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SOME NEW FAMILIES OF EDGE PAIR SUM GRAPHS

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Abstract. The concept of an edge pair sum labeling was introduced in [3]. In this paper we prove that the graphs $(K_2 + mK_1)$, $S_{m,n}$, closed helm graph CH_n, two copies of Petersen graph by a path P_k, $k \geq 5$, two copies of fan graph $F_{1,n}$ by a path P_k, $k \geq 5$ and $K_4 \bigcup K_4$ admit edge pair sum labeling.

1 Introduction

Through out this paper we consider finite, simple and undirected graph $G = (V(G), E(G))$ with p vertices and q edges. G is also called a (p, q) graph. We follow the basic notations and terminology of graph theory as in [2]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling and for a dynamic survey of various graph labeling problems with extensive bibliography one can refer to Gallian [1]. Ponraj [12] introduced the concept of pair sum labeling. An injective map $f : V(G) \to \{\pm 1, \pm 2, \ldots, \pm p\}$ is said to be a pair sum labeling of a graph $G(p, q)$ if the induced edge function $f_e : E(G) \to Z - \{0\}$ defined by $f_e(uv) = f(u) + f(v)$ is one-one and $f_e(E(G))$ is either of the form $\left\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q}{2}}\right\}\}$ or $\left\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q-1}{2}}\right\} \cup \left\{\pm k_{\frac{q+1}{2}}\right\}$ according as q is even or odd. Analogous to pair sum labeling we defined a new labeling called edge pair sum labeling in [3] and further studied in [4-11]. Let $G\,(p, q)$ be a graph. An injective map $f : E(G) \to \{\pm 1, \pm 2, \cdots, \pm q\}$is said to be an edge pair sum labeling if the induced vertex function $f^* : V(G) \to Z - \{0\}$ defined by $f^*(v) = \sum\limits_{e \epsilon E_v} f\,(e)$ is one- one where E_v denotes the set of edges in G that are incident with a vertex v and $f^*(V(G))$ is either of the form $\left\{\pm k_1, \pm k_2, \cdots, \pm k_{\frac{p}{2}}\right\}$ or$\left\{\pm k_1, \pm k_2, \cdots, \pm k_{\frac{p-1}{2}}\right\} \bigcup \left\{\pm k_{\frac{p+1}{2}}\right\}$ according as p is even or odd. A graph with an edge pair sum labeling is called an edge pair sum graph.

We use the following definitions in the subsequent sequel.

Definition 1.1. If G_1 and G_2 are subgraphs of a graph G then union of G_1 and G_2 is denoted by $G_1 \bigcup G_2$ which is the graph consisting of all those vertices which are either in G_1 or in G_2 (or in both) and with edge set consisting of all those edges which are either in G_1 or in G_2 (or in both).

Definition 1.2. A closed helm CH_n is the graph obtained by taking a helm H_n and by adding the edges between the pendant vertices.

Definition 1.3. Generalized Petersen graph, $P(n, k)$ is a graph with $n \geq 5$ and $1 \leq k \leq n$ which has vertex set $\{a_0, a_1, ..., a_{n-1}, b_0, b_1, ..., b_{n-1}\}$ and edge set $\{a_i a_{i+1} : i = 0, 1, ..., n-1\} \bigcup \{a_i b_i : i = 0, 1, ..., n-1\} \bigcup \{b_i b_{i+k} : i = 0, 1, ..., n-1\}$, where all subscripts are taken modulo n. The standard Petersen graph is $P(5, 2)$.

2 Main results

In this section we prove that the graphs $(K_2 + mK_1)$, $S_{m,n}$, closed helm graph CH_n, two copies of Petersen graph by a path P_k, $k \geq 5$, two copies of fan graph $F_{1,n}$ by a path P_k, $k \geq 5$ and $K_4 \bigcup K_4$ admit edge pair sum labeling.

Theorem 2.1. *The graph $(K_2 + mK_1)$ is an edge pair sum graph for m is odd.*

Proof. Let $V((K_2+mK_1)) = \{u, v, u_i : 1 \le i \le m\}$ and $E((K_2+mK_1)) = \{e_1^{'} = uv, e_{2i-1} = uu_i, e_{2i} = vu_i : 1 \le i \le m\}$ are the vertices and edges of the graph $(K_2 + mK_1)$.
Define an edge labeling $f : E((K_2 + mK_1)) \to \{\pm1, \pm2, \pm3, ..., \pm(2m + 1)\}$.
Define $f(e_1^{'}) = -1$, for $1 \le i \le \frac{m-1}{2}$ $f(e_{2i-1}) = (2i + 3)$, $f(e_m) = 2$, $f(e_{m+1}) = -3$,
for $1 \le i \le \frac{m-1}{2}$ $f(e_{2i}) = (m + 2 + 2i) = -f(e_{m+1+2i})$ and $f(e_{m+2i}) = -(2i + 3)$.
The induced vertex labeling are as follows:
$f^*(u) = 1 = -f^*(u_{\frac{m+1}{2}})$, $f^*(v) = -4$,
for $1 \le i \le \frac{m-1}{2}$ $f^*(u_i) = (m + 5 + 4i) = -f^*(u_{\frac{m+1}{2}+i})$.
Then we get $f^*(V((K_2 + mK_1))) = \{\pm1, \pm(m + 9), \pm(m + 13), \pm(m + 17), ..., \pm(3m + 3)\} \bigcup \{-4\}$. Hence f is an edge pair sum labeling.
An example for the edge pair sum graph labeling of $(K_2 + 7K_1)$ is shown in Figure 1. □

