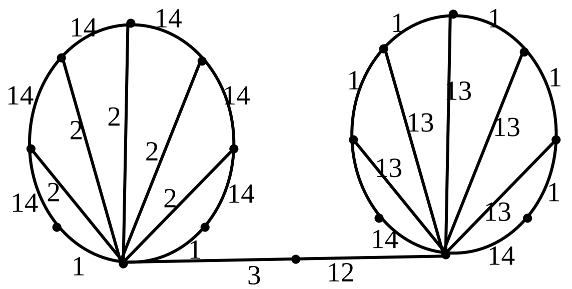

Figure 12: Z_{15}-magic labeling of G

Theorem 2.16. *If the graph G is k magic with magic constant 0 then the splitting graph of G is k-magic.*

Proof. Let $v_1, v_2, \ldots v_n$ be the vertices of graph G. Let $v_1, v_2, \ldots v_n, v_1', v_2', \ldots v_n'$ be the vertices of graph $S'(G)$. Given a graph G is Z_k magic with magic constant 0.

Therefore $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(G)$.

Define the edge labeling $g : E(S'(G)) \to Z_k - \{0\}$ as follows:

$g(e_i) = f(e_i)$ for all $e \in E(G)$,

$g(e_i') = f(N(e)$ for all $e \in E(G)$.

Then the induced vertex labeling $g^+ : V(S'(G)) \to Z_k$ is $g^+(v) \equiv 0(mod\ k)$ for all $v \in V(S'(G))$. Since $S'(G)$ admits Z_k-magic labeling, the graph $S'(G)$ is a k-magic graph. □

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