Palestine Journal of Mathematics

Vol. 6(2)(2017) , 624–637

ASYMPTOTIC STABILITY OF THE COUPLED WAVE EQUATION ON COMPACT MANIFOLDS AND LOCALLY DISTRIBUTED VISCOELASTIC DISSIPATION WITH A DELAY TERM

Mohamed Ferhat and Ali Hakem

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 35L05; 35L15; Secondary 35L70; 93D15.

Keywords and phrases: Wave equation, compact Riemannian manifold, viscoelastic distributed damping.

Abstract. We establish uniform decay rate estimates of the wave equation on a compact Riemannian manifold (M, g) subject to locally distributed viscoelastic effects on a subset $\omega \subset M$. Assuming that the well-known geometric control condition holds and by employing the inverse observability property introduced in [13], we extend the prior results in the literature due to Cavalcanti [1].

1 Introduction

In this paper, we consider the following problem :

$$u_{tt} - k_0\Delta u + \int_0^t g_1(t-s)div\left[a_1(x)\nabla u(s)\right]ds + \sum_{i=1}^{2}\mu_i u_t(x, t-\tau(i)) + f_1(u, v) = 0,\ M \times (0, +\infty), \tag{1.1}$$

$$v_{tt} - k_1\Delta v + \int_0^t g_2(t-s)div\left[a_2(x)\nabla v(s)\right]ds + \sum_{i=1}^{2}\alpha_i v_t(x, t-\tau(i)) + f_2(u, v) = 0,\ M \times (0, +\infty), \tag{1.2}$$

$$u(x,t) = 0,\ v(x,t) = 0, \qquad \partial M \times (0, +\infty), \tag{1.3}$$

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ u_t(x,0) = u_1(x), \qquad x \in M, \tag{1.4}$$

$$u_t(x, t-\tau(2)) = \phi_0(x, t-\tau(2)),\ t \in (0, \tau_2) \qquad x \in M, \tag{1.5}$$

$$v_t(x, t-\tau(2)) = \phi_1(x, t-\tau(2)),\ ,\ \tau(1) = 0,\ \tau(2) = \tau_2, \quad t \in (0, \tau_2), \qquad x \in M. \tag{1.6}$$

Where (M, g) is n-dimensional compact Riemannian manifold with boundary ∂M and g denotes a Riemannian metric of class C^∞. We denote by ∇ the Levi-Civita connection on M and by Δ the Laplace-Beltrami operator on M, where $k_0, k_1 > 0$, $a_1(x) > a_{01} > 0$, $a_2(x) > a_{02} > 0$ in a subset $\omega \subset M$. Assuming that the well-known geometric control condition (ω, T_0) holds and $g_1, g_2 : R^+ \to R^+$, $\phi_i(.,.) : R^2 \to R$ $i = 1, 2$, are given functions which will be specified later, $\tau_2 > 0$ is a time delay, where $\mu_1, \alpha_1, \alpha_2, \mu_2$ are positive real numbers and the initial data $(u_0, u_1, \phi_0), (v_0, v_1, \phi_1)$ belonging to a suitable space. To motivate our work, let us recall some results regarding coupled viscoelastic wave equations. Wenjun [2] proved the energy decay result using the perturbed energy method for the system:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \gamma_1\Delta u_{tt} + \displaystyle\int_0^t g_1(t-s)\Delta u(s)ds + f_1(x, u) = 0, & in\ \Omega \times (0, +\infty), \\ |v_t|^\rho v_{tt} - \Delta v - \gamma_2\Delta v_{tt} + \displaystyle\int_0^t g_2(t-s)\Delta v(s)ds + f_2(x, u) = 0, & in\ \Omega \times (0, +\infty), \end{cases} \tag{1.7}$$

where Ω is a bounded domain in R^n $(n \geq 1)$with a smooth boundary $\partial\Omega, \gamma_1, \gamma_2 \geq 0$ are constants and ρ is a real number such that $0 < \rho < \frac{2}{(n-2)}$ if $n \geq 3$ or $\rho > 0$ if $n = 1, 2$. The functions u_0, u_1, υ_0 and υ_1 are given initial data. The relaxations functions g_1 and g_2 are continuous functions and $f_1(u, \upsilon), f_2(u, \upsilon)$ represent the nonlinear terms. Many authors considered the initial boundary value problem as follows

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g_1(t-s)\Delta u(s)ds + h_1(u_t) = f_1(x,u), & in\ \Omega \times (0, +\infty), \\ \upsilon_{tt} - \Delta \upsilon + \displaystyle\int_0^t g_2(t-s)\Delta \upsilon(s)ds + h_2(\upsilon_t) = f_2(x,u), & in\ \Omega \times (0, +\infty), \end{cases} \tag{1.8}$$

when the viscoelastic terms g_i $(i = 1, 2.)$ are not taken into account in (1.8). Agre and Rammaha [3] obtained several results related to local and global existence of a weak solution. By using the same technique as in [4], they showed that any weak solution blow-up in finite time with negative initial energy. Later Said-Houari [5] extended this blow up result to positive initial energy. Conversely, in the presence of the memory term $(g_i \neq 0$ $(i = 1, 2.)$, there are numerous results related to the asymptotic behavior and blow up of solutions of viscoelastic systems. For example, Liang and Gao [6] studied problem (1.8) with $h_1(u_t) = -\Delta u_t$, $h_2(\upsilon_t) = -\Delta u_t$. They obtained that, under suitable conditions on the functions $g_i, f_i, i = 1, 2$, and certain initial data in the stable set, the decay rate of the energy functions is exponential. On the contrary, for certain initial data in the unstable set, there are solutions with positive initial energy that blow-up in finite time. For $h_1(u_t) = |u_t|^{m-1}u_t$ and $h_2(\upsilon_t) = |\upsilon_t|^{r-1}\upsilon_t$, Hun and Wang [7] established several results related to local existence, global existence and finite time blow-up (the initial energy $E(0) < 0$).
This latter has been improved by Messaoudi and Said-Houari [8] by considering a larger class of initial data for which the initial energy can take positive values, on the other hand, Messaoudi and Tatar [9] considered the following problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g_1(t-s)\Delta u(s)ds + f_1(x,u) = 0, & in\ \Omega \times (0, +\infty), \\ \upsilon_{tt} - \Delta \upsilon + \displaystyle\int_0^t g_2(t-s)\Delta \upsilon(s)ds + f_2(x,u) = 0, & in\ \Omega \times (0, +\infty), \end{cases} \tag{1.9}$$

where the functions f_1 *and* f_2 satisfy the following assumptions

$$|f_1(u, \upsilon)| \leq d(|u|^{\beta_1} + |\upsilon|^{\beta_2}),$$
$$|f_2(u, \upsilon)| \leq d(|u|^{\beta_3} + |\upsilon|^{\beta_4}),$$

for some constant $d > 0$ and $\beta_i \geq 0, \beta_i \leq \frac{n}{(n-2)}$, $i = 1, 2, 3, 4$. They obtained that the solution goes to zero with an exponential or polynomial rate, depending on the decay rate of the relaxation functions $g_i, i = 1, 2$. Muhammad I.M [11] considered the following problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g_1(t-s)\Delta u(s)ds + f_1(\upsilon,u) = 0, & in\ \Omega \times (0, +\infty), \\ \upsilon_{tt} - \Delta \upsilon + \displaystyle\int_0^t g_2(t-s)\Delta \upsilon(s)ds + f_2(\upsilon,u) = 0, & in\ \Omega \times (0, +\infty), \end{cases} \tag{1.10}$$

and proved the well-posedness and energy decay result for wider class of relaxation functions.
Motivated by the previous works, on Riemannian compact manifolds, we need to use the inverse observability property introduced in [14] and a unique continuation property. Indeed, we are assuming that the geometric control condition holds, namely, that the geodesics of $\overline{M}$ have no contact of infinite order with ∂M and that there exists $T_0 > 0$ such that every geodesic traveling

at speed 1 and issued at $t = 0$ meets $\overline{\omega}$ in a time $t < T_0$. Then, we guarantee the existence of constant $C = C(T_0, \omega)$ such that

$$E(0) \leq C \int_0^{T_0} \int_\omega \left\{ |u_t(x,t)|^2 + |v_t(x,t)|^2 \right\} dMdt. \tag{1.11}$$

