BOYD AND WONG'S FIXED POINT THEOREM IN METRIC SPACES ENDOWED WITH A GRAPH

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Abstract. We investigate Boyd-Wong type contractions in metric spaces endowed with a graph. Our main result is the graph version of the well-known Boyd and Wong's fixed point theorem.

1 Introduction and Preliminaries

Suppose that (X, d) is a metric space and P is the range of d. In [3], Boyd and Wong considered a right upper semicontinuous function $\psi : \overline{P} \to [0, \infty)$ satisfying $\psi(t) < t$ for all $t \in \overline{P} \setminus \{0\}$ and made a very interesting generalization of the Banach contraction principle in complete metric spaces. More precisely, they proved that any arbitrary mapping $T : X \to X$ satisfying

$$d(Tx, Ty) \leq \psi\big(d(x, y)\big)$$

for all $x, y \in X$ has a unique fixed point p and $T^n x \to p$ for all $x \in X$. They also presented some examples to support their result. Recently in [1], Aydi et al. proved some theorems for Boyd-Wong type contractions in ordered metric spaces.

In 2008, Jachymski made a graph theory approach to metric fixed point theory in which the underlying metric space is equipped with a directed graph, and formulated contractive mappings in a graph language. Using this interesting idea, he generalized both the usual and the partially ordered versions of the Banach contraction principle simultaneously and from different aspects.

In this paper, following Jachymski's idea, we first formulate Boyd and Wong contractions in metric spaces endowed with a graph and then present two sufficient conditions guaranteeing the existence of fixed points for contractions of this type. Our main theorem is a rewrite of Boyd and Wong's fixed point theorem in metric spaces endowed with a graph.

We recall some basic concepts about graphs and metric fixed point theory used frequently in the paper. For a widespread discussion on graph theory, it is referred to [2].

Suppose that (X, d) is a metric space and G is a directed graph such that the set $V(G)$ of its vertices equals to X and the set $E(G)$ of its edges contains all loops, i.e. $(x, x) \in E(G)$ for all $x \in X$. Assume further that G does not have any parallel edges. If G is such a graph, then it is said that the metric space (X, d) is endowed with the graph G.

The graph $\widetilde{G}$ is an undirected graph obtained from G by ignoring the directions of the edges of G. Therefore, $V(\widetilde{G}) = V(G) = X$ and $(x, y) \in E(\widetilde{G})$ if and only if either $(x, y) \in E(G)$ or $(y, x) \in E(G)$.

Following Petruşel and Rus [5], one can naturally formulate Picard and weakly Picard operators in metric spaces as follows:

Definition 1.1 ([5]). Suppose that (X, d) is a metric space and $T : X \to X$ is an arbitrary mapping.

  i) T is said to be a Picard operator if T has a unique fixed point p and $T^n x \to p$ for all $x \in X$.

  ii) T is said to be a weakly Picard operator if the sequence $\{T^n x\}$ converges to a fixed point of T for all $x \in X$.

We also require the following weaker type of continuity in metric spaces endowed with a graph which was introduced by Jachymski [4]:

Definition 1.2 ([4]). Suppose that (X, d) is a metric space endowed with a graph G and $T : X \to X$ is an arbitrary mapping. The mapping T said to be orbitally G-continuous if $T^{c_n} x \to y$ implies $TT^{c_n} x \to Ty$ for all $x, y \in X$ and all sequences $\{c_n\}$ of positive integers with

$$(T^{c_n} x, T^{c_n+1} x) \in E(G) \qquad n = 1, 2, \ldots .$$

2 The Frattini Subsemigroup of a Finite Semigroup

Suppose that (X, d) is a metric space endowed with a graph G and $T : X \to X$ is an arbitrary mapping. By C_T, it is meant the set of all $x \in X$ such that $(T^m x, T^n x)$ is an edge of $\widetilde{G}$ for all $m, n \geq 0$, i.e.

$$C_T = \left\{ x \in X : (T^m x, T^n x) \in E(\widetilde{G}) \quad m, n = 0, 1, \ldots \right\}.$$

Note that C_T can even be empty.

In this section, we employ a family Ψ consisting of all functions $\psi : [0, \infty) \to [0, \infty)$ with the following properties:

- ψ is right upper semicontinuous, i.e. $t_n \downarrow t \geq 0$ implies

$$\limsup_{n \to \infty} \psi(t_n) \leq \psi(t)$$

  for all sequences (t_n) of nonnegative numbers;

- $\psi(t) < t$ for all $t > 0$.

