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ANTIPODAL OF GRAPH PRODUCTS

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Abstract. The antipodal graph of a graph G, denoted by $A(G)$, is the graph on the same vertices as of G, two vertices being adjacent if the distance between them is equal to the diameter of G. A graph is said to be antipodal if it is the antipodal graph $A(H)$ of some graph H. In this paper, we consider four kinds of graph products and introduce the antipodal graph of each product with respect to the antipodal of their factors.

1 Introduction

Let G be a simple graph with *vertex set* $V = V(G)$ and *edge set* $E = E(G)$. The *order* and *size* of the graph G are $n(G) = |V|$ and $m(G) = |E|$, respectively. If $E = \emptyset$ then G is called *empty graph*. The *open neighborhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V | uv \in E\}$. The degree of a vertex $v \in V(G)$ is $deg_G(v) = |N_G(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Let u and v be two vertices of G. If there exists a uv-path in G, then the *distance* $d_G(u, v)$ between u and v is the length of the shortest uv-path in G. If there is no uv-path in G, then we define $d_G(u, v) = \infty$. The *eccentricity* of a vertex u is $ecc(u) = max\{d_G(u, v)|v \in V(G)\}$, and the *diameter* of G is $diam(G) = max\{ecc(v)|v \in V(G)\}$. We use [15] for terminology and notation not defined here.

In 1971 Smith [13] initiated the concept of *antipodal graph of a graph G* as the graph $A(G)$ having the same vertex set as that of G and such that vertices are adjacent if they are at the distance of $diam(G)$ in G. A graph is *antipodal* if it is the antipodal graph $A(H)$ of some graph H. The conditions on G for $A(G) = G$ and $A(G) = \overline{G}$ and many other results are discussed in [3] and [2]. Acharya et. al. have studied *self-antipodal* graphs in [1].

There is a significant number of graph operations like complement of a graph as a unary operation and graph products as a binary operation. As stated in many references, for example in [5], there are four kinds of fundamental graph products: the *Cartesian product*, the *direct product*, the *strong product*, and the *lexicographic product*. In each case, the product of graphs G and H is a graph $\Gamma(V, E)$ whose vertex set is the Cartesian product $V(G) \times V(H)$ of sets. However, each product has different rules for adjacency. The Cartesian product of G and H is the graph $G \Box H$ for which two vertices (g, h) and (g', h') are adjacent precisely if $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. The graphs G and H are called *factors* of the product $G \Box H$. The direct product of G and H is the graph $G \times H$ for which two vertices (g, h) and (g', h') are adjacent precisely if $gg' \in E(G)$ and $hh' \in E(H)$. The strong product of G and H is the graph $G \boxtimes H$ for which the edge set is defined as $E(G \boxtimes H) = E(G \Box H) \cup E(G \times H)$. Finally, the lexicographic product of G and H is the graph $G \circ H$ for which two vertices (g, h) and (g', h') are adjacent if $gg' \in E(G)$, or $g = g'$ and $hh' \in E(H)$. As shown in [6] there are 256 different possibilities to define a graph product based on different rules for adjacency. For instance, the *co-normal product* of G and H, denoted by $G \odot H$, is that one of them for which vertices (g, h) and (g', h') are adjacent if $gg' \in E(G)$ or $hh' \in E(H)$.

In the literature it is well known how many of the important graph invariants propagate under product formations. These salient structural features have been studied extensively. For example, Specapan [14] has carried out research in connectivity of Cartesian product of graphs, and Li et.

al [11] have shown that the lexicographic product of vertex transitive graphs is vertex transitive, and the lexicographic product of edge transitive graphs is edge transitive. For more information the reader may also consult [4, 5, 8, 9, 12].
Motivated by the above results, in this paper we are concerned with antipodal graphs and their different types of products with respect to the antipodal graphs of their factors. Besides, in some cases we will prove that the antipodal property can be transferred from two factors to their product.
The following theorem has been proved by Aravamudhan et. al in [3].

