Palestine Journal of Mathematics

Vol. 8(2)(2019) , 78–86

$4-$Remainder Cordial Labeling of Some New Families Graphs

Ponraj R., Annathurai K. and Kala R.

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 20M99, 13F10; Secondary 13A15, 13M05.

Keywords and phrases: Wheel, Gear graph, Helm, Closed helm, Fan, Double wheel, Subdivision of comb, Double comb, Splitting of path.

Abstract. Let G be a (p, q) graph. Let f be a function from $V(G)$ to the set $\{1, 2, \ldots, k\}$ where k is an integer $2 < k \leq |V(G)|$. For each edge uv assign the label r where r is the remainder when $f(u)$ is divided by $f(v)$ (or) $f(v)$ is divided by $f(u)$ according as $f(u) \geq f(v)$ or $f(v) \geq f(u)$. The function f is called a k-remainder cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$, for $i, j \in \{1, \ldots, k\}$ where $v_f(x)$ denote the number of vertices labeled with x and $|\eta_e - \eta_o| \leq 1$ where η_e and η_o respectively denote the number of edges labeled with even integers and number of edges labeled with odd integers. A graph with admits a k-remainder cordial labeling is called a k-remainder cordial graph. In this paper we investigate the 4-remainder cordial labeling behavior of wheel, gear graph, helm, closed helm, fan, double wheel, subdivision of comb, double comb, splitting of path of graph.

1 Introduction

Graphs considered here are finite and simple. Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then their join $G_1 + G_2$ is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2 \cup \{uv : u \in V_1 \text{ and } v \in V_2\}$. The graph $W_n = C_n + K_1$ is called a wheel. In a wheel, a vertex of degree 3 is called a rim vertex. A vertex which is adjacent to all the rim vertices is called the central vertex. The edges with one end incident with the rim and the other incident with the central vertex are called spokes. For a graph G, the splitting of G, $S'(G)$ is obtained from G by adding for each vertex v of G a new vertex v' so that v' is adjacent to every vertex that is adjacent to v. Any graph derived from a graph G by a sequence of edge subdivisions is called a subdivision of G or a G-subdivision. Ponraj et al. [4], introduced remainder cordial labeling of graphs and investigate the remainder cordial labeling behavior of path, cycle, star, bistar, complete graph, etc,. and also the concept of k-remainder cordial labeling introduced in [5] and investigate the 4-remainder cordial labeling behavior of certain graphs in [7, 8, 9]. In this paper we investigate the 4-remainder cordial labeling behavior of wheel, gear graph, helm, closed helm, fan, double wheel, subdivision of comb, double comb, splitting of path, etc,. Terms are not defined here follows from Harary [3] and Gallian [2].

2 k-Remainder cordial labeling

Let G be a (p, q) graph. Let f be a function from $V(G)$ to the set $\{1, 2, \ldots, k\}$ where k is an integer $2 < k \leq |V(G)|$. For each edge uv assign the label r where r is the remainder when $f(u)$ is divided by $f(v)$ (or) $f(v)$ is divided by $f(u)$ according as $f(u) \geq f(v)$ or $f(v) \geq f(u)$. The function f is called a k-remainder cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$, for $i, j \in \{1, \ldots, k\}$ where $v_f(x)$ denote the number of vertices labeled with x and $|\eta_e - \eta_o| \leq 1$ where η_e and η_o respectively denote the number of edges labeled with even integers and number of edges labeled with odd integers. A graph with admits a k-remainder cordial labeling is called a k-remainder cordial graph.

First we investigate the $4-$remainder cordial labeling behavior of the wheel.

Theorem 2.1. *If* $n \equiv 0, 1, 2 \pmod 4$*, then the wheel* W_n *is* 4−*remainder cordial.*

Proof. Let $C_n = u_1u_2u_3 \ldots u_nu_1$ be the cycle and $V(W_n) = V(C_n) \cup \{u\}$ and $E(W_n) = E(C_n) \cup \{uu_i : 1 \leq i \leq n\}$. Assign the label 3 to the central vertex u in W_n.

Case(i). $n \equiv 0 \pmod 4$
Consider the rim vertices u_i. First assign the labels $1, 2, 3, 4$ to the vertices u_1, u_2, u_3 and u_4 respectively. Next assign the labels $1, 2, 3, 4$ alternatively to the next four non-labeled vertices u_5, u_6, u_7 and u_8. Proceeding like this until we reach the vertex u_n. In this process the end vertex u_n received the label 4.

