Palestine Journal of Mathematics

Vol. 8(2)(2019) , 40–44

ON DIOPHANTINE EQUATIONS OF NATHANSON

Ho-Hon Leung

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 11D41; Secondary 11D25.

Keywords and phrases: Diophantine equation.

Abstract. For positive integers n and k, we investigate whether the diophantine equation $x^n - y^n = z^{n+k}$ has positive integral solutions.

1 Introduction

Recently, Nathanson [3] constructed infinitely many positive integral solutions for the diophantine equation $x^n - y^n = z^{n+1}$. He proposed to study the diophantine equation

$$x^n - y^n = z^{n+k} \tag{1.1}$$

for any positive integers $n \geq 2$ and $k \geq 2$ (see [3, Section 3]). We initiate the study of (1.1) in this article.

2 Main Results

Let n_1, n_2 and n_3 be positive integers. The diophantine equation $x^{n_1} - y^{n_2} = z^{n_3}$ is called an (n_1, n_2, n_3)*-system*. The triple (a, b, c) of positive integers is called *(n,k)-powerful* if $a > b$ and there exists a positive integer t such that

$$\frac{a^n - b^n}{c^{n+k}} = t^k.$$

We define the following function

$$t_{n,k}(a, b, c) := \frac{a^n - b^n}{c^{n+k}}.$$

The triple (a, b, c) is (n, k)-powerful if and only if $t_{n,k}(a, b, c)$ is a k^{th}-power of a positive integer. The triple (a, b, c) is *relatively prime* if g.c.d.$(a, b, c) = 1$. The following result is a generalization of Theorem 1 in Nathanson's paper [3, Theorem 1].

Theorem 2.1. *Let n and k be positive integers. If the triple (a, b, c) is (n, k)-powerful with $t_{n,k}(a, b, c) = t^k$ for some positive integer t, then the triple $(x, y, z) = (at, bt, ct)$ is a solution of an $(n, n, n + k)$-system. Moreover, every positive integral solution of an $(n, n, n + k)$-system is produced by an unique relatively prime (n, k)-powerful triple.*

Proof. Let (a, b, c) be an (n, k)-powerful triple with $t_{n,k}(a, b, c) = t^k$. We have

$$a^n - b^n = t^k c^{n+k}.$$

Let the triple $(x, y, z) = (at, bt, ct)$. Then,

$$x^n - y^n = (at)^n - (bn)^t = t^n(a^n - b^n) = t^n(t^k c^{n+k}) = (ct)^{n+k}.$$

Hence, (at, bt, ct) is a solution to an $(n, n, n + k)$-system.

If g.c.d.$(a, b, c) = d$, then

$$t_{n,k}(a/d, b/d, c/d) = \frac{(a/d)^n - (b/d)^n}{(c/d)^{n+k}} = d^k\Big(\frac{a^n - b^n}{c^{n+k}}\Big)$$
$$= d^k t_{n,k}(a, b, c) = d^k t^k = (dt)^k.$$

Hence, $(a/d, b/d, c/d)$ is (n, k)-powerful and is relatively prime. We construct a solution to an $(n, n, n + k)$-system by using the (n, k)-powerful triple $(a/d, b/d, c/d)$ as follows:

$$(x, y, z) = ((a/d)dt, (b/d)dt, (c/d)dt) = (at, bt, ct).$$

It is the same solution as the one constructed from the (n, k)-powerful triple (a, b, c).

If (x_1, y_1, z_1) is a positive integral solution of an $(n, n, n + k)$-system, that is,

$$(x_1)^n - (y_1)^n = (z_1)^{n+k},$$

then

$$t_{n,k}(x_1, y_1, z_1) = 1 = 1^k$$

and hence (x_1, y_1, z_1) is (n, k)-powerful. Let $d_1 =$ g.c.d.(x_1, y_1, z_1). The triple $(x_1/d_1, y_1/d_1, z_1/d_1)$ is (n, k)-powerful with

$$t_{n,k}(x_1/d_1, y_1/d_1, z_1/d_1) = (d_1)^k$$

and is relatively prime. So, $(x, y, z) = (x_1, y_1, z_1)$ is a solution to an $(n, n, n + k)$-system produced by the (n, k)-powerful triple $(x_1/d_1, y_1/d_1, z_1/d_1)$. Hence, each positive integral solution of an $(n, n, n + k)$-system can be produced by a relatively prime (n, k)-powerful triple.