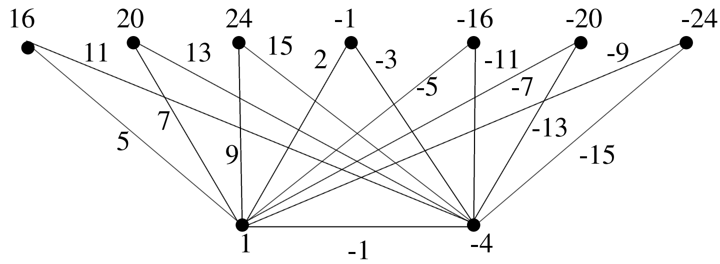

Figure 1

Theorem 2.2. *The graph $S_{m,n}$ is an edge pair sum graph.*

Proof. Let $V(S_{m,n}) = \{v, u_i^j; 1 \le i \le n, 1 \le j \le m\}$ be the vertices of the graph $S_{m,n}$. $E(S_{m,n}) = \{e_i^1 = vu_i^1 : 1 \le i \le n, e_i^{1+j} = u_i^j u_i^{1+j} : 1 \le i \le n, 1 \le j \le (m - 1)\}$ be the edges of the graph $S_{m,n}$.
Define an edge labeling $f : E(S_{m,n}) \to \{\pm1, \pm2, \pm3, .., \pm mn\}$ by considering the following two cases.
Case(i). n is even.
Define $f(e_1^1) = 2$, $f(e_2^1) = -1$, for $1 \le j \le (m - 1)$ $f(e_1^{1+j}) = -(2j + 1) = -f(e_2^{1+j})$ and
for $1 \le i \le \frac{n-2}{2}$, $1 \le j \le m$ $f(e_{2+i}^j) = (2mi + 2j - 1) = -f(e_{\frac{n+2}{2}+i}^j)$.
The induced vertex labeling are as follows:
$f^*(v) = 1 = -f^*(u_1^1)$, $f^*(u_2^1) = 2$, $f^*(u_1^m) = -(2m - 1) = -f^*(u_2^m)$,
for $1 \le i \le (m - 2)$ $f^*(u_1^{1+i}) = -(4 + 4i) = -f^*(u_2^{1+i})$,
for $1 \le i \le \frac{n-2}{2}$, $1 \le j \le m - 1$ $f^*(u_{2+i}^j) = (4m + 4j + (i - 1)4m) = -f^*(u_{\frac{n+2}{2}+i}^j)$ and
for $1 \le i \le \frac{n-2}{2}$ $f^*(u_{2+i}^m) = (2mi + 2m - 1) = -f^*(u_{\frac{n+2}{2}+i}^m)$.
Therefore we get $f^*(V(S_{m,n})) = \{\pm1, \pm(2m - 1), \pm8, \pm12, \pm16, ..., \pm(4m - 4), \pm(4m - 1), \pm(6m - 1), \pm(8m - 1), ..., \pm(mn - 1)$, for $1 \le i \le \frac{n-2}{2}, 1 \le j \le (m - 1) \pm [4m + 4j + (i - 1)4m]\} \bigcup \{2\}$. Hence f is an edge pair sum labeling.
An example for the edge pair sum graph labeling of $S(6, 4)$ is shown in Figure 2.

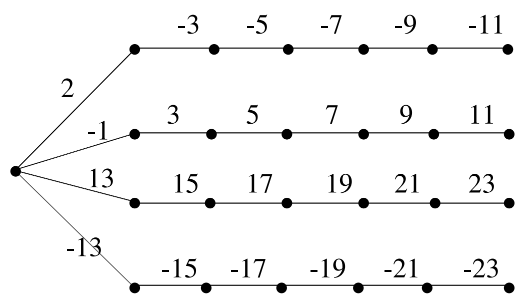

Figure 2

Case(ii). n is odd.
Subcase (i). $m = 2$.
Define $f(e_1^1) = -2$, $f(e_1^2) = 1$ and
for $1 \le i \le \frac{n-1}{2}$, $1 \le j \le 2$ $f(e_{1+i}^j) = [2j + 1 + (i-1)2m] = -f(e_{\frac{n+1}{2}+i}^j)$.
The induced vertex labeling are as follows:
$f^*(v) = -2$, $f^*(u_1^1) = -1 = -f^*(u_1^2)$,
for $1 \le i \le \frac{n-1}{2}$ $f^*(u_{1+i}^1) = 8i = -f^*(u_{\frac{n+1}{2}+i}^1)$,
for $1 \le i \le \frac{n-1}{2}$, $f^*(u_{1+i}^2) = [5 + (i-1)2m] = -f^*(u_{\frac{n+1}{2}+i}^2)$.
Therefore we get $f^*(V(S_{m,n})) = \{\pm1, \pm8, \pm16, \pm24, ..., \pm(4n-4), \pm5, \pm(2m+5), \pm(4m+5), ..., \pm(mn-3m+5)\} \bigcup \{-2\}$. Hence f is an edge pair sum labeling.
Subcase (ii). $m = 3$.
Define $f(e_1^1) = -2$, $f(e_1^2) = -1$, $f(e_1^3) = 3$ and
for $1 \le i \le \frac{n-1}{2}$, $1 \le j \le 3$ $f(e_{1+i}^j) = [2j + 3 + (i-1)2m] = -f(e_{\frac{n+1}{2}+i}^j)$.
The induced vertex labeling are as follows:
$f^*(v) = -2 = -f^*(u_1^2)$, $f^*(u_1^1) = -3 = -f^*(u_1^3)$,
for $1 \le i \le \frac{n-1}{2}$ $f^*(u_{1+i}^3) = [9 + (i-1)2m] = -f^*(u_{\frac{n+1}{2}+i}^3)$,
for $1 \le i \le \frac{n-1}{2}$, $1 \le j \le 2$ $f^*(u_{1+i}^j) = [4j + 8 + (i-1)4m] = -f^*(u_{\frac{n+1}{2}+i}^j)$.
Therefore we get $f^*(V(S_{m,n})) = \{\pm2, \pm3, \pm9, \pm(2m+9), \pm(4m+9), ..., \pm(nm-3m+9)$, for $1 \le i \le \frac{n-1}{2}, 1 \le j \le 2, \pm[4j + 8 + (i-1)4m]\}$.
Hence f is an edge pair sum labeling.
The examples for the edge pair sum graph labeling of $S(2,5)$ and $S(3,3)$ are shown in Figure 3 and Figure 4.

