ON THE STABILITY OF A FUNCTIONAL EQUATION†

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Abstract. In this paper, we study the stability of the functional equation

\[ \sum_{i=1}^{n} \sum_{j=1}^{m} T(p_iq_j) = \sum_{i=1}^{n} T(p_i) \sum_{j=1}^{m} T(q_j) + (m - n)T(0) \sum_{j=1}^{m} T(q_j) + m(n - 1)T(0) \]

in which \( T : I \to \mathbb{R} \), \((p_1, \ldots, p_n) \in \Gamma_n \), \((q_1, \ldots, q_m) \in \Gamma_m \), \( n \geq 3 \), \( m \geq 3 \) being fixed integers.

1 Introduction

For \( n = 1, 2, \ldots \); let \( \Gamma_n = \{(p_1, \ldots, p_n) : p_i \geq 0, \; i = 1, \ldots, n; \sum_{i=1}^{n} p_i = 1 \} \) denote the set of all \( n \)-component discrete probability distributions with nonnegative elements.

A mapping \( a : \mathbb{R} \to \mathbb{R} \) is said to be additive on \( I \) or on the unit triangle \( \Delta = \{(x, y) : 0 \leq x \leq 1, \; 0 \leq y \leq 1, \; 0 \leq x + y \leq 1 \} \) if it satisfies the equation \( a(x + y) = a(x) + a(y) \) for all \( (x, y) \in \Delta; \; I = \{x \in \mathbb{R} : 0 \leq x \leq 1 \} \), \( \mathbb{R} \) denoting the set of all real numbers. It is known [1] that if a mapping \( a : I \to \mathbb{R} \) is additive on the unit triangle \( \Delta \), then there exists one and only one mapping \( A : \mathbb{R} \to \mathbb{R} \) which is an extension of \( a : I \to \mathbb{R} \) in the sense that \( A(x) = a(x) \) for all \( x \in I \) and is additive on \( \mathbb{R} \), that is \( A(x + y) = A(x) + A(y) \) for all \( x, y \in \mathbb{R} \).

A mapping \( M : I \to \mathbb{R} \) is said to be multiplicative if \( M(pq) = M(p)M(q) \) for all \( p \in I \), \( q \in I \).

Suppose a mapping \( T : I \to \mathbb{R} \) satisfies the functional equation

\[ \sum_{i=1}^{n} \sum_{j=1}^{m} T(p_iq_j) = \sum_{i=1}^{n} T(p_i) \sum_{j=1}^{m} T(q_j) + (m - n)T(0) \sum_{j=1}^{m} T(q_j) + m(n - 1)T(0) \]  \hspace{1cm} (1.1)

for all \((p_1, \ldots, p_n) \in \Gamma_n \) and \((q_1, \ldots, q_m) \in \Gamma_m \); \( n \geq 3 \), \( m \geq 3 \) being fixed integers.

The functional equation (1.1) has been considered by Nath and Singh [6]. They determined its general solutions for fixed integers \( n \geq 3 \), \( m \geq 3 \).

The functional equation (1.1) plays an important role in finding the general solutions of several multiplicative and nonmultiplicative type sum form functional equations with at least two unknown mappings (see [6] to [11]). Also, their solutions are related to the Shannon [13] entropy and the entropies of degree \( \alpha \) [2].

Result 1.1 ([6]). Let \( n \geq 3 \), \( m \geq 3 \) be fixed integers. If a mapping \( T : I \to \mathbb{R} \) satisfies the functional equation (1.1) for all \((p_1, \ldots, p_n) \in \Gamma_n \), \((q_1, \ldots, q_m) \in \Gamma_m \), then either

\[ T(p) = a(p) + T(0) \]

where \( a : \mathbb{R} \to \mathbb{R} \) is an additive mapping with

\[ a(1) = \begin{cases} -mT(0) & \text{if } T(1) + (m - 1)T(0) \neq 1 \\ 1 - mT(0) & \text{if } T(1) + (m - 1)T(0) = 1 \end{cases} \]

or

\[ T(p) = M(p) - b(p) + T(0) \]

in which \( b : \mathbb{R} \to \mathbb{R} \) is an additive mapping with \( b(1) = mT(0) \) and \( M : I \to \mathbb{R} \) is a nonadditive multiplicative mapping with \( M(0) = 0 \), \( M(1) = 1 \).