Case(ii). $n \equiv 1 \pmod 4$
Fix the labels $1, 2, 2, 3, 4$ to the first five vertices u_1, u_2, u_3, u_4 and u_5 respectively. Next assign the labels $1, 2, 3, 4$ alternatively to the next four non-labeled vertices u_6, u_7, u_8 and u_9. Then next assign the labels $1, 2, 3, 4$ respectively to the next four non-labeled vertices $u_{10}, u_{11}, u_{12}, u_{13}$ and so on. Continuing like this until we reach the vertex u_n. In this process the end vertex u_n received the label 4.

Case(iii). $n \equiv 2 \pmod 4$
Fix the first six vertices $u_1, u_2, u_3, u_4, u_5, u_6$ by the labels $1, 2, 2, 3, 4, 4$ respectively. Next assign the labels $1, 2, 3, 4$ consecutively to the next four vertices u_7, u_8, u_9, u_{10}. Then next assign the labels $1, 2, 3, 4$ respectively to the next four non-labeled vertices $u_{10}, u_{11}, u_{12}, u_{13}$ and so on. Continuing like this until we reach the vertex u_n. That is the vertex u_n received the label 4. Thus the vertex labeling f is 4-remainder cordial labeling follows form the table 1.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{n}{4}$	$\frac{n}{4}$	$\frac{n}{4}+1$	$\frac{n}{4}$	n	n
$n \equiv 1 \pmod 4$	$\frac{n-1}{4}$	$\frac{n+3}{4}$	$\frac{n+3}{4}$	$\frac{n-1}{4}$	n	n
$n \equiv 2 \pmod 4$	$\frac{n-2}{4}$	$\frac{n+2}{4}$	$\frac{n+2}{4}$	$\frac{n+2}{4}$	n	n

Table 1.

□

Next we investigate the 4−remainder cordial labeling behavior of the gear graph.

The gear graph G_n is a graph which is obtained from the wheel W_n by subdividing each edge of the rim.

Theorem 2.2. *The gear graph* G_n *is* 4−*remainder cordial.*

Proof. Let $W_n = C_n + K_1$ where $C_n = u_1u_2 \ldots u_nu_1$ and $V(K_1) = \{u\}$. The graph G_n is obtained from W_n by subdividing the edge u_iu_{i+1} by a vertex v_i, $1 \leq i \leq n$. We now give a 4−remainder cordial labeling as given below. The proof is divided into four cases depends on the nature of n.

Case(i). $n \equiv 0 \pmod 4$
Assign the label 2 to the central vertex of u. Next assign the label 1 to the vertices $u_1, u_3, u_5, \ldots,$. Assign the label 3 to the vertices $u_2, u_4, u_6, \ldots,$. Finally assign the label 2 to the vertices $v_1, v_3, v_5, \ldots$, and assign the label 4 to the vertices $v_2, v_4, v_6, \ldots,$.

Case(ii). $n \equiv 1 \pmod 4$
As in Case(i), assign the label 4 to the central vertex of u. Next assign the label 3 to the vertices $u_1, u_2, u_3, \ldots, u_{\frac{n+1}{2}}$ and then assign the label 4 to the vertices $u_{\frac{n+1}{2}+1}, u_{\frac{n+1}{2}+2}, u_{\frac{n+1}{2}+3}, \ldots, u_n$. In the similar fashion assign the labels to the vertices v_i as u_i. That is assign the label 2 to the vertices $v_1, v_2, v_3, \ldots, v_{\frac{n+1}{2}}$ and then assign the label 1 to the remaining non-labeled vertices

$v_{\frac{n+1}{2}+1}, v_{\frac{n+1}{2}+2}, v_{\frac{n+1}{2}+3}, \ldots, v_n$.

Case(iii). $n \equiv 2 \pmod 4$
As in the case(i), assign the labels to the vertices of u, u_i and v_i respectively in the case $n \equiv 2 \pmod 4$.

Case(iv). $n \equiv 3 \pmod 4$
As in the case(iii), assign the labels to the vertices of u, u_i and v_i of G_n respectively in the case $n \equiv 3 \pmod 4$. The table 2 shows that this vertex labeling f is a 4−remainder cordial labeling of this cases.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0, 2 \pmod 4$	$\frac{n}{2}$	$\frac{n+2}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 1, 3 \pmod 4$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{n-1}{2}$	$\frac{3n-1}{2}$	$\frac{3n+1}{2}$

Table 2.

□

We now investigate the 4−remainder cordial cordial labeling behavior of helm.

The helm graph H_n is a graph which is obtained from the wheel W_n by adding a pendant vertex between every two rim vertices.