Theorem 1.1. *A graph G is an antipodal graph if and only if it is the antipodal graph of its complement.*

It is easily seen that the next two corollaries could be deduced by Theorem 1.1.

Corollary 1.2. *A graph G is an antipodal graph if and only if $diam(\overline{G}) = 2$ or $\overline{G}$ is disconnected and its components are complete graphs.*

Corollary 1.3. *If G is disconnected, then $A(G)$ is of diameter not bigger than 2 and the components of $\overline{A(G)}$ are complete graphs.*

The proof of the following four theorems can be found in [5], [7] and [10].

Theorem 1.4. *If (g,h) and (g',h') are two vertices of a Cartesian product $\Gamma = G\Box H$, then $d_\Gamma((g,h),(g',h')) = d_G(g,g') + d_H(h,h')$.*

Theorem 1.5. *Suppose (g,h) and (g',h') are two vertices of a direct product $\Gamma = G \times H$, and n is an integer for which G has a gg'-walk of length n and H has a hh'-walk of length n. Then Γ has a walk of length n from (g,h) to (g',h'). The smallest such n (if it exists) equals $d_\Gamma((g,h),(g',h'))$. If no such n exists, then $d_\Gamma((g,h),(g',h')) = \infty$.*

Theorem 1.6. *If (g,h) and (g',h') are two vertices of a strong product $\Gamma = G \boxtimes H$, then $d_\Gamma((g,h),(g',h')) = \max\{d_G(g,g'), d_H(h,h')\}$.*

Theorem 1.7. *If (g,h) and (g',h') are two vertices of a lexicographic product $\Gamma = G \circ H$, then*

$$d_\Gamma((g,h),(g',h')) = \begin{cases} d_G(g,g') & \text{if } g \neq g', \\ d_H(h,h') & \text{if } g = g' \text{ and } deg_G(g) = 0, \\ \min\{2, d_H(h,h')\} & \text{if } g = g' \text{and } deg_G(g) \neq 0. \end{cases}$$

2 Cartesian Product

Let G and H be two simple graphs and $\Gamma = G\Box H$. In this section we present the antipodal graph of Γ by the antipodal graphs of G and H and then we will prove that Γ is an antipodal graph. By Theorem 1.4 the proof of the following proposition is straightforward.

Proposition 2.1. *If both graphs G and H are connected and $\Gamma = G\Box H$, then $diam(\Gamma) = diam(G) + diam(H)$; otherwise, $diam(\Gamma) = \infty$.*

Our next aim is to characterize the antipodal of the Cartesian product of two graphs.

Theorem 2.2. *Let $A(G)$ and $A(H)$ be the antipodal of the graphs G and H, respectively, and let $A(\Gamma)$ be the antipodal of the graph $\Gamma = G\Box H$. If both graphs G and H are connected, then $A(\Gamma) = A(G) \times A(H)$. If only one of the factors of Γ is connected, then $A(\Gamma)$ is the co-normal product of the antipodal of the disconnected factor and the empty graph with the same order of the connected factor. Finally, if both of the factors are disconnected, then $A(\Gamma) = A(G) \odot A(H)$.*

Proof. If both G and H are connected, then they have finite diameters and by Proposition 2.1, $diam(\Gamma) < \infty$. Obviously, two vertices (g,h) and (g',h') of Γ are adjacent in $A(\Gamma)$ if and only if $d_\Gamma((g,h),(g',h')) = diam(G) + diam(H)$. By Theorem 1.4,

$$d_\Gamma((g,h),(g',h')) = d_G(g,g') + d_H(h,h') \leq diam(G) + diam(H).$$

Therefore, two vertices (g,h) and (g',h') are adjacent in $A(\Gamma)$ if and only if $d_G(g,g') = diam(G)$ and $d_H(h,h') = diam(H)$. Hence, $gg' \in E(A(G))$ and $hh' \in E(A(H))$. Thus, $(g,h)(g',h') \in E(A(G) \times A(H))$.