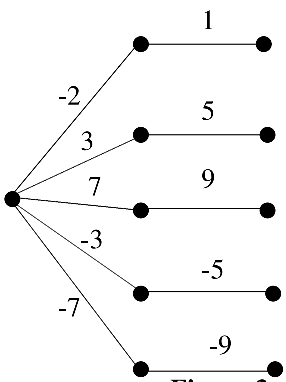

Figure 3

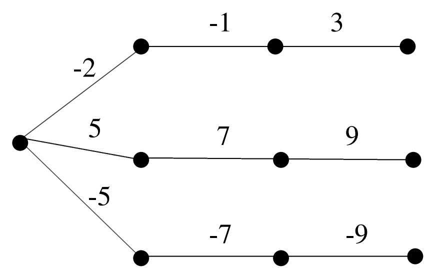

Figure 4

Subcase (iii). m is even and $m \ge 4$.
Define $f(e_1^{\frac{m}{2}}) = 2$, $f(e_1^{\frac{m+2}{2}}) = -1$,
for $1 \le j \le \frac{m-2}{2}$ $f(e_1^j) = -[m + 1 - 2j]$ and $f(e_1^{\frac{m+2}{2}+j}) = (2j+1)$,
for $1 \le i \le \frac{n-1}{2}$, $1 \le j \le m$ $f(e_{1+i}^j) = [m - 1 + 2j + (i-1)2m] = -f(e_{\frac{n+1}{2}+i}^j)$.
The induced vertex labeling are as follows:
$f^*(v) = -(m-1) = -f^*(u_1^m)$, $f^*(u_1^{\frac{m-2}{2}}) = -1 = -f^*(u_1^{\frac{m}{2}})$, $f^*(u_1^{\frac{m+2}{2}}) = 2$,
for $1 \le j \le \frac{m-4}{2}$ $f^*(u_1^j) = -2[m-2j]$ and $f^*(u_1^{\frac{m+2}{2}+j}) = (4+4j)$,
for $1 \le i \le \frac{n-1}{2}$, $1 \le j \le (m-1)$ $f^*(u_{1+i}^j) = [2m + 4j + (i-1)4m] = -f^*(u_{\frac{n+1}{2}+i}^j)$ and
for $1 \le i \le \frac{n-1}{2}$ $f^*(u_{1+i}^m) = [2mi + m - 1] = -f^*(u_{\frac{n+1}{2}+i}^m)$.
Therefore we get $f^*(V(S_{m,n})) = \{\pm1, \pm(m-1), \pm8, \pm12, \pm16, ..., \pm(2m-4), \pm(3m-1), \pm(5m-1), \pm(7m-1), ..., \pm(nm-1)$, for $1 \le i \le \frac{n-1}{2}, 1 \le j \le (m-1), \pm[2m+4j+(i-1)4m]\} \bigcup \{2\}$.
Hence f is an edge pair sum labeling.
An example for the edge pair sum graph labeling of $S(6,3)$ is shown in Figure 5.