Theorem 2.3. *All helms H_n are 4−remainder cordial.*

Proof. Let $W_n = C_n + K_1$ where $C_n = u_1u_2 \ldots u_nu_1$ and $V(K_1) = \{u\}$. Let $V(H_n) = V(W_n) \cup \{v_i : 1 \leq i \leq n\}$ and $E(H_n) = E(W_n) \cup \{u_iv_i : 1 \leq i \leq n\}$. It is easy to verify that H_n has $2n + 1$ vertices and $3n$ edges.

Case(i). $n \equiv 0 \pmod 4$
Assign the label 2 to the central vertex u. Next assign the labels $2, 3$ to the rim vertices u_1 and u_2 respectively. Again assign the labels $2, 3$ to the next two rim vertices u_3 and u_4 respectively. Proceeding like this until reach the vertex u_n. Clearly the vertex u_n received the label 3. Now we move to the pendant vertices v_i. Assign the label 4 to the vertices v_i if u_i received the label 2 and assign the label 1 to the vertices v_i if u_i received the label 3.

Case(ii). $n \equiv 1 \pmod 4$
As in case(i), assign the labels to the vertices of H_n.

Case(iii). $n \equiv 3 \pmod 4$
Assign the label 2 to the central vertex u. Then next assign the labels $3, 2$ to the first two rim vertices u_1 and u_2 respectively. Next assign the labels $3, 2$ to the next two rim vertices u_3 and u_4 respectively. Continuing like this until reach the vertex u_n. Clearly the vertex u_n received the label 3. Next we move to the pendant verticesv_i. Assign the label 4 to the vertices v_i if u_i received the label 2 and assign the label 1 to the vertices v_i if u_i received the label 3.

Case(iii). $n \equiv 3 \pmod 4$
In this case assign the labels to the vertices of H_n as in the case(iii). The table 3, given below establish that this labeling f is a 4− remainder cordial labeling.

□

Now we investigate the closed helm CH_n for the 4−remainder cordial labeling.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0,2 \pmod 4$	$\frac{n}{2}$	$\frac{n}{2}+1$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 1,3 \pmod 4$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{n-1}{2}$	$\frac{3n+1}{2}$	$\frac{3n-1}{2}$

Table 3.

Closed helm CH_n is a graph obtained from the helm H_n with vertex set $V(CH_n) = V(H_n)$ and $E(CH_n) = E(H_n) \cup \{v_i v_{i+1} : 1 \le i \le n-1\} \cup \{v_n v_1\}$.

Theorem 2.4. *The closed helm CH_n is 4−remainder cordial for all values of n.*

Proof. Let f be a 4−remainder cordial labeling of the helm H_n as in theorem **??**. Let g be a vertex labeling of CH_n with $g(u) = f(u)$ and $g(u_i) = f(u_i) : 1 \le i \le n$. Then we define $g(v_i) = 1$ if $f(u_i) = 2$ and $g(v_i) = 4$ if $f(u_i) = 3$. Thus vertex labeling f is a 4− remainder cordial labeling follows from table 4.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0,2 \pmod 4$	$\frac{n}{2}$	$\frac{n}{2}+1$	$\frac{n}{2}$	$\frac{n}{2}$	$2n$	$2n$
$n \equiv 1,3 \pmod 4$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{n-1}{2}$	$2n$	$2n$

Table 4.

□

Here we discus the fan graph F_n.

The graph $P_n + K_1$ is called a fan graph. It is denoted by F_n.

Theorem 2.5. *The fan F_n is 4−remainder cordial for all values of n.*

Proof. LetP_n be the path $u_1 u_2 \dots u_n$ and $V(K_1) = \{u\}$ and $E(F_n) = \{uu_i, u_i u_{i+1} : 1 \le i \le n\}$. It is clearly to verify that F_n has $n+1$ vertices and $2n-1$ edges. Assign the label 3 to the vertex u.

Case(i). $n \equiv 0 \pmod 4$
Assign the labels $1, 2, 3, 4$ to the path vertices u_1, u_2, u_3 and u_4 respectively. Next assign the labels $1, 2, 3, 4$ alternatively to the next four vertices u_5, u_6, u_7 and u_8. Proceeding like this assign the next four vertices and so on. In this process the last vertex u_n received the label 4.

Case(ii). $n \equiv 1 \pmod 4$
As in case(i), assign the labels to the vertices $u_1, u_2, \dots, u_{n-1}$. Finally assign the label 1 to the vertex u_n.