Now, suppose that only one of the graphs is connected. Without loss of generality, let G be a disconnected and H be a connected graph. By Proposition 2.1, $diam(\Gamma) = \infty$ and two vertices (g,h) and (g',h') of Γ are adjacent in $A(\Gamma)$ if and only if $(g,h)(g',h')$-path in Γ does not exist. On the other hand, for all $h,h' \in V(H)$, $d_H(h,h')$ is finite. Therefore, for any two disjoint vertices g and g' in $V(G)$, we have $d_G(g,g') = \infty$ and $gg' \in E(A(G))$. By the definition of the co-normal product, it is easy to verify that $(g,h)(g',h') \in E(A(G) \odot \overline{K_{V(H)}})$.

Finally, let both G and H be disconnected. Clearly, two vertices (g,h) and (g',h') of Γ are adjacent in $A(\Gamma)$ if and only if g and g' are adjacent in $A(G)$ or h and h' are adjacent in $A(H)$. Thus, $(g,h)(g',h') \in E(A(G) \odot A(H))$. □

In the next theorem we give a necessary and sufficient condition when the Cartesian product of two non-empty connected graphs is an antipodal graph.

Theorem 2.3. *Let G and H be two non-empty connected graphs. Then $\Gamma = G \square H$ is an antipodal graph if and only if $n(G) = n(H) = 2$ or $n(G), n(H) \geq 3$.*

Proof. If $n(G) = n(H) = 2$, then $\Gamma \simeq C_4$ and $\overline{\Gamma} \simeq 2P_2$. It is clear that $\Gamma = A(\overline{\Gamma})$. Now, let $n(G), n(H) \geq 3$. If two vertices (g,h) and (g',h') are adjacent in Γ, then

$$d_{\overline{\Gamma}}((g,h),(g',h')) \geq 2.$$

Hence, $diam(\overline{\Gamma}) \geq 2$. By the assumption, since both G and H are graphs of order at least three, there exists a vertex (g'',h'') in $V(\Gamma)$ that is not adjacent to neither (g,h) nor (g',h'). Hence, (g'',h'') is a common neighbor of (g,h) and (g',h') in $\overline{\Gamma}$, and $diam(\overline{\Gamma}) \leq 2$. Thus, $diam(\overline{\Gamma}) = 2$ and Theorem 1.1 completes the proof of this part.
Now, we prove the converse. Let Γ be an antipodal graph. By Theorem 1.1, $\Gamma = A(\overline{\Gamma})$. Since G and H are non-empty graphs, $n(G), n(H) \geq 2$. So, it is enough to show that $n(G) = 2$ if and only if $n(H) = 2$. Let $n(H) = 2$ and on the contrary suppose that $n(G) > 2$. Without loss of generality we may suppose that $n(G) = 3$ i.e. $V(G) = \{g, g', g''\}$ and $V(H) = \{h, h'\}$. There are two general cases:

Case I. $G \simeq P_3$ and $g \sim g' \sim g''$. Then $\overline{\Gamma}$ is a connected graph and $diam(\overline{\Gamma}) = 3$. This is a contradiction by Corollary 1.2.

Case II. $G \simeq K_3$. Then $\overline{\Gamma} \simeq C_6$ and so $diam(\overline{\Gamma}) = 3$. Again this is a contradiction by Corollary 1.2.

□

3 Strong Product

Let G and H be two simple graphs and $\Gamma = G \boxtimes H$ be the strong product of G and H. In this section we present the antipodal graph of Γ by the antipodal graphs of G and H. For avoiding triviality one can assume that the diameters of G and H are not less than two.

Proposition 3.1. *For two connected graphs G and H,*

$$diam(G \boxtimes H) = \max\{diam(G), diam(H)\}.$$

Proof. By Theorem 1.6 the proof is straightforward. □

In the next theorem we characterize the antipodal graph of the strong product of two graphs.