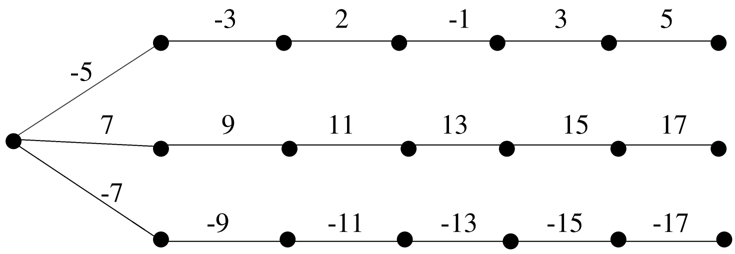

Figure 5

Subcase (iv). m is odd and $m \geq 5$.
Define $f(e_1^{\frac{m-1}{2}}) = -2$, $f(e_1^{\frac{m+1}{2}}) = -1$, $f(e_1^{\frac{m+3}{2}}) = 3$,
for $1 \leq j \leq \frac{m-3}{2}$ $f(e_1^j) = [m+2-2j]$ and $f(e_1^{\frac{m+3}{2}+j}) = -(3+2j)$,
for $1 \leq i \leq \frac{n-1}{2}$, $1 \leq j \leq m$ $f(e_{1+i}^j) = [m+2j+(i-1)2m] = -f(e_{\frac{n+1}{2}+i}^j)$.
The induced vertex labeling are as follows:
$f^*(v) = m = -f^*(u_1^m)$, $f^*(u_1^{\frac{m-3}{2}}) = 3 = -f^*(u_1^{\frac{m-1}{2}})$, $f^*(u_1^{\frac{m+1}{2}}) = 2 = -f^*(u_1^{\frac{m+3}{2}})$,
for $1 \leq j \leq \frac{m-5}{2}$ $f^*(u_1^j) = 2[m+1-2j]$ and $f^*(u_1^{\frac{m+3}{2}+j}) = -(8+4j)$,
for $1 \leq i \leq \frac{n-1}{2}$, $1 \leq j \leq (m-1)$ $f^*(u_{1+i}^j) = [2m+2+4j+(i-1)4m] = -f^*(u_{\frac{n+1}{2}+i}^j)$ and
for $1 \leq i \leq \frac{n-1}{2}$ $f^*(u_{1+i}^m) = [2mi+m] = -f^*(u_{\frac{n+1}{2}+i}^m)$.
Therefore we get $f^*(V(S_{m,n})) = \{\pm 2, \pm 3, \pm m, \pm 12, \pm 16, \pm 20, ..., \pm(2m-2), \pm 3m, \pm 5m,$
$\pm\, 7m, ..., \pm mn$, for $1 \leq i \leq \frac{n-1}{2}, 1 \leq j \leq (m-1), \pm[2m+2+4j+(i-1)4m]\}$.
Hence f is an edge pair sum labeling.
An example for the edge pair sum graph labeling of $S(7,3)$ is shown in Figure 6. □

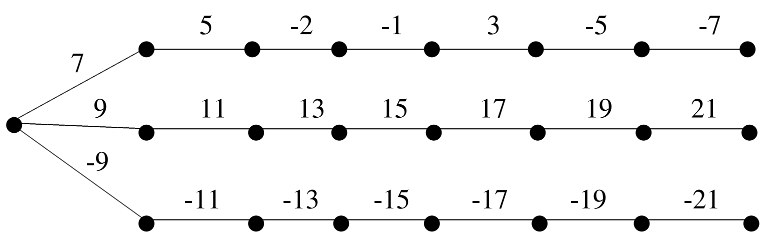

Figure 6

Theorem 2.3. *The closed helm graph CH_n is an edge pair sum graph.*

Proof. Let $V(CH_n) = \{w, u_i, v_i : 1 \leq i \leq n\}$ and $E(CH_n) = \{e_i = wu_i, e_i^{'} = u_iv_i : 1 \leq i \leq n, e_i^{''} = u_iu_{1+i}, e_i^{'''} = v_iv_{1+i} : 1 \leq i \leq (n-1), e_n^{''} = u_nu_1, e_n^{'''} = v_nv_1\}$ are the vertices and edges of the graph CH_n.
Define an edge labeling $f : E(CH_n) \rightarrow \{\pm 1, \pm 2, \pm 3, .., \pm 4n\}$ by considering the two cases.
Case(i). $n = 4, 6, 8, 12, 14, 16, 18, 22, ...$
for $1 \leq i \leq n$, $f(e_i) = (2+2i)$, $f(e_i^{'}) = -(1+i)$ and $f(e_i^{''}) = (n+1+2i) = -f(e_i^{'''})$.
Then the induced vertex labeling are as follows:
$f^*(u_1) = (4n+6) = -f^*(v_1)$,
$f^*(w) = (n^2+3n)$ and for $1 \leq i \leq (n-1)$ $f^*(u_{1+i}) = (2n+6+5i) = -f^*(v_{1+i})$.
Hence we get $f^*(V(CH_n)) = \{\pm(4n+6), \pm(2n+11), \pm(2n+16), \pm(2n+21), ..., \pm(7n+1)\} \bigcup \{(n^2+3n)\}$. Then f is an edge pair sum labeling.
An example for the edge pair sum graph labeling of CH_6 is shown in Figure 7.