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This paper deals with the stability of the sum form functional equation (1.1). For the meaning of stability of a functional equation, see Hyers and Rassias [3]. By the stability problem for the equation (1.1), we mean the following: Let \( n \geq 3, m \geq 3 \) be fixed integers and \( 0 \leq \epsilon \in \mathbb{R} \) be a fixed real number. Find all mappings \( T : I \to \mathbb{R} \) satisfying the functional inequality
\[
| \sum_{i=1}^{n} \sum_{j=1}^{m} T(p_{i}q_{j}) - \sum_{i=1}^{n} T(p_{i}) \sum_{j=1}^{m} T(q_{j}) - (m-n)T(0) \sum_{j=1}^{m} T(q_{j}) - m(n-1)T(0) | \leq \epsilon \] (1.2)
for all \((p_{1}, \ldots, p_{n}), (q_{1}, \ldots, q_{m}) \in \Gamma_{n}\).

Now, we mention below some results needed for the development of the main result of this paper.

**Result 1.2** ([4]). Let \( \epsilon \) be a given real constant. Suppose \( \phi : I \to \mathbb{R} \) is a mapping which satisfies the functional equation \( \sum_{i=1}^{n} \phi(p_{i}) = c \) for all \((p_{1}, \ldots, p_{n}) \in \Gamma_{n}, n \geq 3 \) a fixed integer. Then there exists an additive mapping \( a : \mathbb{R} \to \mathbb{R} \) such that \( \phi(p) = a(p) - \frac{1}{n}a(1) + \frac{c}{n} \) for all \( p \in I \).

**Result 1.3** ([5]). Let \( n \geq 3 \) be a fixed integer and \( \epsilon \) be a fixed nonnegative real number. Suppose a mapping \( \psi : I \to \mathbb{R} \) satisfies the functional inequality \( | \sum_{j=1}^{m} \psi(p_{j}) | \leq \epsilon \) for all \((p_{1}, \ldots, p_{n}) \in \Gamma_{n} \). Then there exist an additive mapping \( A : \mathbb{R} \to \mathbb{R} \) and a bounded mapping \( B : \mathbb{R} \to \mathbb{R} \) satisfying the conditions \( B(0) = 0 \) and \(|B(p)| \leq 18\epsilon \) such that \( \psi(p) - \psi(0) = A(p) + B(p) \) for all \( p \in I \).

## 2 The Main Result

**Theorem 2.1.** Let \( n \geq 3, m \geq 3 \) be fixed integers and \( \epsilon \) be a given nonnegative real constant. Suppose the mapping \( T : I \to \mathbb{R} \) satisfies the inequality (1.2) for all \((p_{1}, \ldots, p_{n}) \in \Gamma_{n} \) and \((q_{1}, \ldots, q_{m}) \in \Gamma_{m} \). Then either
\[
T(p) = a(p) + b(p) \tag{2.1}
\]
for all \( p \in I \) or
\[
T(p) = M(p) - B(p) + T(0) \tag{2.2}
\]
for all \( p \in I \) where \( a : \mathbb{R} \to \mathbb{R} \), \( B : \mathbb{R} \to \mathbb{R} \) are additive mappings; \( B(1) = mT(0) \); \( b : \mathbb{R} \to \mathbb{R} \) is a bounded mapping; and \( M : I \to \mathbb{R} \) is a multiplicative mapping which is not additive and \( M(0) = 0 \), \( M(1) = 1 \).