Theorem 3.2. *Let $A(G)$ and $A(H)$ be the antipodal of the graphs G and H, respectively, and let $A(\Gamma)$ be the antipodal of the graph $\Gamma = G \boxtimes H$. If $diam(G) = diam(H)$, then $A(\Gamma) = A(G) \odot A(H)$; otherwise, the antipodal of Γ is the co-normal product of the antipodal of the factor of Γ with the larger diameter and the empty graph with the same order of the other factor.*

Proof. First, suppose that $diam(G) = diam(H) = d$. Clearly by Proposition 3.1, $diam(\Gamma) = d$. Let (g, h) and (g', h') be two vertices of $V(\Gamma)$. Obviously, $(g, h)(g', h') \in E(A(\Gamma))$ if and only if g is adjacent to g' in $A(G)$ or h is adjacent to h' in $A(H)$. In other words, by the definition of co-normal product, $(g, h)(g', h') \in E(A(G) \odot A(H))$. Now, let $diam(G) \neq diam(H)$. Without loss of generality, suppose that $diam(G) > diam(H)$. Clearly, by Proposition 3.1, $diam(\Gamma) = diam(G)$, and it is easy to verify that two vertices (g, h) and (g', h') of $V(\Gamma)$ are adjacent in $A(\Gamma)$ if and only if $d_\Gamma((g, h), (g', h')) = d_G(g, g')$. Hence, $A(\Gamma) = A(G) \odot \overline{K_{V(H)}}$. □

Theorem 3.3. *Let G and H be two connected graphs and $\Gamma = G \boxtimes H$. If both of the factors of Γ have diameter greater than two, then Γ is an antipodal graph.*

Proof. One can check that if $diam(G) \geq 3$, then for each $a \in V(G)$, there is $b \in V(G)$ such that $d_G(a, b) \geq 2$. Let $(x, y) \nsim (u, v)$ in $\overline{\Gamma}$. Then there is $(\alpha, \beta) \in V(\Gamma)$ such that $d_G(x, \alpha) \geq 2$ and $d_H(v, \beta) \geq 2$. So, $(x, y) \nsim (\alpha, \beta) \nsim (u, v)$ in Γ. Thus, $(x, y) \sim (\alpha, \beta) \sim (u, v)$ in $\overline{\Gamma}$ and hence, $diam(\overline{\Gamma}) = 2$. Therefore, Γ is an antipodal graph. □

Remark 3.4. For each $(g, h) \in V(G \boxtimes H)$,

$$deg_{G \boxtimes H}((g, h)) = deg_G(g) + deg_H(h) + deg_G(g) deg_H(h).$$

Theorem 3.5. *If $\Gamma = G \boxtimes H$ is an antipodal graph, then G or H has no universal vertex.*

Proof. On the contrary, suppose that $deg(g) = |G| - 1$ and $deg(h) = |H| - 1$. By Remark 3.4, $deg((g, h)) = |\Gamma| - 1$, thus (g, h) is an isolated vertex in $\overline{\Gamma}$. Hence, $diam(\overline{\Gamma}) = \infty$. Now, let $g' \in V(G)$ and $h', h'' \in V(H)$ be such that $d_H(h', h'') = 2$. Then $h' \sim h \sim h''$ in H. Thus, $(g', h') \sim (g, h')$ and $(g, h') \nsim (g', h'') \nsim (g', h')$ in Γ. Hence, $(g, h') \sim (g', h'') \sim (g', h')$ in $\overline{\Gamma}$. Thus, $d_{\overline{\Gamma}}((g, h'), (g', h')) = 2 \neq \infty$. This is a contradiction since Γ is an antipodal graph. □

4 Lexicographic Product

Let G and H be two simple graphs and $\Gamma = G \circ H$ be the lexicographic product of G and H. In this section we present the antipodal graph of Γ by the antipodal graphs of G and H. For avoiding triviality one can assume that the diameters of G and H are not less than two. By Theorem 1.7 we have the following proposition.