Under suitable assumptions on the functions $g_i(.)$, $f_i(.,.)(i = 1.2)$, the initial data and the parameters in the equations, we establish asymptotic behavior of solutions to (1.1)-(1.6). Our work is organized as follows. In section 2, we present the preliminaries and some lemmas. In section 3, decay property is derived.

2 Preliminary Results

In this section, we present some material for the proof of our result. For the relaxation function g_i, we assume

(A_0) : The relaxations functions g_1 and g_2 are of class C^1 and satisfy, for $s \geq 0$

$$g_1'(t) \leq -rg_1(t), \ \forall t \geq 0, \quad g_2'(t) \leq -rg_2(t), \ \forall t \geq 0.$$

To obtain the stabilization of problem (1.1)-(1.6), we shall need the following geometrical assumption:

(A_1) : (Geometric control condition). If M is a manifold with boundary, we assume that the geodesics of M have no contact of infinite order with ∂M. Let ω' be an open subset of M and consider that there exists $T_0 > 0$ such that every geodesic traveling at speed 1 and issued at $t = 0$ meets $\overline{\omega'}$ in a time $t < T_0$. We also assume that $a_1, a_2 \in C^\infty(M)$ are nonnegative functions such that

$$a_1(x) \geq a_{01} > 0 \quad in \quad \omega, \tag{2.1}$$

$$a_2(x) \geq a_{02} > 0 \quad in \quad \omega, \tag{2.2}$$

where ω is an open subset verifying $\overline{\omega'} \subset \omega$. If $\partial M \neq \emptyset$ we define $\sum_T = M \times]0, T[$ and we set

$$H_0^1(M) := \left\{ v \in H^1(M); v/_{\partial M} = 0 \right\},$$

which is a Hilbert space with the topology endowed by $H^1(M)$, the condition $v/_{\partial M} = 0$ is required to guarantee the Poincaré's inequality

$$\|u\|_{L^2(M)} \leq (\lambda_1)^{-1} \|\nabla u\|_{L^2(M)} \quad for \quad u \in H_0^1(M), \tag{2.3}$$

where λ_1 is the first eigenvalue of the Laplace-Beltrami operator for the Dirichlet problem.
Take f_1,f_2 as in [10]

$$f_1(u,v) = a|u+v|^{p-1}(u+v) + b|u|^{\frac{p-3}{2}}|v|^{\frac{p+1}{2}}u, \tag{2.4}$$

$$f_2(u,v) = a|u+v|^{p-1}(u+v) + b|v|^{\frac{p-3}{2}}|u|^{\frac{p+1}{2}}v, \tag{2.5}$$

with $a, b > 0$. Further, one can easily verify that

$$uf_1(u,v) + vf_2(u,v) = (p+1)F(u,v), \forall (u,v) \in R^2.$$

Where

$$F(u,v) = \frac{1}{(p+1)}(a|u+v|^{p+1} + 2b|uv|^{\frac{p+1}{2}}), \quad f_1(u,v) = \frac{\partial F}{\partial u}, \quad f_2(u,v) = \frac{\partial F}{\partial v}.$$

(A_2) : There exists $c_0, c_1 > 0$, such that

$$c_0(|u|^{p+1} + |v|^{p+1}) \leq F(u,v) \leq c_1(|u|^{p+1} + |v|^{p+1}), \quad \forall (u,v) \in R^2.$$

and

$$\left|\frac{\partial f_i}{\partial u}(u,v)\right| + \left|\frac{\partial f_i}{\partial v}(u,v)\right| \le C(|u|^{p-1} + |v|^{p-1}), \quad i = 1,2 \quad where \quad 1 \le p < 6.$$

(A_3) :

$$If\ n = 1,2; \quad p \ge 3 \quad if\ n = 3; \quad p = 3. \tag{2.6}$$

Lemma 2.1. [2] *For any $g \in C^1$ and $\varphi \in H^1(0,T)$, we have*

$$\int_0^t \int_M g_i(t-s)\varphi(x,s)\varphi_t(x,t)dxds = -\frac{1}{2}\frac{d}{dt}\left((g_i o\varphi)(t) + \int_0^t g_i(s)ds\|\varphi(t)\|_2^2\right)$$

$$- g_i(t)\|\varphi(t)\|_2^2 + (g_i' o\varphi)(t),$$

where

$$(g_i * \varphi)(t) := \int_0^t g_i(t-s)\varphi(x,s)ds,$$

$$(g_i o\varphi)(t) = \int_0^t g_i(t-s)\int_M |\varphi(x,s) - \varphi(x,t)|^2 dMds,$$

and

$$\|\varphi(t)\|_2^2 = \int_M |\varphi(x,t)|^2 dM.$$

Remark 2.2. Avoiding the complexity of the matter, we take $a = b = 1$ in $(2.4) - (2.5)$.

3 Asymptotic stability

In order to prove our stability result of solutions of problem (1.1)-(1.6), we introduce the new variables z_1, z_2 as in [12]

$$z_1(x,k_1,t) = u_t(x,t-\tau_2 k_1), \quad x \in M,\ k_1 \in (0,1),$$

$$z_2(x,k_2,t) = u_t(x,t-\tau_2 k_2), \quad x \in M,\ k_2 \in (0,1),$$

which implies that

$$\tau_2 z_1'(x,k_1,t) + z_{k_1}(x,k_1,t) = 0, \quad in\ M \times (0,1) \times (0,\infty),$$

$$\tau_2 z_2'(x,k_2,t) + z_{k_2}(x,k_2,t) = 0, \quad in\ M \times (0,1) \times (0,\infty),$$

therefore, problem (1.1)-(1.6) is equivalent to

$$\begin{cases} u_{tt} - k_0\Delta u + \int_0^t g_1(t-s)div[a_1(x)\Delta u(s)]ds & \\ +\mu_1 u_t(x,t) + \mu_2 z_1(x,1,t) + f_1(u,v) = 0, & in \quad M \times (0,\infty), \\ v_{tt} - k_1\Delta v + \int_0^t g_2(t-s)div[a_2(x)\Delta u(s)]ds & \\ +\alpha_1 v_t(x,t) + \alpha_2 z_2(x,1,t) + f_2(u,v) = 0, & in \quad M \times (0,\infty), \\ & \\ \tau_2 z_1'(x,k_1,t) + z_{k_1}(x,k_1,t) = 0, & in \quad M \times (0,1) \times (0,\infty), \\ \tau_2 z_2'(x,k_2,t) + z_{k_2}(x,k_2,t) = 0, & in \quad M \times (0,1) \times (0,\infty), \\ z_1(x,0,t) = u_t(x,t), & x \in M, t > 0, \\ z_2(x,0,t) = v_t(x,t), & x \in M, t > 0, \\ z_1(x,k_1,0) = \phi_0(x,-\tau_2 k_1), & x \in M, \\ z_2(x,k_2,0) = \phi_1(x,-\tau_2 k_2), & x \in M, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in M, \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), & x \in M, \\ u(x,t) = 0, v(x,t) = 0, & x \in \partial M, t \ge 0. \end{cases} \tag{3.1}$$

In the following, we will give sufficient conditions for the well-posedness of problem (3.1) which can be established by using the Fadeo-Galerkin's method.