Case(iii). $n \equiv 2 \pmod 4$
As in case(ii), assign the labels to the vertices $u_1, u_2, \dots, u_{n-1}$. Then finally assign the label 2 to the vertex u_n.

Case(iv). $n \equiv 3 \pmod 4$
Assign the labels to the vertices $u_1, u_2, \dots, u_{n-1}$ as in case(iii). Then finally assign the label 3 to the vertex u_n of fan graph. The table 5 establish that the vertex labeling f is a 4− remainder cordial labeling.

□

Next we investigate the double wheel DW_n.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{n}{4}$	$\frac{n}{4}$	$\frac{n}{4}+1$	$\frac{n}{4}$	$n-1$	n
$n \equiv 1 \pmod 4$	$\frac{n-1}{4}+1$	$\frac{n-1}{4}$	$\frac{n-1}{4}+1$	$\frac{n-1}{4}$	n	$n-1$
$n \equiv 2 \pmod 4$	$\frac{n+2}{4}$	$\frac{n+2}{4}$	$\frac{n+2}{4}$	$\frac{n-2}{4}$	n	$n-1$
$n \equiv 3 \pmod 4$	$\frac{n+1}{4}$	$\frac{n+1}{4}$	$\frac{n+1}{4}$	$\frac{n+1}{4}$	n	$n-1$

Table 5.

Let C_n be the cycle $u_1u_2\dots u_nu_1$. Then the double wheel DW_n is the graph with $V(DW_n) = V(C_n) \cup \{u, v\}$ and $E(DW_n) = E(C_n) \cup \{uu_i, vu_i : 1 \le i \le n\}$.

Theorem 2.6. *If $n \equiv 0, 1, 3 \pmod 4$, then the double wheel DW_n is $4-$remainder cordial.*

Proof. Assign the labels $3, 3$ to the vertices u, v respectively.

Case(i). $n \equiv 0 \pmod 4$
First fix the labels $1, 2, 2$ and 4 respectively to the vertices u_1, u_2, u_3 and u_4. Next consider the vertices u_5, u_6, u_7 and u_8. Assign the labels $1, 2, 3, 4$ to the vertices u_5, u_6, u_7 and u_8. Next assign the labels $1, 2, 3, 4$ to the vertices u_9, u_{10}, u_{11} and u_{12}. Proceeding like this, assign the next four consecutive vertices and so on. In this process 4 is the label of the vertex u_n.

Case(ii). $n \equiv 1 \pmod 4$
In this case, fix the labels $1, 2, 2, 4$ and 4 respectively to the vertices u_1, u_2, u_3, u_4 and u_5. Next consider the vertices u_6, u_7, u_8 and u_9. Assign the labels $1, 2, 3$ and 4 to the vertices u_6, u_7, u_8 and u_9 respectively. Next assign the labels $1, 2, 3$ and 4 to the vertices u_{10}, u_{11}, u_{12} and u_{13}. Proceeding like this, assign the next four consecutive vertices and so on. In this process 4 is the label of the vertex u_n.

Case(iii). $n \equiv 3 \pmod 4$
In this case, fix the seven labels $1, 1, 2, 2, 4, 4$ and 4 respectively to the vertices $u_1, u_2, u_3, u_4, u_5, u_6$ and u_7. Next assign the labels to the vertices as in the patten $1, 2, 3, 4\,; 1, 2, 3, 4\,; \dots, ; 1, 2, 3, 4$. □

Now we discus the graph which is obtained form the Cycle and the Path.

Theorem 2.7. *$C_n * P_n$ is $4-$remainder cordial for all n.*

Proof. Let C_n be the cycle $u_1u_2\dots u_nu_1$ and P_n be the path $v_1v_2\dots v_n$. Then the graph $C_n * P_n$ is the graph with vertex set $V(C_n * P_n) = V(C_n) \cup V(P_n)$ and $E(C_n * P_n) = E(C_n) \cup E(P_n) \cup \{u_1v_i : 1 \le i \le n\}$.

Case(i). $n \equiv 0 \pmod 4$
Assign the label 3 to the vertex u_1. Next assign the labels $1, 2$ and 3 to the vertices u_2, u_3 and u_4 respectively. Next assign the labels $1, 2, 3$ and 4 respectively to the vertices u_5, u_6, u_7 and u_8. Proceeding like this assign the labels $1, 2, 3$ and 4 respectively to the next four vertices u_9, u_{10}, u_{11} and u_{12} and so on. In this process the final vertex u_n of C_n received the label 4. Next move to the path P_n. Assign the labels $1, 2, 3$ and 4 to the four consecutive vertices v_1, v_2, v_3 and v_4 respectively. Next assign the labels $1, 2, 3$ and 4 to the next four consecutive vertices v_5, v_6, v_7 and v_8 respectively and so on. In this process the last vertex v_n of P_n received the label 4.