Next, we show that each positive integral solution of an $(n, n, n + k)$-system is produced by an *unique* relatively prime (n, k)-powerful triple. Let (x, y, z) be a positive integral solution of an $(n, n, n + k)$-system produced by two relatively prime (n, k)-powerful triple (a_1, b_1, c_1) and (a_2, b_2, c_2). Then

$$t_{n,k}(a_1, b_1, c_1) = (t_1)^k,\ t_{n,k}(a_1, b_2, c_2) = (t_2)^k$$

for some positive integers t_1, t_2. Also,

$$(x, y, z) = (a_1t_1, b_1t_1, c_1t_1) = (a_2t_2, b_2t_2, c_2t_2).$$

Let $d' =$ g.c.d.(t_1, t_2). We have

$$a_1(t_1/d') = a_2(t_2/d'),\ b_1(t_1/d') = b_2(t_2/d'),\ c_1(t_1/d') = c_2(t_2/d').$$

Since t_1/d' and t_2/d' are relatively prime, we know that t_1/d' is a common divisor of a_2, b_2, c_2 and t_2/d' is a common divisor of a_1, b_1, c_1. Since g.c.d.$(a_1, b_1, c_1) =$ g.c.d.$(a_2, b_2, c_2) = 1$, we get $t_1/d' = 1$ and $t_2/d' = 1$. Hence, we get that $a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$ as desired. □

Corollary 2.2. *Let m, n be positive integers. There exist infinitely many positive integral solutions to an $(n, n, mn + 1)$-system.*

Proof. Let n be a positive integer. We prove it by induction on m. Let a, b be positive integers such that $a > b$. Let $t = a^n - b^n$. Then $(a, b, 1)$ is $(n, 1)$-powerful with $t_{n,1}(a, b, 1) = t$. By Theorem 2.1, (at, bt, t) is a solution of an $(n, n, n + 1)$-system. It is obvious that there are infinitely many such solutions to an $(n, n, n+1)$-system as there are infinitely many such choices of a and b.

We assume that the statement is true for $m = k$. There exist infinitely many positive integral solutions to an $(n, n, kn + 1)$-system. Let (a', b', c') be a positive integral solution to an $(n, n, kn + 1)$-system.That is,

$$(a')^n - (b')^n = (c')^{kn+1}.$$

Then $(a', b', 1)$ is $(n, kn + 1)$-powerful with $t_{n,kn+1}(a', b', 1) = (c')^{kn+1}$. By Theorem 2.1, $(a'c', b'c', c')$ is a solution of an $(n, n, n + (kn + 1))$-system. It is now clear that there are infinitely many solutions of an $(n, n, (k + 1)n + 1)$-system as there are infinitely many choices of (a', b', c') based on inductive hypothesis. □

Example 2.3. Let $a = 3$, $b = 2$. Then $3^3 - 2^3 = 19$ and the triple $(3, 2, 19)$ is a solution of an $(3, 3, 1)$-system. The triple $(3, 2, 1)$ is $(3, 1)$-powerful with $t_{3,1}(3, 2, 1) = 19$. By Theorem 2.1, $(3 \cdot 19, 2 \cdot 19, 19) = (57, 38, 19)$ is a solution of an $(3, 3, 4)$-system. That is, $57^3 - 38^3 = 19^4$. The triple $(57, 38, 1)$ is $(3, 4)$-powerful with $t_{3,4}(57, 38, 1) = 19^4$ and hence $(57 \cdot 19, 38 \cdot 19, 19) = (1083, 722, 19)$ is a solution of an $(3, 3, 7)$-system by Theorem 2.1. That is, $1083^3 - 722^3 = 19^7$. We can proceed inductively and obtain a solution for an $(3, 3, 3k + 1)$-system for every positive integer k. The solutions of $(3, 3, 3k+1)$-systems for $k = 0, \dots, 9$ induced by the $(3, 1)$-powerful triple $(3, 2, 1)$ are listed as follows:

$$\begin{aligned}
3^3 - 2^3 &= 19^1 \\
57^3 - 38^3 &= 19^4 \\
1083^3 - 722^3 &= 19^7 \\
20577^3 - 13718^3 &= 19^{10} \\
390963^3 - 260642^3 &= 19^{13} \\
7428297^3 - 4952198^3 &= 19^{16} \\
141137643^3 - 94091762^3 &= 19^{19} \\
2681515217^3 - 1787743478^3 &= 19^{22} \\
50950689123^3 - 33967126082^3 &= 19^{25} \\
968063093337^3 - 645375395558^3 &= 19^{28}.
\end{aligned}$$

Corollary 2.4. *There are infinitely many positive integral solutions of an $(2, 2, m)$-system for every positive integer m.*

Proof. For odd m, it is clear due to Corollary 2.2. We prove that there exist infinitely many positive integral solutions of an $(2, 2, m)$-system for even m by induction. If $m = 2$, then a positive integral solution (a, b, c) of an $(2, 2, 2)$-system is a *Pythagoras triple* (b, c, a) such that $b^2 + c^2 = a^2$ and vice versa. It is well known that there are infinitely many Pythagoras triples.

We assume that there are infinitely many solutions of an $(2, 2, m)$-system for an even m. Let (a', b', c') be such solution. Then $(a', b', 1)$ is $(2, m)$-powerful with $t_{2,m}(a', b', 1) = (c')^m$. By Theorem 2.1, $(a'c', b'c', c')$ is a solution of an $(2, 2, 2 + m)$-system. Infinitely many such solutions can be constructed for an $(2, 2, m + 2)$-system as there are infinitely many choices of (a', b', c') by inductive hypothesis. □

Corollary 2.5. *Let n, m, k be positive integers. If there is no positive integral solution of an $(n, n, mn + k)$-system, then there is no positive integral solution of an $(n, n, m'n + k)$-system for $0 \leq m' \leq m$.*

Proof. We prove it by backward induction on m. The base step is clear. We assume that there is no positive integral solution of an $(n, n, k'n + k)$-system for some k' such that $1 \leq k' \leq m$. Let (a, b, c) be a positive integral solution of an $(n, n, (k' - 1)n + k)$-system. The triple $(a, b, 1)$ is $(n, (k' - 1)n + k)$-powerful with $t_{n,(k'-1)n+k}(a, b, 1) = (c)^{(k'-1)n+k}$. By Theorem 2.1, the triple (ac, bc, c) is a solution of an $(n, n, k'n + k)$-system, which is a contradiction. □

For certain positive integers n_1, n_2, n_3, there is no positive integral solution of an (n_1, n_2, n_3)-system. We state two lemmas here.

Lemma 2.6. *Let n_1, n_2, n_3 be positive integers such that g.c.d.$(n_1, n_2, n_3) = d \geq 3$. There is no positive integral solution of an (n_1, n_2, n_3)-system.*

Proof. Let the triple (a, b, c) be a positive integral solution of an (n_1, n_2, n_3)-system. We have

$$a^{n_1} - b^{n_2} = c^{n_3},$$
$$(a^{n_1/d})^d - (b^{n_2/d})^d = (c^{n_3/d})^d.$$

So, the triple $(a^{n_1/d}, b^{n_2/d}, c^{n_3/d})$ is a positive integral solution of an (d, d, d)-system. But such positive integral solutions do not exist by the well known Fermat's Last Theorem [4]. □

Lemma 2.7. *Let k_1, k_2 be positive integers. There is no positive integral solution of an $(4k_1, 4k_1, 2k_2)$-system.*

Proof. Let the triple (a, b, c) be a positive integral solution of an $(4, 4, 2)$-system. We have $a^4 = c^2 + b^4$. But there is no solution for the equation $z^4 = x^2 + y^4$ by the classical method of infinite descent introduced by Euler and Fermat.