Theorem 3.1. *Let* $(u_0, \upsilon_0) \in \left(H_0^1(M) \cap H^2(M)\right)^2$, $(u_1, \upsilon_1) \in (H_0^1(M))^2$ *and* $(\phi_0, \phi_1) \in \left(L^2(M \times (0,1)\right)^2$ *satisfying the compatibility conditions*

$$\phi_0 = (.,0) = u_1, \quad \phi_1 = (.,0) = \upsilon_1.$$

Assume that the hypotheses $(A_0)-(A_3)$ *hold. Then there exists a unique weak solution* $((u, z_1), (\upsilon, z_2))$ *of* (3.1) *such that*

$$u(t), \upsilon(t) \in C\left([-\tau_2, T]; H_0^1(M)\right) \cap C^1\left([-\tau_2, T]; L^2(M)\right),$$

$$u_t(t), \upsilon_t(t) \in L^2\left([-\tau_2(0), T]; H_0^1(\Omega)\right) \cap L^2\left([-\tau_2(0), T] \times \Omega\right),$$

for $T > 0$.

We define the energy of the solution associated with (3.1) by the following formula:

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\upsilon_t(t)\|_2^2 + \frac{1}{2}\left(k_1 - a_1(x)\int_0^t g_1(s)ds\right)\|\nabla u(t)\|_2^2 \\ &+ \frac{1}{2}\left(k_2 - a_2(x)\int_0^t g_2(s)ds\right)\|\nabla \upsilon(t)\|_2^2 + \frac{\xi_1}{2}\int_M\int_0^1 z_1^2(x,k_1,t)dk_1 dM \\ &+ \frac{\xi_2}{2}\int_M\int_0^1 z_2^2(x,k_2,t)dk_2 dM + \int_M F(u,\upsilon)dM + \frac{a_1(x)}{2}(g_1 o \nabla u)(t) \\ &+ \frac{a_2(x)}{2}(g_2 o \nabla \upsilon)(t). \end{aligned} \tag{3.2}$$

Now, let ξ_1, ξ_2 be positive constants such that

$$\tau_2\mu_2 < \xi_1 < \tau_2(2\mu_1 - \mu_2), \tag{3.3}$$

$$\tau_2\alpha_2 < \xi_1 < \tau(2\alpha_1 - \alpha_2). \tag{3.4}$$

Lemma 3.2. *The energy of problem (3.2) satisfies the following inequality*

$$\begin{aligned} E'(t) \leq &-\left(\mu_1 - \frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2}\right)\int_M |u_t|^2 dM - \left(\alpha_1 - \frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2}\right)\int_M |\upsilon_t|^2 dM \\ &-\left(\frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2}\right)\int_M z_1^2(x,1,s)dM - \left(\frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2}\right)\int_M z_2^2(x,1,s)dM \\ &+\int_M \left\{a_1(x)\left\{(g_1' \circ \nabla u) - g_1(t)|\nabla u|^2\right\} + a_2(x)\left\{(g_2' \circ \nabla \upsilon) - g_2(t)|\nabla \upsilon|^2\right\}\right\} dM. \end{aligned} \tag{3.5}$$

Proof. Multiplying the first equation in (3.1) by u_t and the second equation in (3.1) by υ_t, integrating over M, using integration by part and Green's formula, we obtain

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\upsilon_t(t)\|_2^2 + \frac{k_0}{2}\|\nabla u(t)\|_2^2 + \frac{k_1}{2}\|\nabla \upsilon(t)\|_2^2 + \int_M F(u,\upsilon)dM\right] \\ &+ \frac{\mu_1}{2}\|u_t(t)\|_2^2 + \frac{\alpha_1}{2}\|\upsilon_t(t)\|_2^2 + \mu_2\int_M z_1(x,1,t)u_t(x,t)dM \\ &+ \alpha_2\int_M z_2(x,1,t)\upsilon_t(x,t)dM - \int_0^t a_1(x)g_1(t-s)\int_M \nabla u(s)\nabla u_t(t)dMds \\ &- \int_0^t a_2(x)g_2(t-s)\int_M \nabla \upsilon(s)\nabla \upsilon_t(t)dMds = 0. \end{aligned} \tag{3.6}$$

Using a lemma 2.1 and integrating (3.6) over $(0, t)$, we get

$$
\begin{aligned}
&\frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\upsilon_t(t)\|_2^2 + \int_M F(u,\upsilon)dM + \frac{1}{2}\left(k_0 - a_1(x)\int_0^t g_1(s)ds\right)\|\nabla u(t)\|_2^2 \\
&+ \frac{a_1(x)}{2}(g_1 o \nabla u)(t) + \frac{a_2(x)}{2}(g_2 o \nabla \upsilon)(t) + \mu_1 \int_0^t \|u_s(s)\|_2^2 ds + \alpha_1 \int_0^t \|\upsilon_s(s)\|_2^2 ds \\
&+ \mu_2 \int_0^t \int_M z_1(x,1,s)u_s(s)dMds + \alpha_2 \int_0^t \int_M z_2(x,1,s)\upsilon_s(s)dMds \\
&- \frac{a_1(x)}{2}\int_0^t (g_1' o \nabla u)(s)ds + \frac{a_1(x)}{2}\int_0^t g_1(s)\|\nabla u(t)\|_2^2 ds + \frac{a_2(x)}{2}\int_0^t g_2(s)\|\nabla \upsilon(t)\|_2^2 ds \\
&- \frac{a_2(x)}{2}\int_0^t (g_2' o \nabla \upsilon)(s)ds + \frac{1}{2}\left(k_1 - a_2(x)\int_0^t g_2(s)ds\right)\|\nabla \upsilon(t)\|_2^2 = 0,
\end{aligned}
\tag{3.7}
$$

we multiply the third equation in (3.1) by $\frac{\xi_1}{\tau_2} z_{k_1}(t)$ and the forth equation in (3.1) by $\frac{\xi_2}{\tau_2} z_{k_2}(t)$ and integrating the result over $M \times (0,1)$ to obtain

$$
\begin{aligned}
\xi_1 \int_M \int_0^1 z_1' z_1(x,k_1,t)dk_1 dM &= \frac{-\xi_1}{2\tau_2}\int_M \int_0^1 \frac{\partial}{\partial k_1} z_1^2(x,k_1,t)dk_1 dM, \\
&= \frac{-\xi_1}{2\tau_2}\int_M z_1^2(x,1,t) - z_1^2(x,0,t))dM,
\end{aligned}
\tag{3.8}
$$

then

$$
\frac{\xi_1}{2}\frac{d}{dt}\int_M \int_0^1 z_1^2(x,k_1,t)dk_1 dM = -\frac{\xi_1}{2\tau_2}\int_M z_1^2(x,k_1,t)dM + \frac{\xi_1}{2\tau}\|u_t\|_2^2,
\tag{3.9}
$$

in the same manner for the second equation

$$
\frac{\xi_2}{2}\frac{d}{dt}\int_M \int_0^1 z_2^2(x,k_2,t)dk_2 dM = -\frac{\xi_2}{2\tau_2}\int_M z_2^2(x,k_2,t)dM + \frac{\xi_2}{2\tau}\|\upsilon_t\|_2^2.
\tag{3.10}
$$