Case(ii). $n \equiv 1 \pmod 4$
Assign the label 3 to the vertex u_1. Next assign the labels $1, 2, 3$ and 4 to the vertices u_2, u_3, u_3 and u_5 respectively. Next assign the labels $1, 2, 3$ and 4 respectively to the vertices namely u_6, u_7, u_8 and u_9. Proceeding like this assign the labels $1, 2, 3$ and 4 to the next four vertices u_{10}, u_{11}, u_{12} and u_{13} respectively and so on. In this way the final vertex u_n of C_n received the label 4. Next assign the labels to the path P_n with n vertices. As in the case(i), assign the labels to the vertices $v_1, v_2, v_3 \dots v_{n-1}, (1 \le i \le n-1)$. Next assign the label 1 to the last vertex v_n of

path P_n.

Case(iii). $n \equiv 2 \pmod 4$
Assign the label 3 to the vertex u_1. Next assign the labels $1, 2, 3$ and 4 to the vertices u_2, u_3, u_4 and u_5 respectively. Then again assign the labels $1, 2, 3$ and 4 respectively to the next four consecutive vertices namely u_6, u_7, u_8 and u_9. Continuing like this assign the labels $1, 2, 3$ and 4 to the next four vertices $u_{n-4}, u_{n-3}, u_{n-2}$ and u_{n-1} respectively and so on. Finally assign the label 4 to the end vertex u_n of C_n. Next assign the labels to the path P_n with n vertices. As in the case(ii), assign the labels to the vertices namely $v_1, v_2, v_3 \dots v_{n-1}, (1 \leq i \leq n-1)$. Finally assign the label 2 to the last vertex v_n of path P_n.

Case(iv). $n \equiv 3 \pmod 4$
First assign the label 3 to the vertex u_1. Next assign the labels $1, 2, 3$ and 4 to the vertices u_2, u_3, u_4 and u_5 respectively. Then again assign the labels $1, 2, 3$ and 4 respectively to the next four consecutive vertices namely u_6, u_7, u_8 and u_9. Proceeding like this assign the labels $1, 2, 3$ and 4 to the next four vertices $u_{n-5}, u_{n-4}, u_{n-3}$ and u_{n-2} respectively and so on. Finally assign the labels $2, 1$ respectively to the last two vertices u_{n-1}, u_n of C_n. Now assign the labels to the path P_n with v_i vertices, $1 \leq i \leq n$. Assign the labels $1, 2, 3, 4$ to the vertices namely v_1, v_2, v_3 and v_4. Next assign the labels $1, 2, 3, 4$ to the next four vertices v_5, v_6, v_7 and v_8 and so on. Continuing like this until we reach the vertex v_{n-3} of path P_n. Finally assign the labels $4, 3, 2$ to the last three vertices namely v_{n-2}, v_{n-1},and v_n respectively. Thus the vertex labeling f is 4-remainder cordial labeling follows form the table 6.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{2n}{4}$	$\frac{2n}{4}$	$\frac{2n}{4}$	$\frac{2n}{4}$	$\frac{3n}{2}$	$\frac{3n-2}{2}$
$n \equiv 1 \pmod 4$	$\frac{2n+2}{4}$	$\frac{2n-2}{4}$	$\frac{2n+2}{4}$	$\frac{2n-2}{4}$	$\frac{3n-1}{2}$	$\frac{3n-1}{2}$
$n \equiv 2 \pmod 4$	$\frac{2n}{4}$	$\frac{2n}{4}$	$\frac{2n}{4}$	$\frac{2n}{4}$	$\frac{3n}{2}$	$\frac{3n-2}{2}$
$n \equiv 3 \pmod 4$	$\frac{2n-2}{4}$	$\frac{2n+2}{4}$	$\frac{2n+2}{4}$	$\frac{2n-2}{4}$	$\frac{3n-1}{2}$	$\frac{3n-1}{2}$

Table 6.

□

Here we investigate the subdivision of the comb.

Theorem 2.8. $S(P_n \odot K_1)$ *is* 4−*remainder cordial.*

Proof. Let $V(S(P_n \odot K_1)) = \{u_i, w_i, v_i : 1 \leq i \leq n\} \cup \{x_i : 1 \leq i \leq n-1\}$ and $E(S(P_n \odot K_1)) = \{u_i x_i, x_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i w_i, w_i v_i : 1 \leq i \leq n\}$.

