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CONVOLUTION IDENTITIES FOR TRIBONACCI-TYPE NUMBERS WITH ARBITRARY INITIAL VALUES

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Abstract. Tribonacci numbers have been widely studied in relation with Fibonacci numbers and their generalizations. Tribonacci-type numbers $T_n^{(T_0,T_1,T_2)}$ are defined by the recurrence relation $T_n^{(T_0,T_1,T_2)} = T_{n-1}^{(T_0,T_1,T_2)} + T_{n-2}^{(T_0,T_1,T_2)} + T_{n-3}^{(T_0,T_1,T_2)}$ $(n \ge 3)$ with given initial values $T_0^{(T_0,T_1,T_2)} = T_0$, $T_1^{(T_0,T_1,T_2)} = T_1$ and $T_2^{(T_0,T_1,T_2)} = T_2$. When $T_0 = 0$ and $T_1 = T_2 = 1$, $T_n = T_n^{(0,1,1)}$ are ordinary Tribonacci numbers, which sequence is given by $\{T_n\}_{n\ge 0} = 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, \dots$.

In this paper, we give some convolution identities for Tribonacci-type numbers with binomial (multinomial) coefficients.

1 Introduction

Convolution identities for various kinds of numbers (or polynomials) have been studied, with or without binomial (or multinomial) coefficients, including Bernoulli, Euler, Genocchi, Cauchy, Stirling, and Fibonacci numbers ([1, 2, 3, 6, 7, 10, 13]). One typical formula is due to Euler, given by

$$\sum_{k=0}^n \binom{n}{k}\mathcal{B}_k\mathcal{B}_{n-k} = -n\mathcal{B}_{n-1} - (n-1)\mathcal{B}_n \quad (n \ge 0)\,,$$

where $\mathcal{B}_n$ are Bernoulli numbers, defined by

$$\frac{x}{e^x-1} = \sum_{n=0}^\infty \mathcal{B}_n\frac{x^n}{n!} \quad (|x| < 2\pi)\,.$$

In [11], Panda et al. several kinds of the sums of product of two balancing numbers are given. As an application, the sums of the products of two Fibonacci (and Lucas) numbers

$$\sum_{m=0}^n F_{km+r}F_{k(n-m)+r} \quad \text{and} \quad \sum_{m=0}^n L_{km+r}L_{k(n-m)+r}\,,$$

where k and r are fixed integers with $k > r \ge 0$, are given. In [12], Ray et al. consider the higher-order convolution identities for balancing numbers. In addition, let u_n and v_n satisfy the three-term recurrence relations $u_n = au_{n-1} + bu_{n-2}$ $(n \ge 2)$ with $u_0 = 0$ and $u_1 = 1$ and $v_n = av_{n-1} + bv_{n-2}$ $(n \ge 2)$ with $v_0 = 2$ and $v_1 = a$, respectively. Then, explicit formulae for general Fibonacci numbers u_n and Lucas numbers v_n, namely,

$$\sum_{\substack{k_1+\cdots+k_r=n\\ k_1,\dots,k_r\ge 0}} \binom{n}{k_1,\dots,k_r} u_{k_1}\cdots u_{k_r} \quad \text{and} \quad \sum_{\substack{k_1+\cdots+k_r=n\\ k_1,\dots,k_r\ge 0}} \binom{n}{k_1,\dots,k_r} v_{k_1}\cdots v_{k_r}$$

are given, where

$$\binom{n}{k_1,\dots,k_r} = \frac{n!}{k_1!\cdots k_r!}$$

is the multinomial coefficient.

In [8, 9], we studied the convolution identities for the original Tribonacci numbers. In the case of Fibonacci or Lucas numbers, the convolution identities can be expressed in terms of Fibonacci or Lucas numbers only. In the case of balancing or Lucas-balancing numbers, the convolution identities can be expressed in terms of balancing or Lucas-balancing numbers only. However, the convolution identities for Tribonacci numbers need other Tribonacci-type numbers with different initial values. In this paper, we give some convolution identities with binomial (or multinomial) coefficients for Tribonacci-type numbers, generalizing the previous results.

2 Main results

For convenience, we shall introduce Tribonacci-type numbers $T_n^{(T_0,T_1,T_2)}$, satisfying the recurrence relation

$$T_n^{(T_0,T_1,T_2)} = T_{n-1}^{(T_0,T_1,T_2)} + T_{n-2}^{(T_0,T_1,T_2)} + T_{n-3}^{(T_0,T_1,T_2)} \quad (n \ge 3)$$

with given initial values $T_0^{(T_0,T_1,T_2)} = T_0$, $T_1^{(T_0,T_1,T_2)} = T_1$ and $T_2^{(T_0,T_1,T_2)} = T_2$. Hence, $T_n = T_n^{(0,1,1)}$ are ordinary Tribonacci numbers, which sequence is given by

$$\{T_n\}_{n\ge 0} = 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, \dots$$

([14, A000073]).

