Palestine Journal of Mathematics

Vol. 8(2)(2019) , 87–94

Common Fixed Points of Generalized Z-Operators By a Three-Step Iterative Process in CAT(0) Spaces

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Communicated by Hichem Ben-El-Mechaiekh

MSC 2010 Classifications: Primary 54E40, 54H25; Secondary 47H10, 47J25.

Keywords and phrases: CAT(0) spaces, Generalized Z-type operators, Common Fixed Point, Strong convergence.

Abstract. Our purpose in this paper is to approximate common fixed points of two generalized Z-operators by using a three step iterative process (introduced by Karakaya et al [10] in 2013 which is relatively faster than Picard iteration, Mann iteration, S-iteration, Thianwan's iteration and SP-iteration) in CAT(0) spaces. The results obtained in this paper extend and improve the recent ones announced by Saluja, Yildirim et al, Picard, Mann, Thianwan, Phuengrattana and Suantai and many others.

1 Introduction

Let (X, d) be a complete metric space and $T : X \to X$ be a self mapping of X. Suppose $F(T) = \{p \in X : Tp = p\}$ is the set of fixed points of T. In the last four decades, many papers have appeared in the literature on the iteration methods to approximate fixed points for various mapping. In 1890, Picard [14] defined an iterative scheme $\{x_n\}_{n=0}^{\infty}$ as

$$x_{n+1} = Tx_n, n = 0, 1, 2, 3, ...; \tag{1.1}$$

which has been employed to approximate the fixed point of mappings satisfying the inequality

$$d(Tx, Ty) \leq ad(x, y) \tag{1.2}$$

for all $x, y \in X$ and $a \in [0, 1)$. The above condition (1.2) is called Banach's contraction condition.

In 1969, Kannan [11] defined a mapping T called Kannan mapping if there exists $b \in (0, 1/2)$ such that

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \tag{1.3}$$

for all $x, y \in X$.

In 1972, Chatterjea [7] defined a mapping T as a generalization of Kannan mapping called Chatterjea mapping if there exists $c \in (0, 1/2)$ such that

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)] \tag{1.4}$$

for all $x, y \in X$.

In 1972, Zamfirescu [21] obtained the following interesting result for existence of fixed point and convergence of Picard iteration.

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping for which there exist the real numbers a, b, c satisfying $a \in (0, 1)$, $b, c \in (0, 1/2)$ such that for any pair $x, y \in X$, at least one of the following conditions holds:*

(i) $d(Tx, Ty) \leq ad(x, y)$

(ii) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$

(iii) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n$, $n = 0, 1, 2, 3, ...$; converges to p for any arbitrary but fixed $x_0 \in X$.

An operator T which satisfy at least one of the above contractive conditions $(i), (ii), (iii)$ is called a Zamfirescu operator or a Z-operator.

In 1974, Ciric [8] gave an equivalent form for the above conditions $(i), (ii), (iii)$ of Zamfirescu operator by

$$d(Tx, Ty) \leq hmax\{d(x, y), (d(x, Tx) + d(y, Ty))/2, (d(x, Ty) + d(y, Tx))/2\} \tag{1.5}$$

for all $x, y \in X$ and $0 < h < 1$. The mapping satisfying (1.5) is called Ciric quasi-contraction.

In 2004, Berinde [2] proved the strong convergence of Ishikawa iterative process defined by: $x_0 \in C$,

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n Tx_n, \quad \text{for all } n \geq 0; \end{aligned}$$

to approximate fixed points of Zamfirescu operator in an arbitrary Banach space E. While proving the theorem, he made use of the condition,

$$\|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|x - Tx\| \tag{1.6}$$

which holds for any $x, y \in E$ where $0 \leq \delta < 1$.

In this paper, we used a condition introduced in [5] which is more general than condition (1.6) and establish fixed point theorem of three-step iteration scheme in the framework of CAT(0) spaces. The condition is defined as follows: Let C be a nonempty closed convex subset of a CAT(0) space and $T : C \to C$ be a self map of C. There exists a constant $L \geq 0$ such that for all $x, y \in C$, we have

$$d(Tx, Ty) \leq e^{Ld(x,Tx)}(\delta d(x, y) + 2\delta d(x, Tx)) \tag{1.7}$$

where $0 \leq \delta < 1$ and e^x denotes the exponential function of $x \in C$. This condition (1.7) is called generalized Z-type condition.