### Proof.
Let us write (1.2) in the form
\[
| \sum_{i=1}^{n} \left( \sum_{j=1}^{m} T(p_{i}q_{j}) - T(p_{i}) \sum_{j=1}^{m} T(q_{j}) - (m-n)T(0) \sum_{j=1}^{m} T(q_{j}) - m(n-1)T(0)p_{i} \right) | \leq \epsilon.
\]
By Result 1.3, there exist a mapping \( A_{1} : \mathbb{R} \times \Gamma_{m} \to \mathbb{R} \) additive in the first variable and a bounded mapping \( b_{1} : \mathbb{R} \times \Gamma_{m} \to \mathbb{R} \) with \( b_{1}(0; q_{1}, \ldots, q_{m}) = 0 \) and \(|b_{1}(x; q_{1}, \ldots, q_{m})| \leq 18\epsilon \) for all \( x \in \mathbb{R} \) such that
\[
\sum_{j=1}^{m} T(p_{j}q_{j}) - T(p_{j}) \sum_{j=1}^{m} T(q_{j}) - (m-n)T(0) p \sum_{j=1}^{m} T(q_{j}) - m(n-1)T(0)p_{j} \]
\[
- mT(0) + T(0) \sum_{j=1}^{m} T(q_{j}) = A_{1}(p; q_{1}, \ldots, q_{m}) + b_{1}(p; q_{1}, \ldots, q_{m}) \tag{2.3}
\]
for all \( p \in I \). Let \( x \in I \) and \((r_{1}, \ldots, r_{m}) \in \Gamma_{m} \). Putting successively \( p = xr_{t}, t = 1, \ldots, m \) in
\[(2.3)\]; adding the resulting \(m\) equations and using the additivity of \(A_1\), we obtain

\[
\sum_{t=1}^{m} \sum_{j=1}^{m} T(xr_t q_j) - \sum_{t=1}^{m} T(xr_t) \sum_{j=1}^{m} T(q_j) - (m - n) T(0) x \sum_{j=1}^{m} T(q_j)
- m(n - 1)T(0)x - m^2T(0) + mT(0) \sum_{j=1}^{m} T(q_j)
= A_1(x; q_1, \ldots, q_m) + \sum_{t=1}^{m} b_1(xr_t; q_1, \ldots, q_m). \quad (2.4)
\]

Now put \(p = x, q_1 = r_1, \ldots, q_m = r_m\) in \((2.3)\). We obtain

\[
\sum_{t=1}^{m} T(xr_t) = T(x) \sum_{t=1}^{m} T(r_t) + (m - n) T(0) x \sum_{t=1}^{m} T(r_t) + m(n - 1)T(0)x
+ mT(0) - T(0) \sum_{t=1}^{m} T(r_t) + A_1(x; r_1, \ldots, r_m) + b_1(x; r_1, \ldots, r_m).
\quad (2.5)
\]

From \((2.4)\) and \((2.5)\), it follows that

\[
\sum_{t=1}^{m} \sum_{j=1}^{m} T(xr_t q_j) - [T(x) + (m - n) T(0) x - T(0)] \sum_{t=1}^{m} T(r_t) \sum_{j=1}^{m} T(q_j) - m(n - 1)T(0)x
- m^2T(0) = [m(m - 1)T(0)x + A_1(x; r_1, \ldots, r_m) + b_1(x; r_1, \ldots, r_m)] \sum_{j=1}^{m} T(q_j)
+ A_1(x; q_1, \ldots, q_m) + \sum_{t=1}^{m} b_1(xr_t; q_1, \ldots, q_m).
\quad (2.6)
\]

The left hand side of \((2.6)\) is symmetric in \(q_j\) and \(r_t; j = 1, \ldots, m; t = 1, \ldots, m\). So, the right hand side of \((2.6)\) should also be symmetric in \(q_j\) and \(r_t; j = 1, \ldots, m; t = 1, \ldots, m\). This fact gives rise to the equation

\[
[m(m - 1)T(0)x + A_1(x; q_1, \ldots, q_m)] [\sum_{t=1}^{m} T(r_t) - 1]
- [m(m - 1)T(0)x + A_1(x; r_1, \ldots, r_m)] [\sum_{j=1}^{m} T(q_j) - 1]
= b_1(x; r_1, \ldots, r_m) \sum_{j=1}^{m} T(q_j) + \sum_{t=1}^{m} b_1(xr_t; q_1, \ldots, q_m)
- b_1(x; q_1, \ldots, q_m) \sum_{t=1}^{m} T(r_t) - \sum_{j=1}^{m} b_1(xq_j; r_1, \ldots, r_m).
\quad (2.7)
\]