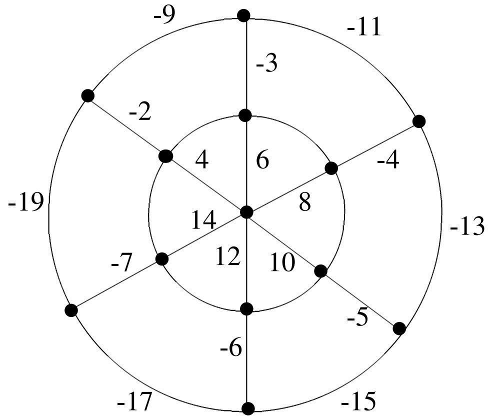

Figure 7

Case(ii). n is odd.
Subcase (i). $n = 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, ...$
For $1 \leq i \leq n$, $f(e_i) = 2i$, $f(e_i^{'}) = -i$ and $f(e_i^{''}) = (n + 2i) = -f(e_i^{'''})$.
Then the induced vertex labeling are as follows:
$f^*(u_1) = (4n + 3) = -f^*(v_1)$, $f^*(w) = n(n + 1)$ and
for $1 \leq i \leq (n - 1)$ $f^*(u_{1+i}) = (2n + 3 + 5i) = -f^*(v_{1+i})$.
Hence we get $f^*(V(CH_n)) = \{\pm(4n + 3), \pm(2n + 8), \pm(2n + 13), \pm(2n + 18), ..., \pm(7n - 2)\} \bigcup \{n(n + 1)\}$. Then f is an edge pair sum labeling.
Subcase (ii). $n \equiv 0(mod 5)$
Define $f(e_i^{''}) = (2n + 2) = -f(e_1^{'''})$, for $1 \leq i \leq n$, $f(e_i) = 2i$, $f(e_i^{'}) = -i$ and
for $1 \leq i \leq (n - 1)$ $f(e_{1+i}^{''}) = (2n + 1 + 2i) = -f(e_{1+i}^{'''})$.
Then the induced vertex labeling are as follows:
$f^*(u_1) = (6n + 2) = -f^*(v_1)$, $f^*(w) = n(n + 1)$, $f^*(u_2) = (4n + 7) = -f^*(v_2)$ and
for $1 \leq i \leq (n - 2)$ $f^*(u_{2+i}) = (4n + 6 + 5i) = -f^*(v_{2+i})$.
Hence we get $f^*(V(CH_n)) = \{\pm(4n + 7), \pm(6n + 2), \pm(4n + 11), \pm(4n + 16), \pm(4n + 21), ..., \pm(9n - 4)\} \bigcup \{n(n + 1)\}$. Then f is an edge pair sum labeling.
The examples for the edge pair sum graph labeling of CH_3 and CH_5 are shown in Figure 8 and Figure 9. □

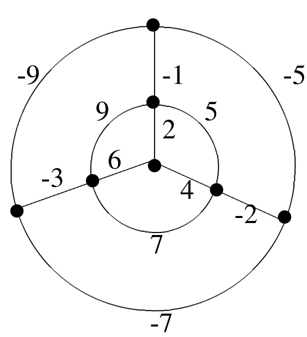

Figure 8

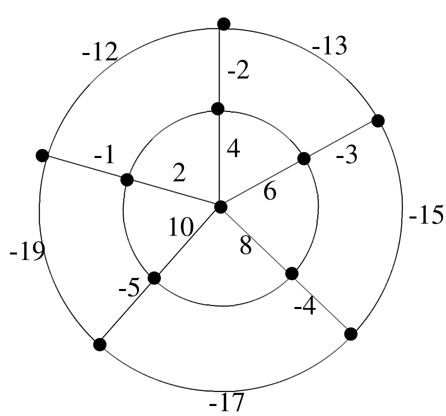

Figure 9

Theorem 2.4. *The graph G obtained by joining two copies of Petersen graph by a path P_k, $k \geq 5$ is edge pair sum graph.*

Proof. Let G be the graph by joining two copies of Petersen graph by a path P_k, $k \geq 5$ of length $(k - 1)$. Let $u_1, u_2, ..., u_5$ and $u_6, u_7, ..., u_{10}$ be the external and internal vertices of first copy