Let the triple (a', b', c') be a positive integral solution of an $(4k_1, 4k_1, 2k_2)$-system, then

$$(a')^{4k_1} - (b')^{4k_1} = (c')^{2k_2},$$
$$((a')^{k_1})^4 - ((b')^{k_1})^4 = ((c')^{k_2})^2$$

and hence $((a')^{k_1}, (b')^{k_1}, (c')^{k_2})$ is a solution of an $(4, 4, 2)$-system, which is a contradiction. □

Corollary 2.8. *Let $k \geq 1$. There exist infinitely many positive integral solutions of an $(3, 3, 3k + 1)$-system. There is no positive integral solution of an $(3, 3, 3k)$-system. There exist at least two solutions of an $(3, 3, 3k + 2)$-system.*

Proof. The first and the second statements are due to Corollary 2.2 and Lemma 2.6 respectively. Only two solutions are known for an $(3, 3, 2)$-system (see Remark 5.3 in Karama's paper [2, Remark 5.3]). Namely,

$$10^3 - 6^3 = 28^2,$$
$$295296^3 - 294528^3 = 14155780^2.$$

Hence, the triples $(10, 6, 1)$ and $(295296, 294528, 1)$ are $(3, 2)$-powerful with $t_{3,2}(10, 6, 1) = 28^2$ and $t_{3,2}(295296, 294528, 1) = 14155780^2$. We can construct two solutions for an $(3, 3, 3k + 2)$-system inductively based on Theorem 2.1. □

There are many open questions on this topic.

Conjecture 2.9. There is no positive integral solution to $x^4 - y^4 = z^7$.

Remark 2.10. If there exists a positive integral solution to $x^4 - y^4 = z^3$, then the triple $(x, y, 1)$ is $(4, 3)$-powerful with $t_{4,3}(x, y, 1) = z^3$ and hence Conjecture 2.9 is false by Theorem 2.1. The existence of positive integral solutions to $x^4 - y^4 = z^3$ is equivalent to the existence of positive integral solutions to $a^2 - b^2 = c^3$ such that both a and b are squares. But the author is not aware of any solution of this form to the latter equation (see the work done by Andrica and Tudor [1] and Karama [2] for the constructions of solutions to the diophantine equation $a^2 - b^2 = c^3$.)

Conjecture 2.11. There is no positive integral solution to $x^6 - y^6 = z^2$.

Remark 2.12. The existence of positive integral solutions to $x^6 - y^6 = z^2$ is equivalent to the existence of positive integral solutions to $a^3 - b^3 = c^2$ such that both a and b are squares. But the author is not aware of any solution of this form to the latter equation.

3 Acknowledgements

The author is grateful to the editor-in-chief and to the referee(s) for carefully reading the paper. The author is supported by Startup Grant 2016 (G00002235) from United Arab Emirates University.

References

[1] D. Andrica and G.M. Tudor, Parametric solutions for some Diophantine equations, *Gen. Math.* **12**, 23–34 (2004).

[2] M.J. Karama, Using summation notation to solve some diophantine equations, *Palest. J. Math.* **5** , Special Issue, 155–158 (2016).

[3] M. Nathanson, On a diophantine equation of M. J. Karama, *Palest. J. Math.* **6**, 524–527 (2017).

[4] A. Wiles, Modular elliptic curves and Fermat's last theorem, *Ann. of Math. (2)*, **141**, 443–551 (1995).

Author information

Ho-Hon Leung, Department of Mathematical Sciences, United Arab Emirates University, Al Ain, 15551, U.A.E..
E-mail: hohon.leung@uaeu.ac.ae

Received: March 10, 2018.
Accepted: July 28, 2018.