For fixed \((q_1, \ldots, q_m) \in \Gamma_m, (r_1, \ldots, r_m) \in \Gamma_m\), the right hand side of \((2.7)\) is a bounded mapping of \(x, x \in I\). On the other hand, the left hand side is \(\alpha\)ditive in \(x, x \in I\). By using
Theorem 1.8 (see p-14 in [12]) and the Definition 1.2 (see p-4 in [12]), we have

\[
\begin{align*}
&[n(m - 1)T(0)x + A_1(x; q_1, \ldots, q_m)][\sum_{t=1}^{m} T(r_t) - 1] \\
&\quad - [n(m - 1)T(0)x + A_1(x; r_1, \ldots, r_m)][\sum_{j=1}^{m} T(q_j) - 1] \\
&= x\left\{[n(m - 1)T(0) + A_1(1; q_1, \ldots, q_m)][\sum_{t=1}^{m} T(r_t) - 1] \\
&\quad - [n(m - 1)T(0) + A_1(1; r_1, \ldots, r_m)][\sum_{j=1}^{m} T(q_j) - 1]\right\}
\end{align*}
\]

which, on simplification, reduces to

\[
A_1(x; q_1, \ldots, q_m) - xA_1(1; q_1, \ldots, q_m)[\sum_{t=1}^{m} T(r_t) - 1] \\
= [A_1(x; r_1, \ldots, r_m) - xA_1(1; r_1, \ldots, r_m)][\sum_{j=1}^{m} T(q_j) - 1].
\]

(2.8)

Now we divide our discussion into two cases:

**Case 1.** \(\sum_{t=1}^{m} T(r_t) - 1\) vanishes identically on \(\Gamma_m\), that is,

\[
\sum_{t=1}^{m} T(r_t) - 1 = 0
\]

for all \((r_1, \ldots, r_m) \in \Gamma_m\). By Result 1.2, there exists an additive mapping \(a : \mathbb{R} \to \mathbb{R}\) such that, for all \(p \in I\),

\[
T(p) = a(p) + T(0)
\]

(2.9)

with \(a(1) = 1 - m T(0)\). The solution (2.9) is included in (2.1) on defining a constant bounded mapping \(b : \mathbb{R} \to \mathbb{R}\) as \(b(p) = T(0)\).

**Case 2.** \(\sum_{t=1}^{m} T(r_t) - 1\) does not vanish identically on \(\Gamma_m\).

In this case, there exists a probability distribution \((r_1^*, \ldots, r_m^*) \in \Gamma_m\) such that

\[
\sum_{t=1}^{m} T(r_t^*) - 1 \neq 0.
\]

(2.10)

Putting \(r_1 = r_1^*, \ldots, r_m = r_m^*\) in (2.8) and making use of (2.10), it follows that

\[
A_1(x; q_1, \ldots, q_m) = A_2(x) \left[\sum_{j=1}^{m} T(q_j) - 1\right] + xA_1(1; q_1, \ldots, q_m)
\]

(2.11)

where \(A_2 : \mathbb{R} \to \mathbb{R}\) is defined as

\[
A_2(x) = \left[\sum_{t=1}^{m} T(r_t^*) - 1\right]^{-1} \left[A_1(x; r_1^*, \ldots, r_m^*) - xA_1(1; r_1^*, \ldots, r_m^*)\right]
\]

(2.12)

for all \(x \in \mathbb{R}\). The mapping \(A_2\) is additive and \(A_2(1) = 0\). Putting \(p = 1\) in (2.3), we obtain

\[
A_1(1; q_1, \ldots, q_m) = [1 - T(1) - (m - n) T(0) + T(0)]\sum_{j=1}^{m} T(q_j) \\
\quad - mn T(0) - b_1(1; q_1, \ldots, q_m).
\]