Remark 1.2. If $L = 0$ in the above condition, then

$$d(Tx, Ty) \leq \delta d(x, y) + 2\delta d(x, Tx),$$

which is the Zamfirescu condition used by Berinde [2] where

$$\delta = max\{a, b/1 - b, c/1 - c\},$$

$0 \leq \delta < 1$, while constants a, b, c are defined in the same manner as in Theorem 1.1.

Since our purpose is to find common fixed points, therefore we need to modify the condition (1.7) to the case of two mappings. One simple way is that we force both of our mappings to satisfy above kind of condition separately. That is, S and T satisfy

$$d(Sx, Sy) \leq e^{Ld(x,Sx)}(\delta d(x, y) + 2\delta d(x, Sx))$$

and

$$d(Tx, Ty) \leq e^{Ld(x,Tx)}(\delta d(x, y) + 2\delta d(x, Tx))$$

respectively.

However, we can modify this to a more general extension as:

$$\begin{aligned} &max\{d(Sx, Sy), d(Tx, Ty)\} \\ &\leq e^{Lmax\{d(x,Sx),d(x,Tx)\}}(\delta d(x, y) + 2\delta max\{d(x, Sx), d(x, Tx)\}) \end{aligned} \tag{1.8}$$

This condition (1.8) reduces to (1.7) as follows when either $S = T$ or one of the mappings is identity.

- The case $S = T$ is obvious.
- When one of the mappings, say S, is identity, then (1.8) reduces to

$$max\{d(x,y), d(Tx,Ty)\} \leq e^{Ld(x,Tx)}(\delta d(x,y) + 2\delta d(x,Tx)). \tag{1.9}$$

- If $max\{d(x,y), d(Tx,Ty)\} = d(Tx,Ty)$, then clearly (1.9) reduces to (1.7).
- If $max\{d(x,y), d(Tx,Ty)\} = d(x,y)$, then (1.9) reduces to

$$d(Tx,Ty) \leq d(x,y) \leq e^{Ld(x,Tx)}(\delta d(x,y) + 2\delta d(x,Tx)).$$

Thus, we conclude that (1.8) reduces to (1.7) when either $S = T$ or one of the mappings is identity.

In 2013, Karakaya et al [10] introduced a new three step iterative process in Banach spaces and they showed by an example that this iteration is much faster than the iteration due to Suantai [18]. Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a mapping. Then the sequence $\{x_n\}$ in C is defined as

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n - \beta_n)y_n + \alpha_n T y_n + \beta_n T z_n, \\ y_n &= (1 - a_n - b_n)z_n + a_n T z_n + b_n T x_n, \\ z_n &= (1 - c_n)x_n + c_n T x_n, \quad \text{for all } n \geq 1, \end{aligned} \tag{1.10}$$

where $\{\alpha_n + \beta_n\}_{n=1}^{\infty}$, $\{a_n + b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ are sequences of positive numbers in $[0, 1]$.

Some special cases of the new three step iterative process given by (1.10), as follows:

(i) If c_n = 1 and α_n = β_n = a_n = b_n = 0 for all $n \geq 1$, then (1.10) is reduced to Picard iteration in [14].

(ii) If α_n = β_n = a_n = b_n = 0 for all $n \geq 1$, then (1.10) is reduced to Mann iteration in [12].

(iii) If c_n = b_n = 0 and $\alpha_n + \beta_n = 1$ for all $n \geq 1$, then (1.10) is reduced to S-iteration in [1].

(iv) If α_n = $a_n = b_n = 0$ for all $n \geq 1$, then (1.10) is reduced to Thianwan's iteration in [19].

(v) If β_n = b_n = 0 for all $n \geq 1$, then (1.10) is reduced to SP-iteration in [13].