Case(i). $n \equiv 0 \pmod 4$ Assign the labels $1, 3$ to the vertices u_1, u_2 respectively. Next assign the labels $1, 3$ to the vertices u_3, u_4. Continue in this way until we reach the vertex u_n. That is the vertices $u_1, u_2; u_3 u_4; \dots; u_{n-1} u_n$ received the labels as $1, 3; 1, 3; \dots; 1, 3$. Next assign the labels $2, 4$ to the vertices x_1, x_2 respectively. Next assign the labels $2, 4$ to the next two non-labelled vertices x_3, x_4. Proceeding like this until we reach the vertex x_{n-2}. Next assign the label 2 to the vertex x_{n-1}. We now consider the vertices w_i. Assign the labels $1, 2, 3, 4$ to the non-labelled vertices w_1, w_2, w_3, w_4 respectively. Next assign the labels $1, 2, 3, 4$ to the next four non-labelled vertices w_5, w_6, w_7, w_8 respectively. Proceeding like this until we reach the vertex w_n. Obviously the vertex w_n received the label 4. Next consider the vertices v_i. Assign the labels $4, , 3, 2, 1$ to the vertices v_1, v_2, v_3, v_4 respectively. Next assign the labels $4, 3, 2, 1$ to the next four non-labelled vertices v_5, v_6, v_7, v_8 respectively. Continuing like this until we reach the vertex v_n. That is the vertex v_n received the label 1.

Case(ii). $n \equiv 1 \pmod 4$ Assign the labels $1, 3$ to the vertices u_1, u_2 respectively. Then next assign the labels $1, 3$ to the next two vertices u_3, u_4. Proceeding like this until we reach the vertex u_{n-1}. That is the vertex u_{n-1} received the label3. Finally assign the label 1 to the end vertex u_n. Next assign the labels $2, 4$ to the first two vertices x_1, x_2 respectively. Next assign the labels $2, 4$ to the next two non-labelled vertices x_3, x_4. Proceeding like this until we reach the vertex x_{n-1}. Now we consider the vertices w_i. Assign the labels $1, 2, 3, 4$ to the non-labelled vertices w_1, w_2, w_3, w_4 respectively. Next assign the labels $1, 2, 3, 4$ respectively to the next four non-labelled vertices w_5, w_6, w_7, w_8. Continuing like this until we reach the vertex w_{n-1}. That is the vertex w_{n-1} received the label 4. Finally assign the label 3 to the end vertex w_n. Next consider the pendant vertices v_i. Assign the labels $4, , 3, 2, 1$ to the vertices v_1, v_2, v_3, v_4 respectively. Next assign the labels $4, 3, 2, 1$ to the next four vertices v_5, v_6, v_7, v_8. Continuing like this until we reach the vertex v_{n-1}. That is the vertex v_{n-1} received the label 1. Finally assign the label 4 to the pendant vertex v_n.

Case(iii). $n \equiv 2 \pmod 4$ As in case(i), assign the labels to the vertices $u_1, u_2, \ldots, u_n$ and $x_1, x_2, \ldots, x_{n-1}$. Next using case(i), assign the labels to the vertices $w_i, 1 \leq i \leq n-2$. Then next assign the labels $1, 2$ to the last two vertices w_{n-1}, w_n respectively. Next also by case(i), assign the labels to the vertices $v_i, 1 \leq i \leq n-2$. Trivially the vertex v_{n-2} received the label 1. Then next assign the labels $4, 3$ to the last two vertices v_{n-1}, v_n respectively.

Case(iv). $n \equiv 3 \pmod 4$ As in case(ii), assign the labels to the vertices $u_i, : 1 \leq i \leq n$ and $x_i : 1 \leq i \leq n-1$. Then we consider the vertices w_i. By case(i), assign the labels to the vertices $w_i, 1 \leq i \leq n-3$. Next assign the labels $1, 2, 3$ to the remaining vertices w_{n-2}, w_{n-1}, w_n respectively. Next we consider the vertices v_i. As in case(i), assign the labels to the vertices $v_i, 1 \leq i \leq n-3$. Then finally assign the labels $4, 3, 2$ to the last three vertices v_{n-2}, v_{n-1}, v_n respectively. Thus the table 7 shows that this vertex labeling f is 4-remainder cordial labeling of the subdivision of the comb graph.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0, 2, 3 \pmod 4$	n	n	n	$n-1$	$2n-1$	$2n-1$
$n \equiv 1 \pmod 4$	n	$n-1$	n	n	$2n-1$	$2n-1$

Table 7.