(2.13)
From (2.7), (2.11) and (2.13), we have

\[ \{ b_1(x; q_1, \ldots, q_m) - x b_1(1; q_1, \ldots, q_m) + x [1 - T(1) - (m - 1)T(0)] \} \sum_{l=1}^{m} T(r_l) \]

\[ = \{ b_1(x; r_1, \ldots, r_m) - x b_1(1; r_1, \ldots, r_m) + x [1 - T(1) - (m - 1)T(0)] \} \sum_{j=1}^{m} T(q_j) \]

\[ + \left[ \sum_{l=1}^{m} b_1(xr_l; q_1, \ldots, q_m) - \sum_{j=1}^{m} b_1(xq_l; r_1, \ldots, r_m) \right] \\
- x [b_1(1; q_1, \ldots, q_m) - b_1(1; r_1, \ldots, r_m)] \] (2.14)

for all \(x \in I, (r_1, \ldots, r_m) \in \Gamma_m\) and \((q_1, \ldots, q_m) \in \Gamma_m\).

**Case 2.1.** The coefficient of \(\sum_{l=1}^{m} T(r_l)\), in (2.14), does not vanish identically on \(I \times \Gamma_m\).

In this case, there exist an element \(x^* \in I\) and a probability distribution \((q_1^*, \ldots, q_m^*) \in \Gamma_m\) such that

\[ \{ b_1(x^*; q_1^*, \ldots, q_m^*) - x^* b_1(1; q_1^*, \ldots, q_m^*) + x^* [1 - T(1) - (m - 1)T(0)] \} \neq 0. \] (2.15)

From (2.14), (2.15) and the boundedness of \(b_1\), it follows that \(| \sum_{l=1}^{m} T(r_l) | \leq \epsilon^*\) for some nonnegative real number \(\epsilon^*\). So, by Result 1.3, there exist an additive mapping \(a : \mathbb{R} \rightarrow \mathbb{R}\) and a bounded mapping \(b_2 : \mathbb{R} \rightarrow \mathbb{R}\) such that \(T(p) - T(0) = a(p) + b_2(p)\) for all \(p \in I\). This solution is included in (2.1) on defining a bounded mapping \(b : \mathbb{R} \rightarrow \mathbb{R}\) as \(b(p) = b_2(p) + T(0)\).

**Case 2.2.** The coefficient of \(\sum_{l=1}^{m} T(r_l)\), in (2.14), vanishes identically on \(I \times \Gamma_m\), that is,

\[ b_1(x; q_1, \ldots, q_m) - x b_1(1; q_1, \ldots, q_m) + x [1 - T(1) - (m - 1)T(0)] = 0 \] (2.16)

for all \(x \in I\) and \((q_1, \ldots, q_m) \in \Gamma_m\).

From (2.11) and (2.13), we obtain

\[ A_1(x; q_1, \ldots, q_m) = A_2(x) \left[ \sum_{j=1}^{m} T(q_j) - 1 \right] + x \{1 - T(1) - (m - n)T(0)\} + T(0) \sum_{j=1}^{m} T(q_j) - mnT(0) - b_1(1; q_1, \ldots, q_m) \]. (2.17)

Now, from (2.3), (2.16) and (2.17), one can derive

\[ \sum_{j=1}^{m} [T(pq_j) + A_2(pq_j) + \{1 - T(1) + T(0)\} pq_j - T(0)] \\
- [T(p) + A_2(p) + \{1 - T(1) + T(0)\} p - T(0)] \\
\times \sum_{j=1}^{m} [T(q_j) + A_2(q_j) + \{1 - T(1) + T(0)\} q_j - T(0)] \\
+ [T(p) + A_2(p) + \{1 - T(1) + T(0)\} p - T(0)] [1 - T(1) - (m - 1)T(0)] = 0. \] (2.18)

The substitution \(p = 1\) in (2.18) gives (using \(A_2(1) = 0\):

\[ 1 - T(1) + T(0) = mT(0). \] (2.19)
Now, equations (2.18) and (2.19) give rise to
\[
\sum_{j=1}^{m} \left[ T(pq_j) + A_2(pq_j) + mT(0)pq_j - T(0) \right] \\
- \left[ T(p) + A_2(p) + mT(0)p - T(0) \right] \sum_{j=1}^{m} \left[ T(q_j) + A_2(q_j) + mT(0)q_j - T(0) \right] = 0.
\] (2.20)