The generating function with binomial coefficients is given by

$$t(x) := c_1 e^{\alpha x} + c_2 e^{\beta x} + c_3 e^{\gamma x} = \sum_{n=0}^\infty T_n^{(s_0,s_1,s_2)} \frac{x^n}{n!}\,, \tag{2.1}$$

where α, β and γ are the roots of $x^3 - x^2 - x - 1 = 0$ and given by

$$\begin{aligned} \alpha &= \frac{\sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}} + 1}{3} = 1.839286755\,,\\ \beta, \gamma &= \frac{2 - (1 \pm \sqrt{-3})\sqrt[3]{19-3\sqrt{33}} - (1 \mp \sqrt{-3})\sqrt[3]{19+3\sqrt{33}}}{6}\\ &= -0.4196433776 \pm 0.6062907292\sqrt{-1}\,, \end{aligned}$$

satisfying

$$\alpha + \beta + \gamma = 1\,,\ \alpha\beta + \beta\gamma + \gamma\alpha = -1\,,\ \alpha\beta\gamma = 1\,. \tag{2.2}$$

Since $c_1 + c_2 + c_3 = T_0$, $c_1\alpha + c_2\beta + c_3\gamma = T_1$ and $c_1\alpha^2 + c_2\beta^2 + c_3\gamma^2 = T_2$, we have

$$\begin{aligned} c_1 &= \frac{T_0\beta\gamma - T_1(\beta+\gamma) + T_2}{(\alpha-\beta)(\alpha-\gamma)}\,,\\ c_2 &= \frac{T_0\gamma\alpha - T_1(\gamma+\alpha) + T_2}{(\beta-\alpha)(\beta-\gamma)}\,,\\ c_3 &= \frac{T_0\alpha\beta - T_1(\alpha+\beta) + T_2}{(\gamma-\alpha)(\gamma-\beta)} \end{aligned}$$

(see e.g., [5]) ,

First, we shall prove the following two lemmas.

Lemma 2.1. *We have*

$$c_1^2 e^{\alpha x} + c_2^2 e^{\beta x} + c_3^2 e^{\gamma x} = \frac{1}{22}\sum_{n=0}^\infty T_n^{(T_0^*,T_1^*,T_2^*)}\frac{x^n}{n!}\,,$$

where

$$T_0^* = 2(4T_0^2 - 6T_1^2 - 2T_2^2 - 3T_0T_1 + 9T_1T_2 + T_2T_0)\,,$$

$$T_1^* = T_0^2 + 4T_1^2 + 5T_2^2 + 2T_0T_1 - 6T_1T_2 - 8T_2T_0\,,$$
$$T_2^* = 2(-2T_0^2 + 3T_1^2 + T_2^2 + 7T_0T_1 + T_1T_2 + 5T_2T_0)\,.$$

Remark 2.2. If $T_0 = 0$ and $T_1 = T_2 = 1$, then $T_0^* = 2$, $T_1^* = 3$ and $T_2^* = 10$.

Proof. For Tribonacci-type numbers s_n, satisfying the recurrence relation $s_n = s_{n-1} + s_{n-2} + s_{n-3}$ $(n \geq 3)$ with given initial values s_0, s_1 and s_2, we have

$$d_1 e^{\alpha x} + d_2 e^{\beta x} + d_3 e^{\gamma x} = \sum_{n=0}^{\infty} T_n^{(s_0,s_1,s_2)} \frac{x^n}{n!}\,. \tag{2.3}$$

Since d_1, d_2 and d_3 satisfy the system of the equations

$$d_1 + d_2 + d_3 = s_0\,,\ d_1\alpha + d_2\beta + d_3\gamma = s_1\,,\ d_1\alpha^2 + d_2\beta^2 + d_3\gamma^2 = s_2\,,$$

we have

$$d_1 = \frac{-s_0\beta\gamma + s_1(\beta+\gamma) - s_2}{(\alpha-\beta)(\gamma-\alpha)}\,,$$
$$d_2 = \frac{-s_0\gamma\alpha + s_1(\gamma+\alpha) - s_2}{(\alpha-\beta)(\beta-\gamma)}\,,$$
$$d_3 = \frac{-s_0\alpha\beta + s_1(\alpha+\beta) - s_2}{(\beta-\gamma)(\gamma-\alpha)}\,.$$