Motivated from [3, (3)] and [4, Definition 2.1], we formulate Boyd and Wong G-contractions in metric spaces endowed with a graph as follows:

Definition 2.1. Suppose that (X, d) is a metric space endowed with a graph G and $T : X \to X$ is an arbitrary mapping. We say that T is a Boyd and Wong G-contraction if

BW1) T preserves the edges of G, i.e. $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;

BW2) there exists a $\psi \in \Psi$ such that $d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$ with $(x, y) \in E(G)$.

We next give some basic examples of Boyd and Wong G-contractions.

Example 2.2. Let (X, d) be a metric space endowed with a graph G. Then every constant mapping from X into itself is a Boyd and Wong G-contraction.

Example 2.3. Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a G-contraction in the sense of Jachymski [4, Definition 2.1], i.e. T preserves the edges of G and there exists an $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) \tag{2.1}$$

for all $x, y \in X$ with $(x, y) \in E(G)$. Define a function $\psi : [0, \infty) \to [0, \infty)$ by the rule $\psi(t) = \alpha t$ for all $t \geq 0$. Then $\psi \in \Psi$ and (2.1) shows that T satisfies (BW2). Therefore, T is a Boyd and Wong G-contraction. Hence Boyd and Wong G-contractions are a generalization of G-contractions in metric spaces endowed with a graph.

Example 2.4. Let (X, d) be a metric space and let a mapping $T : X \to X$ satisfy

$$d(Tx, Ty) \leq \psi\big(d(x, y)\big) \qquad (x, y \in X), \tag{2.2}$$

for all $x, y \in X$, where the function $\psi : [0, \infty) \to [0, \infty)$ is right upper semicontinuous and satisfies $\psi(t) < t$ for all $t > 0$. Assume that (X, d) is endowed with a graph G_0 defined by

$$V(G_0) = X \quad \text{and} \quad E(G_0) = X \times X.$$

Evidently, T preserves the edges of G_0 and (2.2) ensures that T satisfies (BW2). Therefore, T is a Boyd and Wong G_0-contraction. Hence Boyd and Wong G-contractions are a generalization of Boyd-Wong type contractions from metric spaces to metric spaces endowed with a graph.

Example 2.5. Let $(X, \preceq)$ be a poset and d be a metric on X. Recall that two elements $x, y \in X$ are said to be comparable if either $x \preceq y$ or $y \preceq x$. Consider a graph G_1 defined by

$$V(G_1) = X \quad \text{and} \quad E(G_1) = \big\{(x, y) \in X \times X : x \preceq y\big\}$$

and assume that (X, d) is endowed with G_1. Then a mapping $T : X \to X$ preserves the edges of G_1 if and only if T is nondecreasing with respect to $\preceq$, i.e. $x \preceq y$ implies $Tx \preceq Ty$ for all $x, y \in X$. Moreover, T satisfies (BW2) if and only if there exists a $\psi \in \Psi$ such that

$$d(Tx, Ty) \le \psi\big(d(x, y)\big) \tag{2.3}$$

for all comparable elements $x, y \in X$.

Example 2.6. Let $(X, \preceq)$ be a poset and d be a metric on X. Consider a graph G_2 defined by

$$V(G_2) = X \quad \text{and} \quad E(G_2) = \big\{(x, y) \in X \times X : x \preceq y \ \vee \ y \preceq x\big\}$$

and assume that (X, d) is endowed with G_2. Then a mapping $T : X \to X$ preserves the edges of G_2 if and only if T maps comparable elements of $(X, \preceq)$ onto comparable elements, and T satisfies (BW2) if and only if (2.3) holds. In particular, if T is a Boyd and Wong G_1-contraction, then T is also a Boyd and Wong G_2-contraction. Hence Boyd and Wong G-contractions are a generalization of ordered Boyd-Wong type contractions from partially ordered metric spaces to metric spaces endowed with a graph.

Example 2.7. Let (X, d) be a metric space and $\varepsilon > 0$ be a fixed number. Recall that two elements $x, y \in X$ are said to be ε-close if $d(x, y) < \varepsilon$. Now assume that (X, d) is endowed with a graph G_3 defined by

$$V(G_3) = X \quad \text{and} \quad E(G_3) = \big\{(x, y) \in X \times X : d(x, y) < \varepsilon\big\}. \tag{2.4}$$

Then a mapping $T : X \to X$ preserves the edges of G_3 if and only if T maps ε-close elements of (X, d) onto ε-close elements, and T satisfies (BW2) if and only if there exists a $\psi \in \Psi$ such that

$$d(Tx, Ty) \le \psi\big(d(x, y)\big)$$

for all ε-close elements $x, y \in X$.