Summing (3.7), (3.9) and (3.10), we get

$$
\begin{aligned}
&E(t) + \left(\mu_1 - \frac{\xi_1}{2\tau_2}\right)\int_0^t \|u_s\|_2^2 ds + \left(\alpha_1 - \frac{\xi_2}{2\tau_2}\right)\int_0^t \|\upsilon_s\|_2^2 ds \\
&+ \frac{\xi_2}{2}\int_0^t \int_M z_2^2(x,1,s)dMds + \mu_2 \int_M \int_0^t z_2^2(x,1,s)\upsilon_s(x,t)dMds \\
&+ \alpha_2 \int_M \int_0^t z_2^2(x,1,s)\upsilon_s(x,s)dMds + \frac{\xi_1}{2}\int_0^t \int_M z_1^2(x,1,s)dMds \\
&+ \int_0^t \int_M a_1(x)\left\{(g_1' \circ \nabla u) - g_1(t)|\nabla u|^2\right\} dMdt \\
&+ \int_0^t \int_M a_2(x)\left\{(g_2' \circ \nabla \upsilon) - g_2(t)|\nabla \upsilon|^2\right\} dMdt = E(0),
\end{aligned}
\tag{3.11}
$$

using Young and Cauchy-Schwartz inequalities, we obtain

$$
\begin{aligned}
&E(t) + \left(\mu_1 - \frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2}\right)\int_0^t \|u_s\|_2^2 ds + \left(\alpha_1 - \frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2}\right)\int_0^t \|\upsilon_s\|_2^2 ds \\
&+ \left(\frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2}\right)\int_0^t \int_M z_1^2(x,1,s)dMds + \left(\frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2}\right)\int_0^t \int_M z_2^2(x,1,s)dMds \\
&+ \int_0^t \int_M a_1(x)\left\{(g_1' \circ \nabla u) - g_1(t)|\nabla u|^2\right\} dMdt \\
&+ \int_0^t \int_M a_2(x)\left\{(g_2' \circ \nabla \upsilon) - g_2(t)|\nabla \upsilon|^2\right\} dMdt = E(0).
\end{aligned}
\tag{3.12}
$$

This completes the proof of lemma 3.1. □

Remark 3.3. Due to the conditions (3.8), (3.9) we have

$$\left(\mu_1 - \frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2}\right) > 0, \ \left(\alpha_1 - \frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2}\right) > 0, \ \left(\frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2}\right) > 0, \ \left(\frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2}\right) > 0.$$

As a consequence of Lemma 3.1, every solution of (3.1) satisfies the following identity

$$\begin{aligned} E(t_2) - E(t_1) \le &- \int_{t_1}^{t_2} \left\{ \left(\mu_1 - \frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2}\right) \int_M |u_t|^2 dM - \left(\alpha_1 - \frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2}\right) \int_M |\upsilon_t|^2 dM \right\} dt \\ &- \int_{t_1}^{t_2} \left\{ \left(\frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2}\right) \int_M z_1^2(x,1,s) dM - \left(\frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2}\right) \int_M z_2^2(x,1,s) dM \right\} dt \\ &+ \int_{t_1}^{t_2} \int_M a_1(x) \left\{ (g_1' \circ \nabla u) - g_1(t) |\nabla u|^2 \right\} dM dt \\ &+ \int_0^t \int_M a_2(x) \left\{ (g_2' \circ \nabla \upsilon) - g_2(t) |\nabla \upsilon|^2 \right\} dM dt, \quad \forall t_2 > t_2 \ge 0. \end{aligned} \tag{3.13}$$

Note that

$$0 < k_0 - \|a_1\|_{L^\infty} \int_0^\infty g_1(s) ds \le k_0 - \|a_1\|_{L^\infty} \int_0^t g_1(s) ds \le k_0, \quad \forall (x,t) \in M \times R_+, \tag{3.14}$$

$$0 < k_1 - \|a_2\|_{L^\infty} \int_0^\infty g_2(s) ds \le k_1 - \|a_2\|_{L^\infty} \int_0^t g_2(s) ds \le k_1, \quad \forall (x,t) \in M \times R_+. \tag{3.15}$$

Theorem 3.4. *Suppose that $\mu_2 < \mu_1$, $\alpha_2 < \alpha_1$, (A_0)-(A_3) hold. Assume that $((u_0, u_1), (\upsilon_0, \upsilon_1)) \in (H_0^1(M))^2$ and $(\phi_0, \phi_1) \in (L^2(M \times (0,1)))^2$. Then we have the following decay property*

$$E(t) \le C_0\, e^{-\gamma t}\, E(0), \quad \forall t \ge T_0,$$

where c, ω, are positive constants, independent of the initial data and $E_k(0) \le L_1$.

Our goal is to prove the inequality below

$$\begin{aligned} E(T) \le &\int_0^T \left\{ \left(\mu_1 - \frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2}\right) \int_M |u_t|^2 dM + \left(\alpha_1 - \frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2}\right) \int_M |\upsilon_t|^2 dM \right\} dt \\ &+ \int_0^T \left\{ \left(\frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2}\right) \int_M z_1^2(x,1,s) dM + \left(\frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2}\right) \int_M z_2^2(x,1,s) dM \right\} dt \\ &+ \int_0^T \int_M a_1(x) \left\{ (g_1' \circ \nabla u) - g_1(t) |\nabla u|^2 \right\} dM dt \\ &+ \int_0^t \int_M a_2(x) \left\{ (g_2' \circ \nabla \upsilon) - g_2(t) |\nabla \upsilon|^2 \right\} dM dt, \quad \forall\, T > T_0, \end{aligned} \tag{3.16}$$

where the initial (u_0, u_1, z_0) data are taken in bounded sets of $H_0^1(M) \times L^2(M)$. To do this, we must prove the lemma:

Lemma 3.5. *Let us assume hypotheses (A_0). For all $T > T_0$ and $\forall\, L_1 > 0$, there exists a positive* *constant $C = C(T, L_1)$, such that*

$$
\begin{aligned}
E(0) \le & \int_0^T \left\{ \left(\mu_1 - \frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2} \right) \int_M |u_t|^2 dM + \left(\alpha_1 - \frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2} \right) \int_M |v_t|^2 dM \right\} dt \\
& + \int_0^T \left\{ \left(\frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2} \right) \int_M z_1^2(x,1,s) dM + \left(\frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2} \right) \int_M z_2^2(x,1,s) dM \right\} dt \\
& + \int_0^T \int_M a_1(x) \left\{ (g_1' \circ \nabla u) - g_1(t) |\nabla u|^2 \right\} dM dt \\
& + \int_0^t \int_M a_2(x) \left\{ (g_2' \circ \nabla v) - g_2(t) |\nabla v|^2 \right\} dM dt, \quad \forall\, T > T_0,
\end{aligned}
\tag{3.17}
$$

holds for every solution $((u, z_1), (v, z_2))$ of problem (3.1) provided that the initial data satisfies

$$
E(0) \le L_1. \tag{3.18}
$$

Proof. We argue by contradiction by using compactness-uniqueness argument. Suppose that (3.17) is not verified and let $(u_{k0}, u'_{k0}, v_{k0}, v'_{k0})$ be a sequence of initial data where the corresponding solutions $((u_k, z_{1k}), (v_k, z_{2k}))_{k \in N}$ of (3.1) with $E_k(0)$, assumed uniformly bounded in k, verifies

$$
\lim_{k \to +\infty} \frac{E_k(0)}{\left(\begin{array}{c} \displaystyle\int_0^T \left\{ \left(\mu_1 - \frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2} \right) \int_M |u_{tk}|^2 dM + \left(\alpha_1 - \frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2} \right) \int_M |v_{tk}|^2 dM \right\} dt \\ \displaystyle + \int_0^T \left\{ \left(\frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2} \right) \int_M z_{1k}^2(x,1,s) dM + \left(\frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2} \right) \int_M z_{2k}^2(x,1,s) dM \right\} dt \\ \displaystyle + \int_0^T \int_M a_1(x) \left\{ (-g_1' \circ \nabla u_k) + g_1(t) |\nabla u_k|^2 \right\} dM dt \\ \displaystyle + \int_0^T a_2(x) \left\{ (-g_2' \circ \nabla v_k) + g_2(t) |\nabla v_k|^2 \right\} dM dt \end{array} \right)} = +\infty
\tag{3.19}
$$

$$
\lim_{k \to +\infty} \frac{\left(\begin{array}{c} \displaystyle\int_0^T \left\{ \left(\mu_1 - \frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2} \right) \int_M |u_{tk}|^2 dM + \left(\alpha_1 - \frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2} \right) \int_M |v_{tk}|^2 dM \right\} dt \\ \displaystyle + \int_0^T \left\{ \left(\frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2} \right) \int_M z_{1k}^2(x,1,s) dM + \left(\frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2} \right) \int_M z_{2k}^2(x,1,s) dM \right\} dt \\ \displaystyle + \int_0^T \int_M a_1(x) \left\{ (-g_1' \circ \nabla u_k) + g_1(t) |\nabla u_k|^2 \right\} dM dt \\ \displaystyle + \int_0^T a_2(x) \left\{ (-g_2' \circ \nabla v) + g_2(t) |\nabla v_k|^2 \right\} dM dt \end{array} \right)}{E_k(0)} = 0
\tag{3.20}
$$

Since $E_k(t) \le E_k(0) \le L_1$ where L_1 is positive constant, we obtain a subsequence, still denoted by $\{u_k\}$, $\{v_k\}$, $\{z_{1k}\}$, $\{z_{2k}\}$. We observe that there exists a subsequence $(u_k, z_{1k}), (v_k, z_{2k})$ such that

$$
u_k \to u \ \textit{weakly star in}\ L^\infty(0,T; H^2(M) \cap H_0^1(M)), \tag{3.21}
$$

$$
v_k \to v \ \textit{weakly star in}\ L^\infty(0,T; H^2(M) \cap H_0^1(M)), \tag{3.22}
$$

$$
u_{tk} \to u_t \ \textit{weakly star in}\ L^\infty(0,T; H_0^1(M)), \tag{3.23}
$$

$$
v_{tk} \to v_t \ \textit{weakly star in}\ L^\infty(0,T; H_0^1(M)), \tag{3.24}
$$

$$
z_{1k}(x,1,t) \to \psi_1 \ \textit{weakly star in}\ L^2(M \times (0,T)), \tag{3.25}
$$

$$
z_{2k}(x,1,t) \to \psi_2 \ \textit{weakly star in}\ L^2(M \times (0,T)). \tag{3.26}
$$

Further, by Aubin's lemma [15], it follows from (3.21) and (3.22) that there exists a subsequence (u_k, υ_k) still represented by the same notation, such that

$$u_k \to u \ strongly \ in \ L^2(0,T;L^2(M)), \tag{3.27}$$

$$\upsilon_k \to \upsilon \ strongly \ in \ L^2(0,T;L^2(M)), \tag{3.28}$$

Then

$$u_k \to u \ and \ \upsilon_k \to \upsilon \ a.e \ in \ (0,T)\times M, \tag{3.29}$$

and

$$u_{tk} \to u_t \ and \ \upsilon_{tk} \to \upsilon_t \ a.e \ in \ (0,T)\times M. \tag{3.30}$$

(i) **Analysis of nonlinear term 1**

$$\begin{aligned}
\|f_1(u_k,\upsilon_k)\|_{L^2(M\times(0,T))} &= \int_0^T \int_M (|u_k(s)|^p + |\upsilon_k(s)|^p + |u_k(s)|^{\frac{p-1}{2}}|\upsilon_k(s)|^{\frac{p+1}{2}})dsdM,\\
&\le c_s^p \int_0^T \|\nabla u_k(s)\|^p ds + c_s^p \int_0^T \|\nabla \upsilon_k(s)\|^p ds,\\
&+ c_s^{\frac{p-1}{2}} \int_0^T \|\nabla u_k(s)\|^{\frac{p-1}{2}} ds + c_s^{\frac{p+1}{2}} \int_0^T \|\nabla \upsilon_k(s)\|^{\frac{p+1}{2}} ds,\\
&\le 2c_s^p T L_1^P + c_s^{\frac{p-1}{2}} T L_1^{\frac{p-1}{2}} T L_1^{\frac{p-1}{2}} + c_s^{\frac{p+1}{2}} T L_1^{\frac{p+1}{2}} T L_1^{\frac{p+1}{2}} = C.
\end{aligned} \tag{3.31}$$

In the same way for $f_2(u_k,\upsilon_k)$

$$\|f_2(u_k,\upsilon_k)\|_{L^2(M\times(0,T))} \le C. \tag{3.32}$$

From the (3.31) and (3.32) we deduce that

$$\begin{aligned}
f_1(u_k,\upsilon_k) &\to f_1(u,\upsilon) \ weakly \ in \ L^2(0,T;L^2(M)),\\
f_1(u_k,\upsilon_k) &\to f_1(u,\upsilon) \ weakly \ in \ L^2(0,T;L^2(M)).
\end{aligned} \tag{3.33}$$

For suitable functions $u,\upsilon \in L^\infty(0,T;H_0^1(M))$, $z_1,z_2 \in L^\infty(0,T;L^2(M\times(0.1))$, $\psi_1,\psi_2 \in L^2(M\times(0,T))$. At this point we will divide our proof into two cases, namely $u=0$, $\upsilon=0$ and $u\neq 0$, $\upsilon \neq 0$.