Since $d_1 = 22c_1^2$, we have

$$\frac{-s_0\beta\gamma + s_1(\beta+\gamma) - s_2}{(\alpha-\beta)(\gamma-\alpha)} = 22\left(\frac{T_0\beta\gamma - T_1(\beta+\gamma) + T_2}{(\alpha-\beta)(\alpha-\gamma)}\right)^2\,.$$

By using the relations (2.2), we have

$$\alpha^2 = 2 - (\beta+\gamma) + \beta\gamma\,,$$
$$(\beta+\gamma)^2 = 1 + (\beta+\gamma) + \beta\gamma\,,$$
$$\beta^2\gamma^2 = -(\beta+\gamma) - \beta\gamma\,,$$
$$\beta\gamma(\beta+\gamma) = \beta\gamma - 1\,.$$

Thus,

$$\begin{aligned}
&\big(-s_0\beta\gamma + s_1(\beta+\gamma) - s_2\big)(\alpha-\beta)(\gamma-\alpha)\\
&= s_0\big(-2(\beta+\gamma) + \beta^2\gamma^2 + 1\big) + s_1\big(-2(\beta+\gamma) - 2\beta\gamma(\beta+\gamma) + 2\big)\\
&\quad + s_2\big(-(\beta+\gamma) + 3\beta\gamma + 3\big)\\
&= (-s_0 - 2s_1 + 3s_2)\beta\gamma + (-3s_0 - 2s_1 - s_2)(\beta+\gamma) + (s_0 + 4s_1 + 3s_2)\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\big(T_0\beta\gamma - T_1(\beta+\gamma) + T_2\big)^2\\
&= (2T_0T_1 + 2T_0T_2 - T_0^2 + T_1^2)\beta\gamma + (-2T_1T_2 - T_0^2 + T_1^2)(\beta+\gamma)\\
&\quad + (2T_0T_1 + T_1^2 + T_2^2)\,.
\end{aligned}$$

By solving the system

$$-s_0 - 2s_1 + 3s_2 = 2T_0T_1 + 2T_0T_2 - T_0^2 + T_1^2\,,$$

$$-3s_0 - 2s_1 - s_2 = -2T_1T_2 - T_0^2 + T_1^2\,,$$
$$s_0 + 4s_1 + 3s_2 = 2T_0T_1 + T_1^2 + T_2^2\,,$$

we obtain that

$$s_0 = \frac{4T_0^2 - 6T_1^2 - 2T_2^2 - 3T_0T_1 + 9T_1T_2 + T_2T_0}{11}\,,$$
$$s_1 = \frac{T_0^2 + 4T_1^2 + 5T_2^2 + 2T_0T_1 - 6T_1T_2 - 8T_2T_0}{22}\,,$$
$$s_2 = \frac{-2T_0^2 + 3T_1^2 + T_2^2 + 7T_0T_1 + T_1T_2 + 5T_2T_0}{11}\,.$$

By replacing s_0, s_1 and s_2 by $s_0/22$, $s_1/22$ and $s_2/22$, respectively, we get the result. It is similar for $d_2 = 22c_2^2$ and $d_3 = 22c_3^2$. □

Lemma 2.3. *We have*

$$c_2c_3e^{\alpha x} + c_3c_1e^{\beta x} + c_1c_2e^{\gamma x} = \frac{1}{22}\sum_{n=0}^\infty T_n^{(\hat T_0,\hat T_1,\hat T_2)}\frac{x^n}{n!}\,,$$

where

$$\hat T_0 = 7T_0^2 + 6T_1^2 + 2T_2^2 + 5T_0T_1 - T_0T_2 - 9T_1T_2\,,$$
$$\hat T_1 = 8T_0^2 + 10T_1^2 + 7T_2^2 + T_0T_1 - 9T_0T_2 - 15T_1T_2\,,$$
$$\hat T_2 = 17T_0^2 + 24T_1^2 + 8T_2^2 + 9T_0T_1 - 15T_0T_2 - 25T_1T_2\,.$$

Remark 2.4. If $T_0 =$ and $T_1 = T_2 = 1$, then $\hat T_0 = -1$, $\hat T_1 = 2$ and $\hat T_2 = 7$.

Proof. Similarly to the proof of Lemma 2.1, we need to have $d_1 = 22c_2c_3$. In this case, we can obtain that $d_2 = 22c_3c_1$ and $d_3 = 22c_1c_2$. □

By using Lemma 2.1 and Lemma 2.3, we get the sum of the products of two Tribonacci-type numbers with initial values T_0, T_1 and T_2 with the binomial coefficient.