of Petersen graph respectively. Similarly let $w_1, w_2, ..., w_5$ and $w_1, w_2, ..., w_5$ be the external and internal vertices of second copy of Petersen graph respectively.
Let $v_1, v_2, ..., v_k$ be successive vertices of path P_k with $u_1 = v_1$ and $w_1 = v_k$.
Let $E(G) = \{e_i = v_iv_{i+1} : 1 \leq i \leq (k-1), e_i^{'} = u_iu_{i+1} : 1 \leq i \leq 4, e_5^{'} = u_5u_1, e_{5+i}^{'} = w_5w_1, e_{5+i}^{''} = w_iw_{i+1} : 1 \leq i \leq 4, e_{10}^{''} = w_5w_{10}, e_{11}^{'} = u_9u_6, e_{12}^{'} = u_6u_8, e_{13}^{'} = u_8u_{10}, e_{14}^{'} = u_{10}u_7, e_{15}^{'} = u_7u_9, e_{11}^{''} = w_9w_6, e_{12}^{''} = w_6w_8, e_{13}^{''} = w_8w_{10}, e_{14}^{''} = w_{10}w_7, e_{15}^{''} = w_7w_9\}$ are the edges of the graph.
Define an edge labeling $f : E(G) \to \{\pm 1, \pm 2, \pm 3, ..., \pm(k+29)\}$ by considering the two cases.
Case(i). k is odd, $k \geq 5$.
$f(e_{\frac{k+1}{2}}) = 1$, $f(e_{\frac{k-1}{2}}) = 2$, $f(e_{\frac{k-3}{2}}) = -5 = -f(e_{\frac{k+3}{2}})$,
for $1 \leq i \leq \frac{k-5}{2}$ $f(e_i) = -(k+2-2i)$,
for $\frac{k+5}{2} \leq i \leq (k-1)$ $f(e_i) = (-k+2+2i)$,
for $1 \leq i \leq 5$ $f(e_{10+i}^{'}) = -(k+2i) = -f(e_{10+i}^{''})$,
for $1 \leq i \leq 3$ $f(e_{5+i}^{'}) = -(k+14+2i) = -f(e_{5+i}^{''})$,
for $1 \leq i \leq 2$ $f(e_{8+i}^{'}) = -(k+10+2i) = -f(e_{8+i}^{''})$, $f(e_{3+i}^{'}) = -(k+20+2i) = -f(e_{3+i}^{''})$ and $f(e_i^{'}) = -(k+24+2i) = -f(e_i^{''})$, $f(e_3^{'}) = -4 = -f(e_3^{''})$.
The induced vertex labeling are as follows:
$f^*(v_1) = -2(2k+33) = -f^*(v_k)$,
for $1 \leq i \leq \frac{k-5}{2}$ $f^*(v_{1+i}) = 2(-k-1+2i)$ and $f^*(v_{\frac{k+3}{2}+i}) = (8+4i)$,
$f^*(v_{\frac{k-1}{2}}) = -3 = -f^*(v_{\frac{k+1}{2}})$, $f^*(v_{\frac{k+3}{2}}) = 6$, $f^*(u_2) = -3(k+24) = -f^*(w_2)$,
$f^*(u_3) = -2(k+26) = -f^*(w_3)$, $f^*(u_4) = -2(k+19) = -f^*(w_4)$,
$f^*(u_5) = -3(k+20) = -f^*(w_5)$, $f^*(u_6) = -(3k+22) = -f^*(w_6)$,
$f^*(u_7) = -3(k+12) = -f^*(w_7)$, $f^*(u_8) = -3(k+10) = -f^*(w_8)$,
$f^*(u_9) = -3(k+8) = -f^*(w_9)$ and $f^*(u_{10}) = -(3k+28) = -f^*(w_{10})$.
Then $f^*(V(G)) = \{\pm 3, \pm 12, \pm 16, \pm 20, ..., \pm(2k-2), \pm 2(2k+33), \pm 2(k+19), \pm 2(k+26), \pm(3k+22), \pm(3k+28), \pm 3(k+8), \pm 3(k+10), \pm 3(k+12), \pm 3(k+12), \pm 3(k+20), \pm 3(k+24)\} \bigcup \{6\}$. Hence f is an edge pair sum labeling.
Case(ii). k is even, $k \geq 6$.
$f(e_{\frac{k-2}{2}}) = -2$, $f(e_{\frac{k}{2}}) = -1$, $f(e_{\frac{k+2}{2}}) = 3$,
for $1 \leq i \leq \frac{k-4}{2}$ $f(e_i) = -(k+1-2i)$,
for $\frac{k+4}{2} \leq i \leq (k-1)$ $f(e_i) = (k-1-2i)$,
for $1 \leq i \leq 5$ $f(e_{10+i}^{'}) = (k-1+2i) = -f(e_{10+i}^{''})$,
for $1 \leq i \leq 3$ $f(e_{5+i}^{'}) = (k+13+2i) = -f(e_{5+i}^{''})$, $f(e_i^{'}) = (k+23+2i) = -f(e_i^{''})$,
for $1 \leq i \leq 2$ $f(e_{8+i}^{'}) = (k+9+2i) = -f(e_{8+i}^{''})$ and $f(e_{3+i}^{'}) = (k+19+2i) = -f(e_{3+i}^{''})$.
The induced vertex labeling are as follows:
$f^*(v_1) = 2(2k+31) = -f^*(v_k)$,
for $1 \leq i \leq \frac{k-6}{2}$ $f^*(v_{1+i}) = (2k-4i)$ and $f^*(v_{\frac{k+4}{2}+i}) = -(8+4i)$,
$f^*(v_{\frac{k-2}{2}}) = 3 = -f^*(v_{\frac{k}{2}})$, $f^*(v_{\frac{k+2}{2}}) = 2 = -f^*(v_{\frac{k+4}{2}})$,
$f^*(u_2) = 3(k+23) = -f^*(w_2)$, $f^*(u_3) = 3(k+25) = -f^*(w_3)$,
$f^*(u_4) = (3k+61) = -f^*(w_4)$, $f^*(u_5) = 3(k+19) = -f^*(w_5)$,
$f^*(u_6) = (3k+19) = -f^*(w_6)$, $f^*(u_7) = 3(k+11) = -f^*(w_7)$,
$f^*(u_8) = 3(k+9) = -f^*(w_8)$, $f^*(u_9) = 3(k+7) = -f^*(w_9)$ and
$f^*(u_{10}) = (3k+25) = -f^*(w_{10})$.
Then $f^*(V(G)) = \{\pm 2, \pm 3, \pm 12, \pm 16, \pm 20, ..., \pm(2k-4), \pm 2(2k+31), \pm(3k+19), \pm(3k+25), \pm(3k+61), \pm(3k+28), \pm 3(k+7), \pm 3(k+9), \pm 3(k+11), \pm 3(k+19), \pm 3(k+23), \pm 3(k+25)\}$. Hence f is an edge pair sum labeling.
An example for the edge pair sum graph labeling of two copies of Petersen graph by a path P_6 is shown in Figure 10. □