Case 1: $u\neq 0$, $\upsilon\neq 0$: We observe that $E_k(0)\le L_1, \quad \forall k\in N$

$$\lim_{k\to+\infty}\left(\begin{array}{c}
\int_0^T \left\{\left(\mu_1 - \frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2}\right)\int_M |u_{tk}|^2 dM + \left(\alpha_1 - \frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2}\right)\int_M |\upsilon_{tk}|^2 dM\right\} dt\\
+\int_0^T \left\{\left(\frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2}\right)\int_M z_{1k}^2(x,1,s)dM + \left(\frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2}\right)\int_M z_{2k}^2(x,1,s)dM\right\} dt\\
+\int_0^T\int_M a_1(x)\left\{(-g_1'\circ\nabla u_k) + g_1(t)|\nabla u_k|^2\right\} dMdt\\
+\int_0^T a_2(x)\left\{(-g_2'\circ\nabla \upsilon_k) + g_2(t)|\nabla \upsilon_k|^2\right\} dMdt
\end{array}\right) = 0. \tag{3.34}$$

From (3.34), we deduce that

$$g_1(t)a_1(x)|\nabla u_k|^2 + \int_0^t (-g_1'(t-s)a_1(.)|\nabla u_k(.,t) - \nabla u_k(.,s)|^2 ds \to 0 \ in \ L^1(M\times(0,T)), \tag{3.35}$$

$$g_2(t)a_2(x)|\nabla v_k|^2 + \int_0^t (-g_2'(t-s)a_2(.)|\nabla v_k(.,t) - \nabla v_k(.,s)|^2 ds \to 0 \ in \ L^1(M \times (0,T)). \tag{3.36}$$

Since

$$a_1(x) > a_{01} > 0 \quad in\ \omega, \tag{3.37}$$

$$a_2(x) > a_{02} > 0 \quad in\ \omega, \tag{3.38}$$

$$g_1(t) \geq g_1(T), \quad \forall t \in [0,T], \tag{3.39}$$

$$g_2(t) \geq g_2(T), \quad \forall t \in [0,T]. \tag{3.40}$$

Following the same ideas in [1], we deduce that

$$\begin{aligned}
&\int_0^T \int_M a_1(x) g_1(t) |\nabla u_k|^2 dM dt \\
&= \int_0^T \int_{M\setminus\omega} a_1(x) g_1(t) |\nabla u_k|^2 dM dt + \int_0^T \int_\omega a_1(x) g_1(t) |\nabla u_k|^2 dM dt \\
&\geq \int_0^T \int_\omega a_1(x) g_1(t) |\nabla u_k|^2 dM dt \\
&\geq a_{01} g_1(T) \int_0^T \int_\omega |\nabla u_k|^2 dM dt,
\end{aligned} \tag{3.41}$$

in the same way for the second equation

$$\int_0^T \int_M a_2(x) g_2(t) |\nabla v_k|^2 dM dt \geq a_{02} g_2(T) \int_0^T \int_\omega |\nabla v_k|^2 dM dt, \tag{3.42}$$

and

$$\int_0^T \left\{ \left(\mu_1 - \frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2}\right) \int_M |u_{tk}|^2 dM + \left(\alpha_1 - \frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2}\right) \int_M |v_{tk}|^2 dM \right\} dt = 0, \tag{3.43}$$

$$\int_0^T \left\{ \left(\frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2}\right) \int_M z_{1k}^2(x,1,s) dM + \left(\frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2}\right) \int_M z_{2k}^2(x,1,s) dM \right\} dt = 0. \tag{3.44}$$

(3.41)-(3.42), yields

$$\lim_{k\to+\infty} \int_0^T \int_\omega |\nabla u_k|^2 dM\ dt = 0, \tag{3.45}$$

$$\lim_{k\to+\infty} \int_0^T \int_\omega |\nabla v_k|^2 dM dt = 0. \tag{3.46}$$

Combining (3.27)-(3.28) and (3.45)-(3.46) we deduce that $\nabla u = 0$ and $\nabla v = 0$ in $L^2(0,T; L^2(\omega))$ consequently $u(x,t) = C_1(t)$ and $v(x,t) = C_2(t)$ a.e in $\omega \times (0,T)$. Since $u(t) = 0$ a.e ∂M, $v(t) = 0$ a.e ∂M, we infer that $C_1(t) = 0$ a.e in $(0,T)$, $C_2(t) = 0$ a.e in $(0,T)$ which implies that $u = 0$,$u_t = 0$, $v_t = 0$ and $v = 0$ a.e in $\omega \times (0,T)$ which is a contradiction.

Case 2: $u = 0$, $\upsilon = 0$: We define,

$$c_k := [E_k(0)]^{\frac{1}{2}}$$

and

$$\overline{u_k} = \frac{1}{c_k} u_k, \quad \overline{\upsilon_k} = \frac{1}{c_k} \upsilon_k, \quad \overline{z_{1k}} = \frac{1}{c_k} z_{1k}, \quad \overline{z_{2k}} = \frac{1}{c_k} z_{2k}.$$

(ii) **Analysis of nonlinear term 2**. From (A_2), we deduce

$$\begin{aligned}
\|F(u_k, \upsilon_k)\|_{L^2(M\times(0,T))} &\le c_1 \int_0^T \int_M \left(|u_k(s)|^{p+1} + |\upsilon_k(s)|^{p+1}\right) dsdM \\
&\le c_1 c_s^{p+1} \int_0^T \int_M \left(|\nabla u_k(s)|^{p+1} + |\nabla \upsilon_k(s)|^{p+1}\right) ds \\
&\le c_1 c_s^{p+1} L_1 T \\
&= C,
\end{aligned} \tag{3.47}$$

where C is a positive constant.

$$\begin{aligned}
E_k(t) &= \frac{1}{2}\|\overline{u_{tk}}(t)\|_2^2 + \frac{1}{2}\|\overline{\upsilon_{tk}}(t)\|_2^2 + \frac{1}{2}\left(k_1 - a_1(x)\int_0^t g_1(s)ds\right)\|\nabla\overline{u_k}(t)\|_2^2 \\
&+ \frac{1}{2}\left(k_2 - a_2(x)\int_0^t g_2(s)ds\right)\|\nabla\overline{\upsilon_k}(t)\|_2^2 + \frac{\xi_1}{2}\int_M\int_0^1 \overline{z_{1k}^2}(x, k_1, t)dk_1 dM \\
&+ \frac{\xi_2}{2}\int_M\int_0^1 \overline{z_{2k}^2}(x, k_2, t)dk_2 dM + \int_M F(\overline{u_k}, \overline{\upsilon_k})dM + \frac{a_1(x)}{2}(g_1 \, o \, \nabla\overline{u_k})(t) \\
&+ \frac{a_2(x)}{2}(g_2 \, o \, \nabla\overline{\upsilon_k})(t).
\end{aligned} \tag{3.48}$$