The next proposition is an immediate consequence of the definition of Boyd and Wong G-contractions.

Proposition 2.8. *Suppose that* (X, d) *is a metric space endowed with a graph* G *and* $T : X \to X$ *is an arbitrary mapping.*

  i) *If* T *preserves the edges of* G, *then* T *also preserves the edges of* $\widetilde{G}$.

 ii) *If* T *satisfies* (BW2) *for the graph* G *and the function* $\psi \in \Psi$, *then* T *also satisfies* (BW2) *for the graph* $\widetilde{G}$ *and the same function* ψ.

iii) *If* T *is a Boyd and Wong* G-*contraction, then* T *is also a Boyd and Wong* $\widetilde{G}$-*contraction.*

To prove the existence of fixed points for Boyd and Wong G-contractions in complete metric spaces endowed with a graph, we need the following two lemmas:

Lemma 2.9. *Suppose that* (X, d) *is a metric space endowed with a graph* G *and* $T : X \to X$ *is a Boyd and Wong* G-*contraction. Then* $d(T^n x, T^{n+1} x) \to 0$ *for all* $x \in X$ *with* $(x, Tx) \in E(\widetilde{G})$.

Proof. Let $x \in X$ be given such that $(x, Tx) \in E(\widetilde{G})$. If there exists $N > 0$ such that $T^N x = T^{N+1} x$, then $d(T^n x, T^{n+1} x) = 0$ for all $n \ge N$ and it remains nothing to prove. So assume that no two successive iterates of T at x coincide, i.e. no term of the sequence $\{d(T^n x, T^{n+1} x)\}$ is zero. From Proposition 2.8(iii), T is a Boyd and Wong $\widetilde{G}$-contraction. Since $(x, Tx) \in E(\widetilde{G})$, it follows by induction that $(T^n x, T^{n+1} x) \in E(\widetilde{G})$ for all $n \ge 0$. On the other hand, we have

$$d(T^{n+1} x, T^{n+2} x) = d(TT^n x, TT^{n+1} x) \le \psi\big(d(T^n x, T^{n+1} x)\big) < d(T^n x, T^{n+1} x)$$

for all $n \geq 0$, where $\psi \in \Psi$ is as in (BW2). Therefore, $\{d(T^n x, T^{n+1} x)\}$ is a nonincreasing sequence of nonnegative numbers and so it converges, say to $\eta \geq 0$. In particular, $d(T^n x, T^{n+1} x) \downarrow \eta$. Using (BW2) again as well as the right upper semicontinuity of ψ, we get

$$\eta = \lim_{n \to \infty} d(T^{n+1} x, T^{n+2} x) = \lim_{n \to \infty} d(TT^n x, TT^{n+1} x) \leq \limsup_{n \to \infty} \psi\big(d(T^n x, T^{n+1} x)\big) \leq \psi(\eta)$$

which is a contradiction unless $\eta = 0$. Hence $d(T^n x, T^{n+1} x) \to 0$. □

Lemma 2.10. *Suppose that (X, d) is a metric space endowed with a graph G and $T : X \to X$ is a Boyd and Wong G-contraction. Then the sequence $\{T^n x\}$ is Cauchy for all $x \in C_T$.*

Proof. Let $x \in C_T$ be given and suppose on the contrary that $\{T^n x\}$ fails to be Cauchy. Then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that

$$m_k > n_k \geq k \quad \text{and} \quad d(T^{m_k} x, T^{n_k} x) \geq \varepsilon \qquad k = 1, 2, \ldots.$$

Without loss of generality and using the well-ordering principle if necessary, assume that for all $k \geq 1$, the integer m_k is the smallest one greater than n_k with $d(T^{m_k} x, T^{n_k} x) \geq \varepsilon$. In this case, we have $d(T^{m_k - 1} x, T^{n_k} x) < \varepsilon$ for all $k \geq 1$. So we get

$$\varepsilon \leq d(T^{m_k} x, T^{n_k} x) \leq d(T^{m_k} x, T^{m_k - 1} x) + d(T^{m_k - 1} x, T^{n_k} x) < d(T^{m_k} x, T^{m_k - 1} x) + \varepsilon$$

for all $k \geq 1$. Since from Lemma 2.9 we have $d(T^{m_k} x, T^{m_k - 1} x) \to 0$, it follows from the squeeze theorem that $d(T^{m_k} x, T^{n_k} x) \to \varepsilon$. In particular, $d(T^{m_k} x, T^{n_k} x) \downarrow \varepsilon$.