□

Next we show the splitting of path is 4−remainder cordial.

Theorem 2.9. *The $S(P_n)$ is 4−remainder cordial.*

Proof. Let P_n be the path $u_1u_2 \ldots u_nu_1$. Let $V(S(P_n)) = V(P_n) \cup \{v_i : 1 \leq i \leq n\}$ and $E(S(P_n)) = E(P_n) \cup \{v_iu_{i+1}, v_{i+1}u_i : 1 \leq i \leq n-1\}$.

Case(i). $n \equiv 0 \pmod 4$
First we consider the vertices u_i of path P_n. Assign the labels $1, 4, 3, 2$ to the vertices u_1, u_2, u_3, u_4 respectively. Next assign the labels $1, 4, 3, 2$ to the four consecutive non-labelled vertices u_5, u_6, u_7, u_8. Continue in this way, assign the labels to the four consecutive non-labelled vertices and so on. That is the vertices $u_1, u_2, u_3, u_4; \ldots ; u_{n-3}, u_{n-2}, u_{n-1}, u_n$ received the labels as $1, 4, 3, 2; \ldots ; 1, 4, 3, 2$. Next move the vertices v_i. Assign the labels $3, 2, 1, 4$ to the non-labelled vertices v_1, v_2, v_3, v_4 respectively. Next assign the labels $3, 2, 1, 4$ to the next four non-labelled vertices v_5, v_6, v_7, v_8 and so on. Proceeding like this until we reach the vertex v_n. Obviously the vertex v_n received the label 4.

Case(ii). $n \equiv 1 \pmod 4$
As in case(i), assign the labels to the vertices $u_i, v_i : 1 \leq i \leq n-1$. Then finally assign the labels $1, 3$ to the vertices u_n and v_n respectively.

Case(iii). $n \equiv 2 \pmod 4$
As in case(ii), assign the labels to the vertices $u_i, v_i : 1 \leq i \leq n-1$. Then finally assign the labels $4, 2$ to the vertices u_n and v_n respectively. That is the vertex u_n received the label 4 and

the vertex v_n received the label 2.

Case(iv). $n \equiv 3 \pmod 4$
As in case(iii), assign the labels to the vertices $u_i, v_i : 1 \leq i \leq n-1$. Next assign the labels $3, 1$ to the vertices u_n and v_n respectively. That is the vertex u_n received the label 3 and the vertex v_n received the label 1. Thus the table 8 shows that this vertex labeling f is 4-remainder cordial labeling of the splitting of the comb graph.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	n	$n+1$
$n \equiv 1, 3 \pmod 4$	$\lceil \frac{n}{2} \rceil$	$\lfloor \frac{n}{2} \rfloor$	$\lceil \frac{n}{2} \rceil$	$\lfloor \frac{n}{2} \rfloor$	$\frac{3n-3}{2}$	$\frac{3n-3}{2}$
$n \equiv 2 \pmod 4$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{3n-2}{2}$	$\frac{3n-4}{2}$

Table 8.

□

Finally we investigate the double comb.

Theorem 2.10. *The double comb $P_n + 2K_1$ is 4−remainder cordial.*

Proof. Let P_n be the path $u_1 u_2 \ldots u_n u_1$. Let v_i, w_i be the vertices adjacent to u_i.

Case(i). $n \equiv 0 \pmod 4$
Consider the vertices u_i of path P_n. Assign the labels $1, 2, 3, 4$ to the vertices u_1, u_2, u_3, u_4 respectively. Next assign the labels $1, 2, 3, 4$ to the four consecutive non-labelled vertices u_5, u_6, u_7, u_8. Continuing like this assign the labels to the four consecutive non-labelled vertices and so on. It is easy to verify that the vertices $u_1, u_2, u_3, u_4; \ldots; u_{n-3}, u_{n-2}, u_{n-1}, u_n$ received the labels as $1, 2, 3, 4; \ldots; 1, 2, 3, 4$. Next move the pendant vertices v_i. Assign the labels $4, 3, 2, 1$ to the non-labelled vertices v_1, v_2, v_3, v_4 respectively. Next assign the labels $4, 3, 2, 1$ to the next four non-labelled vertices v_5, v_6, v_7, v_8 and so on. Proceeding like this until we reach the vertex v_n. Obviously the vertex v_n received the label 1. Finally assign the labels to the vertices w_i as in v_i vertices.

Case(ii). $n \equiv 1 \pmod 4$
As in case(i), assign the labels to the vertices $u_i, v_i, w_i : 1 \leq i \leq n-1$. Then next assign the labels $4, 3, 1$ to the last non-labelled vertices u_n, v_n and w_n respectively.