Then

$$\begin{aligned}
E_k(0) &= \frac{1}{2}\|\overline{u_{tk}}(0)\|_2^2 + \frac{1}{2}\|\overline{\upsilon_{tk}}(0)\|_2^2 + \frac{1}{2}\|\nabla\overline{u_k}(0)\|_2^2 \\
&+ \frac{1}{2}\|\nabla\overline{\upsilon_k}(0)\|_2^2 + \frac{\xi_1}{2}\int_M\int_0^1 \overline{z_{1k}^2}(x, k_1, 0)dk_1 dM \\
&+ \frac{\xi_2}{2}\int_M\int_0^1 \overline{z_{2k}^2}(x, k_2, 0)dk_2 dM + \int_M F(\overline{u_k}(0), \overline{\upsilon_k}(0))dM \\
&= \frac{1}{2}\frac{\|u_{tk}(0)\|_2^2}{c_k^2} + \frac{1}{2}\frac{\|\upsilon_{tk}(0)\|_2^2}{c_k^2} + \frac{1}{2}\frac{\|\nabla u_k(0)\|_2^2}{c_k^2} \\
&+ \frac{1}{2}\frac{\|\nabla\upsilon_k(0)\|_2^2}{c_k^2} + \frac{\xi_1}{2}\int_M\int_0^1 \frac{z_{1k}^2(x, k_1, 0)}{c_k^2}dk_1 dM \\
&+ \frac{\xi_2}{2}\int_M\int_0^1 \frac{z_{2k}^2(x, k_2, 0)}{c_k^2}dk_2 dM + \int_M \frac{F(u_k(0), \upsilon_k(0))}{c_k^2}dM \\
&= \frac{1}{E_k(0)}E_k(0) = 1,
\end{aligned} \tag{3.49}$$

we deduce that

$$\overline{u_k} \to \overline{u} \ weakly\ star\ in\ L^\infty(0,T;H_0^1(M)), \tag{3.50}$$

$$\overline{u_k}' \to \overline{u}' \ weakly\ star\ in\ L^\infty(0,T;L^2(M)), \tag{3.51}$$

$$\overline{u_k} \to \overline{u} \ strongly\ in\ L^2(0,T;L^2(M)), \tag{3.52}$$

$$\overline{\upsilon_k} \to \overline{\upsilon} \ weakly\ star\ in\ L^\infty(0,T;H_0^1(M)), \tag{3.53}$$

$$\overline{\upsilon_k}' \to \overline{\upsilon}' \ weakly\ star\ in\ L^\infty(0,T;L^2(M)), \tag{3.54}$$

$$\overline{\upsilon_k} \to \overline{\upsilon} \ strongly\ in\ L^2(0,T;L^2(M)), \tag{3.55}$$

$$\overline{z_{1k}} \to \overline{z_1} \ weakly\ star\ in\ L^\infty(0,T;L^2(M)\times(0,1)), \tag{3.56}$$

$$\overline{z_{2k}} \to \overline{z_2} \ weakly\ star\ in\ L^\infty(0,T;L^2(M)\times(0,1)), \tag{3.57}$$

$$F(\overline{u_k},\overline{\upsilon_k}) \to F(\overline{u},\overline{\upsilon}) \ weakly\ star\ in\ L^2(0,T;L^2(M)), \tag{3.58}$$

We observe that from (3.5), we can deduce by applying the same idea used in **case 1**

$$\begin{aligned}
E(T)-E(0) \le & -\int_0^T \left\{ \left(\mu_1 - \frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2}\right) \int_M |u_t|^2 dM - \left(\alpha_1 - \frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2}\right) \int_M |\upsilon_t|^2 dM \right\} dt \\
& - \int_0^T \left\{ \left(\frac{\xi_1}{2\tau_2} - \frac{\mu_2}{2}\right) \int_M z_1^2(x,1,s) dM - \left(\frac{\xi_2}{2\tau_2} - \frac{\alpha_2}{2}\right) \int_M z_2^2(x,1,s) dM \right\} dt \\
& + \int_0^T \int_M a_1(x) \left\{ (g_1' \circ \nabla u) - g_1(t)|\nabla u|^2 \right\} dM dt \\
& + \int_0^t \int_M a_2(x) \left\{ (g_2' \circ \nabla \upsilon) - g_2(t)|\nabla \upsilon|^2 \right\} dM dt,
\end{aligned} \tag{3.59}$$

and considering (3.35)-(3.36), (3.49) and (3.50)-(3.58), we deduce that

$$\lim_{k\to\infty} E_k(t) = 1, \quad \forall t \in [0,T]. \tag{3.60}$$

Let $\epsilon > 0$ be small enough such that $T-\epsilon > T_0$ and let us consider a cutoff function $\psi \in C_0^\infty(0,T)$ such that $\psi = 1$ in $[\epsilon, T-\epsilon]$, $0 \le \psi \le 1$ in addition, consider another cutoff function $\phi \in C_0^\infty(M)$ such that $\phi = 1$ in $\overline{\omega'} \subset\subset \omega$, $\phi = 0$ in $M\backslash_\omega$, $0 \le \phi \le 1$, we get

$$\begin{aligned}
& -\int_0^T \int_M \overline{u_k}'\overline{u_k}\psi\phi' dM dt - \int_0^T \int_M |\overline{u_k}'|^2 \psi\phi dM dt + \int_0^T \int_M |\nabla \overline{u_k}|^2 \psi\phi dM dt \\
& + \int_0^T \int_M (\nabla \overline{u_k}.\nabla\psi)\overline{u_k}\phi dM dt + \int_0^T \int_M div\{a_1(x) g_1 * \nabla \overline{u_k}\}\overline{u_k}\psi\phi dM dt \\
& + \mu_1 \int_0^T \int_M \overline{u_k}'\overline{u_k}\psi\phi dM dt + \mu_2 \int_0^T \int_M \overline{u_k}'(x,t-\tau_2)\overline{u_k}\psi\phi dM dt \\
& -\int_0^T \int_M \overline{\upsilon_k}'\overline{\upsilon_k}\psi\phi' dM dt - \int_0^T \int_M |\overline{\upsilon_k}'|^2 \psi\phi dM dt + \int_0^T \int_M |\nabla \overline{\upsilon_k}|^2 \psi\phi dM dt \\
& + \int_0^T \int_M (\nabla \overline{\upsilon_k}.\nabla\psi)\overline{\upsilon_k}\phi dM dt + \int_0^T \int_M div\{a_2(x) g_1 * \nabla \overline{\upsilon_k}\}\overline{\upsilon_k}\psi\phi dM dt \\
& + \alpha_1 \int_0^T \int_M \overline{\upsilon_k}'\overline{\upsilon_k}\psi\phi dM dt + \alpha_2 \int_0^T \int_M \overline{\upsilon_k}'(x,t-\tau_2)\overline{\upsilon_k}\psi\phi dM dt \\
& \int_0^T \int_M (|\overline{u_k} + \overline{\upsilon_k}|^{p+1} + 2|\overline{u_k}\overline{\upsilon_k}|^{\frac{p+1}{2}}\psi\phi dM dt = 0.
\end{aligned} \tag{3.61}$$