Theorem 2.5. *For $n \ge 0$,*

$$\sum_{k=0}^n \binom{n}{k} T_k^{(T_0,T_1,T_2)} T_{n-k}^{(T_0,T_1,T_2)} = \frac{1}{22}\left(2^n T_n^{(T_0^*,T_1^*,T_2^*0)} + 2\sum_{k=0}^n \binom{n}{k}(-1)^k T_k^{(\hat T_0,\hat T_1,\hat T_2)}\right).$$

Proof. First, by Lemmas 2.1 and 2.3,

$$\begin{aligned}
&(c_1e^{\alpha x} + c_2e^{\beta x} + c_3e^{\gamma x})^2\\
&= (c_1^2e^{2\alpha x} + c_2^2e^{2\beta x} + c_3^2e^{2\gamma x}) + 2(c_1c_2e^{(\alpha+\beta)x} + c_2c_3e^{(\beta+\gamma)x} + c_3c_1e^{(\gamma+\alpha)x})\\
&= (c_1^2e^{2\alpha x} + c_2^2e^{2\beta x} + c_3^2e^{2\gamma x}) + 2(c_1c_2e^{(1-\gamma)x} + c_2c_3e^{(1-\alpha)x} + c_3c_1e^{(1-\beta)x})\\
&= \frac{1}{22}\sum_{n=0}^\infty T_n^{(T_0^*,T_1^*,T_2^*)}\frac{(2x)^n}{n!} + 2\sum_{i=0}^n \frac{x^i}{i!}\frac{1}{22}\sum_{k=0}^\infty T_k^{(\hat T_0,\hat T_1,\hat T_2)}\frac{(-x)^k}{k!}\\
&= \frac{1}{22}\sum_{n=0}^\infty T_n^{(T_0^*,T_1^*,T_2^*)}\frac{(2x)^n}{n!} + \sum_{n=0}^\infty \frac{2}{22}\sum_{k=0}^n \binom{n}{k}(-1)^k T_k^{(\hat T_0,\hat T_1,\hat T_2)}\frac{x^n}{n!}\,.
\end{aligned}$$

On the other hand,

$$\left(\sum_{n=0}^\infty T_n^{(T_0,T_1,T_2)}\frac{x^n}{n!}\right)^2 = \sum_{n=0}^\infty\sum_{k=0}^n \binom{n}{k} T_k^{(T_0,T_1,T_2)} T_{n-k}^{(T_0,T_1,T_2)}\frac{x^n}{n!}\,.$$

Comparing the coefficients on both sides, we get the desired result. □

It would be possible to obtain higher-order convolution identities, but the forms seem to become more complicated. We present the results for the sum of the products of three and four Tribonacci-type numbers with multinomial coefficients.

Theorem 2.6. *For* $n \ge 0$,

$$\sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\ge 0}} \binom{n}{k_1,k_2,k_3} T_{k_1}^{(T_0,T_1,T_2)} T_{k_2}^{(T_0,T_1,T_2)} T_{k_3}^{(T_0,T_1,T_2)}$$

$$= \frac{1}{22}\left(3\sum_{k=0}^{n}\binom{n}{k}2^{n-k}T_{n-k}^{(T_0^*,T_1^*,T_2^*)}T_k^{(T_0,T_1,T_2)} - 3^n T_n^{(\check{T}_0,\check{T}_1,\check{T}_2)} + 3\tilde{T}\right).$$

Theorem 2.7. *For* $n \ge 0$,

$$\sum_{\substack{k_1+k_2+k_3+k_4=n\\ k_1,k_2,k_3,k_4\ge 1}} \binom{n}{k_1,k_2,k_3,k_4} T_{k_1}^{(T_0,T_1,T_2)} T_{k_2}^{(T_0,T_1,T_2)} T_{k_3}^{(T_0,T_1,T_2)} T_{k_4}^{(T_0,T_1,T_2)}$$

$$= \frac{1}{484}\left(44\sum_{k=0}^{n}\binom{n}{k}3^{n-k}T_{n-k}^{(\check{T}_0,\check{T}_1,\check{T}_2)}T_k - 3\cdot 4^n T_n^{(\bar{T}_0,\bar{T}_1,\bar{T}_2)}\right.$$

$$\left. +6\sum_{k=0}^{n}\binom{n}{k}\left(\sum_{i=0}^{k}\binom{k}{i}(-1)^i T_i^{(\hat{T}_0,\hat{T}_1,\hat{T}_2)}\right)\left(\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j T_j^{(\hat{T}_0,\hat{T}_1,\hat{T}_2)}\right)\right).$$

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