On the other hand, from Proposition 2.8(iii), T is a Boyd and Wong $\widetilde{G}$-contraction. Thus,

$$
\begin{aligned}
d(T^{m_k} x, T^{n_k} x) &\leq d(T^{m_k} x, T^{m_k + 1} x) + d(T^{m_k + 1} x, T^{n_k + 1} x) + d(T^{n_k + 1} x, T^{n_k} x) \\
&= d(T^{m_k} x, T^{m_k + 1} x) + d(TT^{m_k} x, TT^{n_k} x) + d(T^{n_k + 1} x, T^{n_k} x) \\
&\leq d(T^{m_k} x, T^{m_k + 1} x) + \psi\big(d(T^{m_k} x, T^{n_k} x)\big) + d(T^{n_k + 1} x, T^{n_k} x)
\end{aligned}
$$

for all $k \geq 1$, where $\psi \in \Psi$ is as in (BW2). Now letting $k \to \infty$ and using the right upper semicontinuity of ψ as well as Lemma 2.9, we get

$$
\begin{aligned}
\varepsilon &= \lim_{k \to \infty} d(T^{m_k} x, T^{n_k} x) \\
&= \limsup_{k \to \infty} d(T^{m_k} x, T^{n_k} x) \\
&\leq \limsup_{k \to \infty} \Big(d(T^{m_k} x, T^{m_k + 1} x) + \psi\big(d(T^{m_k} x, T^{n_k} x)\big) + d(T^{n_k + 1} x, T^{n_k} x) \Big) \\
&\leq \underbrace{\limsup_{k \to \infty} d(T^{m_k} x, T^{m_k + 1} x)}_{=0} + \limsup_{k \to \infty} \psi\big(d(T^{m_k} x, T^{n_k} x)\big) + \underbrace{\limsup_{k \to \infty} d(T^{n_k + 1} x, T^{n_k} x)}_{=0} \\
&= \limsup_{k \to \infty} \psi\big(d(T^{m_k} x, T^{n_k} x)\big) \\
&\leq \psi(\varepsilon),
\end{aligned}
$$

which is a contradiction. Hence $\{T^n x\}$ is Cauchy. □

We now are going on our main theorem about the existence of fixed points for Boyd and Wong G-contractions in complete metric spaces endowed with a graph.

Theorem 2.11. *Suppose that (X, d) is a complete metric space endowed with a graph G and $T : X \to X$ is a Boyd and Wong G-contraction. Then the restriction of T to C_T is a weakly Picard operator if one of the following assertions holds:*

 i) *T is orbitally $\widetilde{G}$-continuous;*

 ii) *The function ψ in (BW2) vanishes at zero and the triple (X, d, G) satisfies the following property:*

($\star$) *If $x_n \to x$ and $(x_n, x_{n+1}) \in E(\widetilde{G})$ for all $n \geq 1$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(\widetilde{G})$ for all $k \geq 1$.*

In particular, whenever (i) *or* (ii) *holds,* $\mathrm{Fix}(T) \neq \emptyset$ *if and only if $C_T \neq \emptyset$.*

Proof. If $C_T = \emptyset$, then it remains nothing to prove. Otherwise, given any $x \in C_T$, since $(T^m x, T^n x) \in E(\widetilde{G})$ for all $m, n \geq 0$, it follows that $Tx \in C_T$. So $T(C_T) \subseteq C_T$, i.e. T maps C_T into itself.

Now let again that $x \in C_T$ be given. From Lemma 2.10, the sequence $\{T^n x\}$ is Cauchy and because (X, d) is complete, it converges to some point in X which depends on x, say p. We show that p is a fixed point for T.

To this end, note first that from $x \in C_T$, we have $(T^n x, T^{n+1} x) \in E(\widetilde{G})$ for all $n \geq 0$. If T is orbitally $\widetilde{G}$-continuous, then from $T^n x \to p$, we find $T^{n+1} x \to Tp$, and therefore, $Tp = p$.