Kipping in mind $\overline{u_k}\equiv 0$, $\overline{v_k}\equiv 0$, $\overline{z_{1k}}\equiv 0$,$\overline{z_{2k}}\equiv 0$, by combining (3.50)-(3.58) and (3.61)

$$\lim_{k\to+\infty}\left\{-\int_0^T\int_M|\overline{u_k}'|^2\psi\phi dMdt+\int_0^T\int_M|\nabla\overline{u_k}|^2\psi\phi dMdt\right\}=0, \tag{3.62}$$

$$\lim_{k\to+\infty}\left\{-\int_0^T\int_M|\overline{v_k}'|^2\psi\phi dMdt+\int_0^T\int_M|\nabla\overline{v_k}|^2\psi\phi dMdt\right\}=0, \tag{3.63}$$

from the properties of the function ψ, we deduce

$$\lim_{k\to+\infty}\left\{-\int_0^T\int_\omega|\overline{u_k}'|^2\psi\phi dMdt+\int_0^T\int_\omega|\nabla\overline{u_k}|^2\psi\phi dMdt\right\}=0, \tag{3.64}$$

$$\lim_{k\to+\infty}\left\{-\int_0^T\int_\omega|\overline{v_k}'|^2\psi\phi dMdt+\int_0^T\int_\omega|\nabla\overline{v_k}|^2\psi\phi dMdt\right\}=0. \tag{3.65}$$

Combining (3.61)with (3.45)-(3.46), yields

$$\lim_{k\to+\infty}\int_0^T\int_\omega|\overline{u_k}'|^2\psi\phi dMdt=0, \tag{3.66}$$

$$\lim_{k\to+\infty}\int_0^T\int_\omega|\overline{v_k}'|^2\psi\phi dMdt=0. \tag{3.67}$$

We deduce that

$$\lim_{k\to+\infty}\int_\epsilon^{T-\epsilon}\int_{\omega'}|\overline{u_k}'|^2\psi\phi dMdt=0,\quad \lim_{k\to+\infty}\int_\epsilon^{T-\epsilon}\int_{\omega'}|\nabla\overline{u_k}|^2\psi\phi dMdt=0, \tag{3.68}$$

$$\lim_{k\to+\infty}\int_\epsilon^{T-\epsilon}\int_{\omega'}|\overline{v_k}'|^2\psi\phi dM\ dt=0,\quad \lim_{k\to+\infty}\int_\epsilon^{T-\epsilon}\int_{\omega'}|\nabla\overline{v_k}|^2\psi\phi dM\ dt=0. \tag{3.69}$$

From (3.68)-(3.69) and taking (A_0); we have also $E_k(t)\le E_k(0)=1$, $\forall t\in[0,T]$, $k\in N$ as in (1.11)

$$1=E_k(0)\le C_{T_0,\omega',\epsilon}\int_\epsilon^{T-\epsilon}\int_{\omega'}\left\{|\overline{u_k}'|^2+|\overline{v_k}'|^2\right\}\psi\phi dMdt\to 0\ as\ k\to\infty,$$

$$E(T)\le -C[E(T)-E(0)],\quad \forall\, T>T_0, \tag{3.70}$$

as in Mu$\tilde{n}$oz-Rivera and Salvatierra [13]

$$E(T)\le\frac{C}{C+1}E(0)=\frac{1}{1+\frac{1}{C}}E(0),\quad \forall\, T>T_0. \tag{3.71}$$

Repeating the above process from T to $2T$ we obtain

$$E(2T)\le\frac{1}{\left(1+\frac{1}{C}\right)^2}E(0),\quad \forall\, T>T_0. \tag{3.72}$$

In general

$$E(nT)\le\frac{1}{\left(1+\frac{1}{C}\right)^n}E(0),\quad \forall\, T>T_0, \tag{3.73}$$

t can take the form $t=nT+r$ where $0\le r<T$ and since $E(t)$ is a decreasing function, we get

$$E(t)\le E(t-r)\le\frac{1}{\left(1+\frac{1}{C}\right)^{\frac{t-r}{T}}}E(0)=\beta e^{-\alpha t}E(0),\quad \forall\, T>T_0, \tag{3.74}$$

where $\beta=e^{\frac{r}{T}\ln(+\frac{1}{C})}$, $\alpha=\frac{\ln(+\frac{1}{C})}{T}$. This completes the proof. □

Remark 3.6. Our functional energy formula is different from the one used in [1]. Our problem contains some nonlinear and delay terms. This work can be viewed as a continuation of the works of M. Cavalcanti, V D. Cavalcanti and F. A Nascimento [1].

References

[1] M.Cavalcanti, V. D.Cavalcanti and F.F. Nascimento, *Asymptotic stability of the wave equation on compact manifolds and locally distributed viscoelastic dissipation*, Proceeding of the American Mathematical Society. **141**, 3183–3193 (2013).

[2] Wenjun, *Global and asymptotic behavior for a coupled quasilinear system with a viscoelastic term*, SIAM J. Math. Anal. **5**, 138–146 (2011).

[3] Mohammad.A. Rammaha, S.Sakuntasathien, *Global existence and blow-up of solution to systems of nonlinear wave equation with degenerate damping and source terms* , Nonlinear Analysis. 2658–2683 (2010).

[4] K. ONO, *Global existence, decay and blow up of solutions for some mildly degenerate nonlinear Kirchhoff string*, J. Differential Equations. **137**, 273–301 (1997).

[5] B. Said Houari, *Global nonexistence of a positive initial energy solutions of a system of nonlinear wave equation with dumping and source terms*, J. Differential Integral Equations. **23**, 79–92 (2010).

[6] F.Liang, H, J.Gao, *Exponential energy decay and blow-up of solution for a system of nonlinear viscoelastic wave equations with strong dumping,* Bound.Value. Probl.(2011)22.

[7] X.S. Han, M.X. Wang, *Global existence and blow-up of solutions for a system of nonlinear viscoelastic wave equation with dumping and source,* Nonlinear Analysis, TMA7, 5427–5450 (2009).

[8] S.A. Messaoudi, Tatar, *Global existence and uniform stability of solutions for a quasilinear viscolastic problem*, Math. Method. Appl. Sci. **30**, 665–680 (2007).

[9] A. Guesmia, *Nouvelles inegalites integrales et applications a la stabilisation des systemes distribues non dissipatifs*, Math. Method. Appl. Sci. **336**, 801–804(2003).

[10] S.T. Wu, *On decay and blow up of solutions for a system of nonlinear wave equations,* J. Math. Anal. Appl. **394**, 360–377 (2012).

[11] M.I. Mustafa, *Well posedness and asymptotic behavior of a coupled system of nonlinear viscoelastic equations*, Nonlinear Analysis: Real World app. **13**(7), 452–463(2012).

[12] S. Nicaise and C. Pignotti, *Stabilization of the wave equation with boundary or internal distributed delay,* Differential Integral Equations. **21**, 935–958(2008).

[13] J. E. Munoz Rivera and A. Peres Salvatierra, *Asymptotic behaviour of the energy in partially viscoelastic materials*, Quart. Appl. Math. **59** (3), 557–578(2001). MR1848535 (2002f:35038)

[14] C. Bardos, G. Lebeau and J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim. **30** (5), 1024–1065(1992). MR1178650 (94b:93067)

[15] J.-L. Lions, *Quelques méthodes de résolution des problémes aux limites non linéaires*, Dunod, 1969.

Author information

Mohamed Ferhat and Ali Hakem, Laboratory ACEDP,
Djillali Liabes University, 22000 Sidi Bel Abbes, ALGERIA.
E-mail: ferhat22@hotmail.fr; hakemali@yahoo.com

Received: June 26, 2015.
Accepted: November 3, 2015.