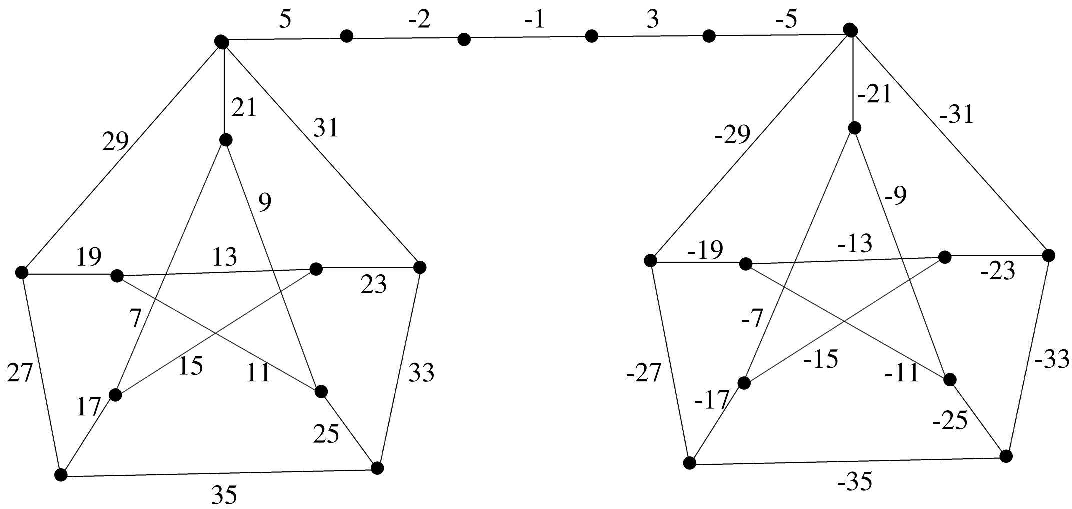

Figure 10

Theorem 2.5. *The graph G obtained by joining the center vertices of two copies of fan graph $F_{1,n}$ by a path P_k, $k \geq 5$ is edge pair sum graph.*

Proof. Let G be the graph by joining two copies of fan graph by a path P_k, $k \geq 5$ of length $(k-1)$. Let $u, u_i : 1 \leq i \leq n$ be the vertices of the first copy of the graph $F_{1,n}$.
Let $v, v_i : 1 \leq i \leq n$ be the vertices of the second copy of the graph $F_{1,n}$.
Let $w_1, w_2, ..., w_5$ and $w_1, w_2, ..., w_k$ be successive vertices of path P_k with $w_1 = u$ and $w_k = v$.
Let $E(G) = \{e_i = uu_i, e_i^{'} = vv_i : 1 \leq i \leq n, e_{n+i} = u_iu_{i+1}, e_{n+i}^{'} = v_iv_{i+1} : 1 \leq i \leq (n-1), e_i^{''} = w_iw_{i+1} : 1 \leq i \leq (k-1)\}$.
Define an edge labeling $f : E(G) \rightarrow \{\pm1, \pm2, \pm3, ..., \pm(4n+k-3)\}$ by considering two cases.
Case(i). k is even, $k \geq 6$.
$f(e_{\frac{k-2}{2}}^{''}) = -2$, $f(e_{\frac{k}{2}}^{''}) = -1$, $f(e_{\frac{k+2}{2}}^{''}) = 3$,
for $1 \leq i \leq \frac{k-4}{2}$ $f(e_i^{''}) = -(k+1-2i)$,
for $\frac{k+4}{2} \leq i \leq (k-1)$ $f(e_i^{''}) = (k-1-2i)$ and
for $1 \leq i \leq n$ $f(e_i) = (k-2+2i) = -f(e_i^{'})$,
for $1 \leq i \leq (n-1)$ $f(e_{n+i}) = (k-1+2i) = -f(e_{n+i}^{'})$.
The induced vertex labeling are as follows:
$f^*(u) = (n^2+nk-n+k-1) = -f^*(v)$,
for $1 \leq i \leq \frac{k-6}{2}$ $f^*(w_{1+i}) = (2k-4i)$ and $f^*(w_{\frac{k+4}{2}+i}) = -(8+4i)$,
$f^*(w_{\frac{k-2}{2}}) = 3 = -f^*(w_{\frac{k}{2}})$, $f^*(w_{\frac{k+2}{2}}) = 2 = -f^*(w_{\frac{k+4}{2}})$,
$f^*(u_1) = (2k+1) = -f^*(v_1)$, $f^*(u_n) = (2k+4n-5) = -f^*(v_n)$ and
for $1 \leq i \leq (n-2)$ $f^*(u_{1+i}) = (3k+6i) = -f^*(v_{1+i})$.
Then $f^*(V(G)) = \{\pm2, \pm3, \pm12, \pm16, \pm20, ..., \pm(2k-4), \pm(2k+1), \pm(2k+4n-5), \pm(n^2+nk-n+k-1), \pm(3k+6), \pm(3k+12), \pm(3k+18), ..., \pm(3k+6n-12)\}$.
Hence f is an edge pair sum labeling.
Case(ii). k is odd, $k \geq 5$.
$f(e_{\frac{k+1}{2}}^{''}) = 2$, $f(e_{\frac{k-1}{2}}^{''}) = 1$, $f(e_{\frac{k-3}{2}}^{''}) = 5 = -f(e_{\frac{k+3}{2}}^{''})$,
for $1 \leq i \leq \frac{k-5}{2}$ $f(e_i^{''}) = (k+2-2i)$,
for $\frac{k+5}{2} \leq i \leq (k-1)$ $f(e_i^{''}) = (k-2-2i)$,
for $1 \leq i \leq n$ $f(e_i) = (k-1+2i) = -f(e_i^{'})$ and
for $1 \leq i \leq (n-1)$ $f(e_{n+i}) = (k+2i) = -f(e_{n+i}^{'})$.
The induced vertex labeling are as follows:
$f^*(u) = (n^2+nk+k) = -f^*(v)$
for $1 \leq i \leq \frac{k-5}{2}$ $f^*(w_{1+i}) = 2(k+1-2i)$ and $f^*(w_{\frac{k+3}{2}+i}) = -(8+4i)$,