Approximating common fixed points has its own importance as it has a direct link with the minimization problem. Keeping this in mind, we extend (1.10) to two mappings case as follows:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n - \beta_n)y_n + \alpha_n T y_n + \beta_n S z_n, \\ y_n &= (1 - a_n - b_n)z_n + a_n S z_n + b_n T x_n, \\ z_n &= (1 - c_n)x_n + c_n S x_n. \end{aligned} \tag{1.11}$$

This process (1.11) reduces to (1.10) if we take $S = T$. And all the above cases also hold good for this process also.

Our aim of this paper is to establish strong convergence results for the above iterative process (1.11) for two generalized Z-type operators satisfying (1.8) to approximate common fixed points in CAT(0) space setting. Our result extend and improve many results in the existing literature due to Agarwal et al. [1], Karakaya et al [10], Rathee and Ritika [15], Ritika and Rathee [16], Suantai [18], Thianwan [19] and many others.

2 Preliminaries

Let us recall some definitions and known results in the existing literature in this concept.

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset R$ to X such that $c(0) = x$,

$c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\Delta(x_1, x_2, x_2)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) = \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = (x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom:

Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$, $d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y})$.

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2 \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [6]. In fact, a geodesic space is a CAT(0) space if and only if it satisfy (CN) inequality.

Lemma 2.1 ([9]). *Let X be a CAT(0) space. Then*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z)$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

Let C be a nonempty subset of a CAT(0) space X and $T : C \to C$ be a mapping. A point $x \in C$ is called a fixed point of T if $Tx = x$. Denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$.

Now we convert Karakaya et al [10] iterative process given by (1.11) to the CAT(0) space setting as: Let C be a nonempty closed convex subset of a complete CAT(0) space X and $S, T : C \to C$ be two mappings. Then the sequence $\{x_n\}$ in C is defined as

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n - \beta_n)y_n \oplus \alpha_n T y_n \oplus \beta_n S z_n, \\ y_n &= (1 - a_n - b_n)z_n \oplus a_n S z_n \oplus b_n T x_n, \\ z_n &= (1 - c_n)x_n \oplus c_n S x_n, \quad \text{for all } n \geq 1, \end{aligned} \tag{2.1}$$

where $\{\alpha_n + \beta_n\}_{n=1}^{\infty}$, $\{a_n + b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ are sequences of positive numbers in $[0, 1]$.

3 Main Results

We need the following useful lemma to prove our main results in this paper.

Lemma 3.1 ([3]). *Suppose that $\{p_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=0}^{\infty}$ are three sequences of nonnegative real numbers satisfying the following condition: $p_{n+1} \leq (1 - s_n)p_n + q_n + r_n$, $n = 0, 1, 2, 3, ...$,where $\{s_n\}_{n=0}^{\infty} \subset [0, 1]$. If $\sum_{n=0}^{\infty} s_n = \infty$, $\lim\limits_{n\to\infty} q_n = O\{s_n\}$ and $\sum_{n=0}^{\infty} r_n < \infty$, then $\lim\limits_{n\to\infty} p_n = 0$.*

Theorem 3.2. *Let C be a nonempty closed convex subset of a complete CAT(0) space X and $S, T : C \to C$ be self-mappings satisfying generalized Z-type condition given by (1.8) with $F(S) \cap F(T) \neq \phi$. For $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (2.1). If $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are the sequences in $[0, 1]$, $\{\alpha_n + \beta_n\}_{n=1}^{\infty} \in [0, 1]$, $\{a_n + b_n\}_{n=1}^{\infty} \in [0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof. Assume that $F(S) \cap F(T) \neq \phi$. Let $p \in F(S) \cap F(T)$. Since S and T satisfy the generalized Z-type condition given by (1.8). Taking $x = p$ and $y = y_n$, we see from (1.8) that

$$\begin{aligned}
&d(Ty_n, p) \\
&\leq max\{d(Sy_n, p), d(Ty_n, p)\} \\
&\leq e^{L(0)}\{\delta d(p, y_n) + 2\delta d(p, Tp)\},
\end{aligned}$$