Proposition 4.1. *If G and H are two simple graphs and $\Gamma = G \circ H$ is the lexicographic product of G and H, then*

$$diam(\Gamma) = \begin{cases} diam(G) & \textit{if } diam(G) \geq diam(H) \textit{ or } \delta(G) \geq 1, \\ diam(H) & \textit{otherwise.} \end{cases}$$

The next theorem describes the antipodal graph of the lexicographic product of two graphs.

Theorem 4.2. *If $\Gamma = G \circ H$, then*

$$A(\Gamma) = \begin{cases} A(G) \odot \overline{K_{V(H)}} & \textit{if } diam(G) > diam(H) \geq 2 \textit{ or } \delta(G) \geq 1, \\ G[S] \times A(H) & \textit{if } diam(H) > diam(G) \geq 2 \textit{ and } \delta(G) = 0, \\ (A(G) \odot \overline{K_{V(H)}}) \cup (G[S] \times A(H)) & diam(G) = diam(H) > 2, \end{cases}$$

where $S \subseteq V(G)$ is the set of isolated vertices in $V(G)$, and $G[S]$ is the induced subgraph by S.

Proof. Let (g, h) and (g', h') be two vertices of $V(\Gamma)$. If $diam(G) > diam(H)$, then by Proposition 4.1, $(g, h)(g', h') \in E(A(\Gamma))$ if and only if two vertices g and g' are adjacent in $A(G)$, and h and h' can be any two arbitrary vertices in $V(H)$. Hence, by the definition of co-normal product, $(g, h)(g', h') \in E(A(G) \odot \overline{K_{V(H)}})$. If $\delta(G) \geq 1$, then, by Proposition 4.1, $(g, h)(g', h') \in E(A(\Gamma))$ if and only if $d_\Gamma((g, h), (g', h')) = diam(G)$. In this case, $g \neq g'$ because if $g = g'$, then the distance between (g, h) and (g', h') would not be more than two and could not be equal to $diam(G)$, which is a contradiction. Thus, $d_\Gamma((g, h), (g', h')) = d_G(g, g') = diam(G)$ if and only if $g \neq g'$ and hence, $(g, h)(g', h') \in E(A(G) \odot \overline{K_{V(H)}})$. Now, let $diam(G) < diam(H)$ and let the graph G have some isolated vertices, i.e. $S \neq \emptyset$. Then, by Proposition 4.1, two vertices (g,h) and (g',h') of $V(\Gamma)$ are adjacent in $A(\Gamma)$ if and only if $g = g'$ is an isolated vertex and $d_H(h,h') = diam(H)$. Hence, $(g,h)(g',h') \in E(G[S] \times A(H))$. Finally, if $diam(G) = diam(H)$, then by Proposition 4.1, $(g,h)(g',h') \in E(A(\Gamma))$ if and only if $g = g'$ is an isolated vertex in $V(G)$ and $d_H(h,h') = diam(H)$, or $g \neq g'$ and $d_G(g,g') = diam(G)$. Hence, $(g,h)(g',h') \in E((A(G) \odot \overline{K_{V(H)}}) \cup (G[S] \times A(H)))$. This completes the proof. □

Theorem 4.3. *Let G and H be two connected graphs and $\Gamma = G \circ H$. If $diam(G)$ is greater than three, then Γ is an antipodal graph.*

Proof. First, we show that $diam(\overline{\Gamma}) = 2$. Let $(a,b) \nsim (u,v)$ be two vertices in $\overline{\Gamma}$. Then $(a,b) \sim (u,v)$ in Γ. According to the definition of lexicographic product, there are two cases:

Case I. Let $a = u$ and $b \sim v$ in H. Since $diam(G) > 3$, for any vertex $a \in V(G)$ there is some vertex $\alpha \in V(G)$ such that $d(a,\alpha) \geq 2$. Hence, $u = a \nsim \alpha$. Also, $(a,b) \nsim (\alpha,b) \nsim (a,v) = (u,v)$ in Γ. Thus $(a,b) \sim (\alpha,b) \sim (a,v) = (u,v)$ in $\overline{\Gamma}$ and therefore $d_{\overline{\Gamma}}((a,b),(u,v)) = 2$.