$f^*(w_{\frac{k-1}{2}}) = 6$, $f^*(w_{\frac{k+1}{2}}) = 3 = -f^*(w_{\frac{k+3}{2}})$,
$f^*(u_1) = (2k+3) = -f^*(v_1)$, $f^*(u_n) = (2k+4n-3) = -f^*(v_n)$ and
for $1 \leq i \leq (n-2)$ $f^*(u_{1+i}) = (3k+3+6i) = -f^*(v_{1+i})$.
Then $f^*(V(G)) = \{\pm 3, \pm 12, \pm 16, \pm 20, ..., \pm(2k-2), \pm(2k+3), \pm(2k+4n-3), \pm(n^2+nk+k), \pm(3k+9), \pm(3k+15), \pm(3k+21), ..., \pm(3k+6n-9)\} \bigcup \{6\}$.
Hence f is an edge pair sum labeling.
An example for the edge pair sum graph labeling of two copies of fan graph $F_{1,3}$ by a path P_7 is shown in Figure 11. □

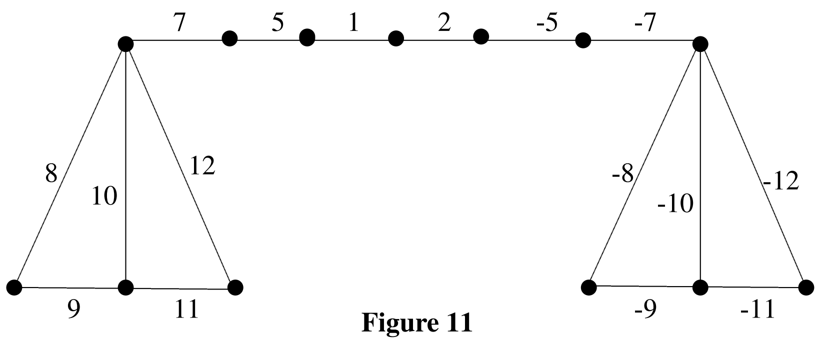

Figure 11

Corollary 2.6. *For $n \geq 3$, the graph $C_n \bigcup C_n$ is an edge pair sum graph* [3].

Corollary 2.7. *for $n, m \geq 2$, the graph $K_{1,n} \bigcup K_{1,m}$ is an edge pair sum graph* [3].

Corollary 2.8. *The complete graph K_4 is not an edge pair sum graph* [3].

Theorem 2.9. *The complete graph $K_4 \bigcup K_4$ is an edge pair sum graph.*

Proof. Let $u_i : 1 \leq i \leq 4$ be the first copy of the graph K_4. Let $v_i : 1 \leq i \leq 4$ be the second copy of the graph K_4. Let $E(K_4 \bigcup K_4) = \{e_i = u_i u_{i+1} : 1 \leq i \leq 3, e_4 = u_4 u_1, e_5 = u_1 u_3, e_6 = u_2 u_4, e_i^{'} = v_i v_{i+1} : 1 \leq i \leq 3, e_4^{'} = v_4 v_1, e_5^{'} = v_1 v_3, e_6^{'} = v_2 v_4\}$ be the edges of the graph $K_4 \bigcup K_4$.
Define an edge labeling $f : E(K_4 \bigcup K_4) \to \{\pm 1, \pm 2, \pm 3, ..., \pm 12\}$.
for $1 \leq i \leq 3$ $f(e_i) = (2i-1) = -f(e_i^{'})$,
$f(e_5) = 2 = -f(e_5^{'})$ and $f(e_6) = 4 = -f(e_6^{'})$.
Then the induced vertex labeling are as follows:
$f^*(u_1) = -4 = -f^*(v_1)$, $f^*(u_2) = 8 = -f^*(v_2)$,
$f^*(u_3) = 10 = -f^*(v_3)$ and $f^*(u_4) = 2 = -f^*(v_4)$.
$f^*(V(K_4 \bigcup K_4)) = \{\pm 2, \pm 4, \pm 8, \pm 10\}$. Hence f is an edge pair sum labeling.
An example for the edge pair sum graph labeling of $K_4 \bigcup K_4$ is shown in Figure 12. □

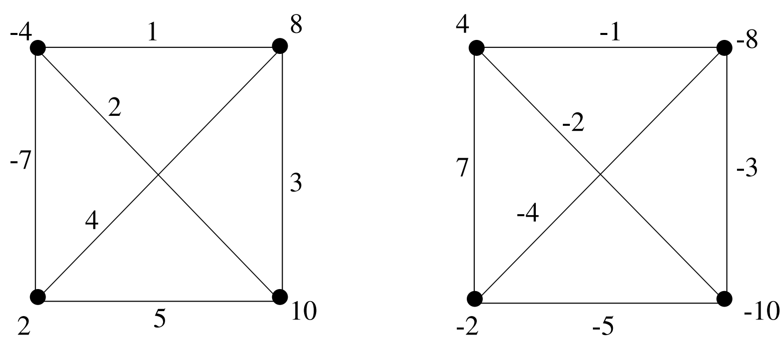

Figure 12

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