which gives

$$d(Ty_n, p) \leq \delta d(y_n, p) \tag{3.1}$$

Similarly, by taking $x = p$ and $y = x_n, z_n$ in (1.8), we have

$$d(Sx_n, p) \leq \delta d(x_n, p) \tag{3.2}$$

and

$$d(Sz_n, p) \leq \delta d(z_n, p) \tag{3.3}$$

and

$$d(Tx_n, p) \leq \delta d(x_n, p) \tag{3.4}$$

Now, using (2.1), we have

$$\begin{aligned}
d(x_{n+1}, p) &= d((1 - \alpha_n - \beta_n)y_n \oplus \alpha_n Ty_n \oplus \beta_n Sz_n, p) \\
&\leq (1 - \alpha_n - \beta_n)d(y_n, p) + \alpha_n d(Ty_n, p) + \beta_n d(Sz_n, p)
\end{aligned}$$

From (3.1) and (3.3), we have

$$\begin{aligned}
d(x_{n+1}, p) &\leq (1 - \alpha_n - \beta_n)d(y_n, p) + \alpha_n \delta d(y_n, p) + \beta_n \delta d(z_n, p) \\
&= (1 - \alpha_n(1 - \delta) - \beta_n)d(y_n, p) + \beta_n \delta d(z_n, p) \qquad (3.5)
\end{aligned}$$

But

$$\begin{aligned}
d(y_n, p) &= d((1 - a_n - b_n)z_n \oplus a_n Sz_n \oplus b_n Tx_n, p) \\
&\leq (1 - a_n - b_n)d(z_n, p) + a_n d(Sz_n, p) + b_n d(Tx_n, p)
\end{aligned}$$

From (3.3) and (3.4), we have

$$\begin{aligned}
d(y_n, p) &\leq (1 - a_n - b_n)d(z_n, p) + a_n \delta d(z_n, p) + b_n \delta d(x_n, p) \\
&= (1 - a_n(1 - \delta) - b_n)d(z_n, p) + b_n \delta d(x_n, p) \qquad (3.6)
\end{aligned}$$

But

$$\begin{aligned}
d(z_n, p) &= d((1 - c_n)x_n \oplus c_n Sx_n, p) \\
&\leq (1 - c_n)d(x_n, p) + c_n d(Sx_n, p)
\end{aligned}$$

From (3.2), we have

$$\begin{aligned}
d(z_n, p) &\leq (1 - c_n)d(x_n, p) + c_n \delta d(x_n, p) \\
&= (1 - c_n(1 - \delta))d(x_n, p) \qquad (3.7)
\end{aligned}$$

Therefore, using (3.7) in (3.6), we get

$$\begin{aligned}
d(y_n, p) &\leq (1 - a_n(1 - \delta) - b_n)(1 - c_n(1 - \delta))d(x_n, p) + b_n \delta d(x_n, p) \\
&\leq (1 - a_n(1 - \delta) - b_n(1 - \delta))(1 - c_n(1 - \delta))d(x_n, p) \\
&\leq (1 - (a_n + b_n)(1 - \delta))(1 - c_n(1 - \delta))d(x_n, p) \\
&\leq \delta(1 - c_n(1 - \delta))d(x_n, p) \qquad (3.8)
\end{aligned}$$

Thus, using (3.7) and (3.8) in (3.5), we get

$$\begin{aligned}
d(x_{n+1},p) &\le (1-\alpha_n(1-\delta)-\beta_n)d(y_n,p)+\beta_n\delta d(z_n,p)\\
&\le (1-\alpha_n(1-\delta)-\beta_n)\delta(1-c_n(1-\delta))d(x_n,p)+\beta_n\delta(1-c_n(1-\delta))d(x_n,p)\\
&= \delta(1-c_n(1-\delta))(1-\alpha_n(1-\delta)-\beta_n+\beta_n)d(x_n,p)\\
&= \delta(1-c_n(1-\delta))(1-\alpha_n(1-\delta))d(x_n,p)\\
&\le (1-\{1-\delta\}^3\alpha_n)d(x_n,p)\\
&\le (1-B_n)d(x_n,p);
\end{aligned}$$