On the other hand, if $\psi(0) = 0$ and the triple (X, d, G) satisfies ($\star$), then $\{T^n x\}$ has a subsequence $\{T^{n_k} x\}$ with $(T^{n_k} x, p) \in E(\widetilde{G})$ for all $k \geq 0$. Since from Proposition 2.8(iii), T is also a Boyd and Wong $\widetilde{G}$-contraction, we have

$$d(T^{n_k+1} x, Tp) = d(T T^{n_k} x, Tp) \leq \psi\big(d(T^{n_k} x, Tp)\big) \leq d(T^{n_k} x, p) \to 0.$$

Thus, $Tp = p$.

Finally, it is clear that $\mathrm{Fix}(T)$ is contained in C_T and so $p \in C_T$. Hence $T \mid_{C_T} : C_T \to C_T$ is a weakly Picard operator.   □

We next have three special forms of Theorem 2.11. Firstly, we put $G = G_0$ in Theorem 2.11 and we get a new version of Boyd and Wong's fixed point theorem (see [3, Theorem 1]) in complete metric spaces.

Corollary 2.12. *Suppose that (X, d) is a complete metric space and a mapping $T : X \to X$ satisfies* (2.2). *Then T is a Picard operator.*

Proof. We first show that T is continuous. To this end, let $\{x_n\}$ be a sequence in X converging to an $x \in X$. Assume without loss of generality that all x_n's are distinct from x. Then from (2.2) we get

$$d(Tx_n, Tx) \leq \psi\big(d(x_n, x)\big) < d(x_n, x) \to 0.$$

Thus, T is continuous. In particular, T is orbitally $\widetilde{G_0}$-continuous.

On the other hand, the set C_T coincides with the whole set X and in particular, $C_T \neq \emptyset$. So it follows from Theorem 2.11 that $\mathrm{Fix}(T) \neq \emptyset$ and $T = T \mid_{C_T}$ is a weakly Picard operator. To complete the proof, it suffices to show that T has a unique fixed point in X. Assume that $p, q \in X$ are two fixed points for T. If $p \neq q$, then using (2.2) once more we have

$$d(p, q) = d(Tp, Tq) \leq \psi\big(d(p, q)\big) < d(p, q)$$

which is a contradiction. Therefore, $p = q$. Hence the fixed point of T is unique and T is a Picard operator.   □

Secondly, we suppose that the metric space (X, d) is equipped with a partial order and we put $G = G_1$ or $G = G_2$ in Theorem 2.11. In this case, we obtain an ordered version of Boyd and Wong's fixed point theorem in complete metric spaces equipped with a partial order as follows:

Corollary 2.13. *Suppose that $(X, \preceq)$ is a partially ordered set and d is a metric on X such that the metric space (X, d) is complete. Suppose further that an arbitrary mapping $T : X \to X$ maps comparable elements of $(X, \preceq)$ onto comparable elements and satisfies* (2.3). *Then the restriction of T to*

$$\big\{ x \in X : T^m x \text{ and } T^n x \text{ are comparable} \quad m, n = 0, 1, \dots \big\}$$

is a weakly Picard operator if one of the following assertions holds:

  i) *T is orbitally G_2-continuous;*

ii) *The function ψ vanishes at zero and the triple $(X, d, \preceq)$ satisfies the following property:*

> *If $x_n \to x$ and the successive terms of $\{x_n\}$ are pairwise comparable, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that x_{n_k} and x are comparable for all $k \geq 1$.*

In particular, whenever (i) *or* (ii) *holds,* $\text{Fix}(T) \neq \emptyset$ *if and only if there exists an $x \in X$ such that $T^m x$ and $T^n x$ are comparable for all $m, n \geq 0$.*

Finally, given any arbitrary $\varepsilon > 0$, if we put $G = G_3$ in Theorem 2.11, then it is not hard to see that the triple (X, d, G_3) satisfies ($\star$). Thus, we get a new version of Boyd and Wong's fixed point theorem in complete metric spaces.

Corollary 2.14. *Suppose that (X, d) is a complete metric space (X, d) and $\varepsilon > 0$ is a fixed number. Suppose further that an arbitrary mapping $T : X \to X$ maps ε-close elements of (X, d) onto ε-close elements and satisfies* (2.4). *Then the restriction of T to*

$$\left\{ x \in X : d(T^m x, T^n x) < \varepsilon \quad m, n = 0, 1, \dots \right\}$$

is a weakly Picard operator if one of the following assertions holds:

i) *T is orbitally G_3-continuous;*

ii) *$\psi(0) = 0$.*

In particular, whenever (i) *or* (ii) *holds,* $\text{Fix}(T) \neq \emptyset$ *if and only if there exists an $x \in X$ such that $T^m x$ and $T^n x$ are ε-close for all $m, n \geq 0$.*

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