Case(iii). $n \equiv 2 \pmod 4$
As in case(i), assign the labels to the vertices $u_i, v_i, w_i : 1 \leq i \leq n-2$. Next assign the labels in the pattern as $4, 3; 1, 2; 1, 2$ to the remaining non-labelled vertices $u_{n-1}, u_n; v_{n-1}, v_n$; and w_{n-1}, w_n respectively. Observation that the vertices u_{n-1}, u_n received the labels $4, 3$,the vertices v_{n-1}, v_n received the labels $1, 2$ and also the vertices w_{n-1}, w_n received the labels $1, 2$.

Case(iv). $n \equiv 3 \pmod 4$
As in case(i), assign the labels to the vertices $u_i, v_i, w_i : 1 \leq i \leq n-3$. Next assign the labels in the pattern as $1, 3, 4; 1, 4, 2; 1, 2, 3$ to the remaining three non-labelled vertices $u_{n-2}, u_{n-1}, u_n; v_{n-2}, v_{n-1}, v_n$; and w_{n-2}, w_{n-1}, w_n respectively. Note that the vertices u_{n-2}, u_{n-1}, u_n received the labels $1, 3, 4$,the vertices v_{n-2}, v_{n-1}, v_n received the labels $1, 4, 2$ and also the vertices w_{n-2}, w_{n-1}, w_n received the labels $1, 2, 3$. Thus the table 9 shows that this vertex labeling f is 4-remainder cordial labeling of the double comb graph.

□

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n-2}{2}$	$\frac{3n}{2}$
$n \equiv 1 \pmod 4$	$\frac{3n+1}{4}$	$\frac{3n}{4}$	$\frac{3n+1}{4}$	$\frac{3n+1}{4}$	$\frac{3n-1}{2}$	$\frac{3n-1}{2}$
$n \equiv 2 \pmod 4$	$\frac{3n-2}{4}$	$\frac{3n-2}{4}$	$\frac{3n+2}{4}$	$\frac{3n+2}{4}$	$\frac{3n}{2}$	$\frac{3n-2}{2}$
$n \equiv 3 \pmod 4$	$\frac{3n+3}{4}$	$\frac{3n-1}{4}$	$\frac{3n-1}{4}$	$\frac{3n-1}{4}$	$\frac{3n-1}{2}$	$\frac{3n-1}{2}$

Table 9.

References

[1] Cahit, I., Cordial Graphs : A weaker version of Graceful and Harmonious graphs, *Ars combin.*, **23** (1987), 201–207.

[2] Gallian, J.A., A Dynamic survey of graph labeling, *The Electronic Journal of Combinatorics.*, **19**, (2016).

[3] Harary, F., Graph theory, *Addision wesley*, New Delhi, 1969.

[4] Ponraj, R., Annathurai, K., and Kala, R., Remainder cordial labeling of graphs, *Journal of Algorithms and Computation*, **Vol.49**, (2017), 17-30.

[5] Ponraj, R. Annathurai, K., and Kala, R., k-Remainder cordial graphs,*Journal of Algorithms and Computation*,**Vol.49(2)**,(2017), 41-52.

[6] Ponraj, R. Annathurai, K., and Kala, R., Remainder cordiality of some graphs, *Palestine Journal of Mathematics*, **Vol.8**(2019), 367–372.

[7] Ponraj, R. Annathurai, K., and Kala, R., 4-Remainder cordial labeling of some special graphs,*International Journal of Pure and Applied Mathematics*, **Vol. 118**, No.6 (2018), 399 - 405.

[8] Ponraj, R. Annathurai, K., and Kala, R., 4-Remainder cordial labeling of some graphs,*International Journal of Mathematical Combinatorics*,**Vol.1**, (March - 2018), 138 - 145.

[9] Ponraj, R., Annathurai, K., and Kala, R., 4-Reminder cordial labeling graphs obtained from ladder,*International Journal of Mathematical Combinatorics*, Special issue, Vol 1(2018), 114–117.

Author information

Ponraj R., Department of Mathematics, Sri Paramakalyani College, Alwarkurichi–627 412, India..
E-mail: ponrajmaths@gmail.com

Annathurai K., Department of Mathematics, Thiruvalluvar College, Papanasam–627 425, India..
E-mail: kannathuraitvcmaths@gmail.com

Kala R., Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli– 627 012, India..
E-mail: karthipyi91@yahoo.co.in

Received: September 5, 2017.
Accepted: September 27, 2018.