where $B_n=\{1-\delta\}^3\alpha_n$, since $0\le\delta<1$ and by assumption of the theorem $\sum_{n=1}^{\infty}\alpha_n=\infty$, it follows that $\sum_{n=1}^{\infty}B_n=\infty$, therefore from Lemma 3.1, we get that $\lim_{n\to\infty} d(x_n,p)=0$. Thus $\{x_n\}$ converges strongly to a common fixed point of S and T.
To show the uniqueness of the fixed point p, assume that $p_1,p_2\in F(S)\cap F(T)$ and $p_1\neq p_2$. Applying the generalized Z-type condition given by (1.8) and using the fact that $0\le\delta<1$, we obtain

$$\begin{aligned}
d(p_1,p_2) &= d(Tp_1,Tp_2)\\
&\le max\{d(Sp_1,Sp_2),d(Tp_1,Tp_2)\}\\
&\le e^{L(0)}\{\delta d(p_1,p_2)+2\delta(0)\}\\
&= \delta d(p_1,p_2)\\
&< d(p_1,p_2);
\end{aligned}$$

which is a contradiction. Therefore, $p_1=p_2$. Thus, $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique common fixed point of S and T. □

Corollary 3.3. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T:C\to C$ be a self-mapping satisfying generalized Z-type condition given by (1.7) with $F(T)\neq\phi$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by Karakaya's iteration (1.10) in $CAT(0)$ space setting. Then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. Put $S=T$ in the above theorem. □

Corollary 3.4. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T:C\to C$ be a self-mapping satisfying generalized Z-type condition given by (1.7) with $F(T)\neq\phi$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by SP-iteration in $CAT(0)$ space setting. Then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. The proof follows by taking $S=T$ and $\beta_n=b_n=0$ for all $n\ge 0$ in above Theorem. This completes the proof. □

Corollary 3.5. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T:C\to C$ be a self-mapping satisfying generalized Z-type condition given by (1.7) with $F(T)\neq\phi$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by Thianwan's iteration in $CAT(0)$ space setting. Then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. The proof immediately follows by taking $S=T$ and $\alpha_n=a_n=b_n=0$ in above Theorem. This completes the proof. □

Corollary 3.6. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T:C\to C$ be a self-mapping satisfying generalized Z-type condition given by (1.7) with $F(T)\neq\phi$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by S-iteration in $CAT(0)$ space setting. Then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. The proof is similar to the proof of the above Theorem. □

Corollary 3.7. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be a self-mapping satisfying generalized Z-type condition given by (1.7) with $F(T) \neq \phi$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by Mann iteration in $CAT(0)$ space setting. Then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. The proof of this corollary immediately follows by taking $S = T$ and $\alpha_n = \beta_n = a_n = b_n = 0$ in above Theorem. This completes the proof. □

Corollary 3.8. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be a self-mapping satisfying generalized Z-type condition given by (1.7) with $F(T) \neq \phi$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by Picard iteration in $CAT(0)$ space setting. Then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. The proof of this corollary immediately follows by taking $S = T$ and $c_n = 1$ and $\alpha_n = \beta_n = a_n = b_n = 0$ in above Theorem. This completes the proof. □

In conclusion we can say that the class of operators given by (1.8) is wider than the class of Zamfirescu operators, Chatterjea's contractive condition (1.3), Kannan's contractive condition (1.4). Our results are also true for Suantai's iteration [18], Picard iteration [14], Mann iteration [12], S-iteration [1], Thianwan's iteration [19], SP-iteration [13] for generalized Z-operators and also for Zamfirescu operators in $CAT(0)$ space setting. Thus the convergence results of SP-iteration for generalized Z-operators in $CAT(0)$ spaces proven by Saluja [17] are hold as a special case of our result. The results obtained in this paper are good improvement and generalization of several known results in the contemporary literature (see, e.g., [1, 3, 4, 5, 12, 13, 14, 17, 18, 19, 20, 21] and some others).

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Received: April 22, 2018.
Accepted: August 13, 2018.