Define a mapping \( M : I \to \mathbb{R} \) as
\[
M(x) = T(x) + A_2(x) + mT(0)x - T(0)
\] (2.21)
for all \( x \in I \). Putting \( x = 0 \) and \( x = 1 \) respectively in (2.20) and using the fact that \( A_2(1) = 0 \), we obtain
\[
M(0) = 0, M(1) = 1.
\] (2.22)

Also (2.20) and (2.21) give
\[
\sum_{j=1}^{m} [M(pq_j) - M(p)M(q_j)] = 0.
\]

By Result 1.2, there exists a mapping \( E : I \times \mathbb{R} \to \mathbb{R} \), additive in second variable, such that
\[
M(pq) - M(p)M(q) = E(p; q)
\] (2.23)
for all \( p \in I, q \in I \) and \( E(p; 1) = 0 \). The symmetry of the left hand side of (2.23), in \( p \) and \( q \), gives \( E(p; q) = E(q; p) \) for all \( p \in I, q \in I \). Consequently, \( E \) is also additive in the first variable. We may suppose that \( E(\cdot; q) \) has been extended additively to the whole of \( \mathbb{R} \).

For all \( p, q, r \in I \), (2.23) gives
\[
M(pqr) - M(p)M(q)M(r) = E(pqr; r) + M(r)E(p; q) \\
= E(qr; p) + M(p)E(q; r).
\] (2.24)

Now, we prove that \( E(p; q) \equiv 0 \) on \( I \times I \). To the contrary, suppose that \( E(p; q) \neq 0 \) on \( I \times I \). Then, there exist \( p^* \in I \) and \( q^* \in I \) such that \( E(p^*; q^*) \neq 0 \). Substituting \( p = p^* \), \( q = q^* \) in (2.24) and using \( E(p^*; q^*) \neq 0 \), it follows that
\[
M(r) = [E(p^*; q^*)]^{-1} \left[ E(q^* r; p^*) + M(p^*)E(q^*; r) - E(p^* q^*; r) \right]
\] (2.25)
for all \( r \in I \). The right hand side of (2.25) is additive. Hence \( M \) is also additive. Now, making use of (2.10), (2.21), (2.22) and the fact that \( A_2(1) = 0 \), we have
\[
1 \neq \sum_{i=1}^{m} T(r_i) = \sum_{i=1}^{m} M(r_i) - A_2(1) - mT(0) + mT(0) = M(1) = 1,
\]
a contradiction. Hence our supposition \( \text{“} E(p; q) \neq 0 \text{ on } I \times I \text{”} \) is false. So, \( E(p; q) = 0 \) for all \( p \in I, q \in I \). Making use of this fact in (2.23), we conclude that \( M \), defined by (2.21), is multiplicative with \( M(0) = 0 \) and \( M(1) = 1 \).

From (2.21), we have \( T(x) = M(x) = A_2(x) + mT(0)x + T(0) \). Define a mapping \( B : \mathbb{R} \to \mathbb{R} \) as \( B(x) = A_2(x) + mT(0)x \) for all \( x \in I \). Then \( B \) is additive with \( B(1) = mT(0) \). Thus, we have obtained the solution (2.2).

If the multiplicative mapping \( M : I \to \mathbb{R} \), with \( M(0) = 0 \), \( M(1) = 1 \), appearing in the solution (2.2), is also additive, then \( M \) is only of the form \( M(p) = p \) for all \( p \in I \). So, (2.2) reduces to \( T(p) = p - B(p) + T(0) \). Making use of (2.10), we have
\[
1 \neq \sum_{i=1}^{m} T(r_i) = \sum_{i=1}^{m} [r_i + B(r_i) + T(0)] = 1 - B(1) + mT(0) = 1,
\]
a contradiction. Hence \( M \) is not additive. This completes the proof of the theorem.\[\square\]
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