Case II. Let $a \sim u$ in G. If there is vertex $\alpha \in V(G)$ such that $d_G(a,\alpha), d_G(u,\alpha) \geq 2$, then $(a,b) \nsim (\alpha,b) \nsim (u,v)$ in Γ and our claim follows. Otherwise, for each $\beta \in V(G) \setminus \{a,u\}$, $\beta \in N_G(a) \cup N_G(u)$. Hence, $diam(G) \leq 3$, which is a contradiction. Thus, $d_{\overline{\Gamma}}((a,b),(u,v)) = 2$.

Therefore, $diam(\overline{\Gamma}) = 2$. Hence, the distance between any two non-adjacent vertices in $\overline{\Gamma}$ is equal to the diameter of $\overline{\Gamma}$ and the same two vertices are adjacent in Γ. Thus, by Theorem 1.1, Γ is an antipodal graph. □

Theorem 4.4. *Let G be a connected graph. If $\Gamma = G \circ H$ is an antipodal graph, then $diam(G) \neq 3$ or G has at least one cycle.*

Proof. Let Γ be an antipodal graph. Then, by Theorem 1.2, $diam(\Gamma) = 2$ or $\overline{\Gamma}$ is a disconnected graph. If $\overline{\Gamma}$ is not a connected graph, then by Theorem 1.3, $diam(A(\overline{\Gamma})) \leq 2$, hence $diam(\Gamma) \leq 2$. By Proposition 4.1, $diam(G) = diam(\Gamma) = 2$. Let $diam(\overline{\Gamma}) = 2$. If G has at least one cycle, then then there is nothing to prove. So, let us assume that $diam(G) = 3$ and let $a \sim b$ in G. Then for each $h \in H$, $(a,h) \nsim (b,h)$ in $\overline{\Gamma}$. Since $diam(\overline{\Gamma}) = 2$, there is a vertex $(x,y) \in N_{\overline{\Gamma}}((a,h),(b,h))$. Hence, $(a,h) \nsim (x,y) \nsim (b,h)$ in Γ. As a consequence, $a \nsim x \nsim b$ in G. Since G is a tree of diameter 3, there is a path of length two between a,x or b,x. Let $a \sim b \sim u \sim x$ in G. Then $(b,h) \sim (x,h) \sim (a,h) \sim (u,h)$ in $\overline{\Gamma}$. Since $diam(\overline{\Gamma}) = 2$, there is $(g,l) \in N_{\overline{\Gamma}}((b,h),(u,h))$. Again we have $g \notin N_G(b) \cup N_G(u)$. Thus, there is just one path of length at least two between b,g or u,g. Therefore, $d_G(a,g) \geq 4$ or $d_G(x,g) \geq 4$. This contradiction completes the proof. □

5 Co-normal Product

Throughout this section, we assume that G and H are two simple fnite graphs and we are going to describe the antipodal graph of $\Gamma = G \odot H$.

Proposition 5.1. *If Γ is the co-normal product of two connected graphs G and H, then*

$$diam(\Gamma) = \begin{cases} 1 & \text{if } G \text{ and } H \text{ are complete graphs,} \\ 2 & \text{otherwise.} \end{cases}$$

Proof. First, suppose that G and H are complete graphs. Then for each $g \neq g' \in V(G)$ and $h \neq h' \in V(H)$, $(g,h) \sim (g',h')$ and $(g,h) \sim (g,h')$. Hence, Γ is a complete graph. If $G \simeq K_n$ and H is not a complete graph, then for each $g \neq g' \in V(G)$ and $h \nsim h' \in V(H)$, we have $(g,h) \nsim (g,h')$. But $(g,h) \sim (g',h) \sim (g,h')$ in Γ. So $diam(\Gamma) = 2$. Now, let $G \not\simeq K_n$ and $H \not\simeq K_m$. Let also $g \neq g' \in V(G)$ and $h \neq h' \in V(H)$ be such that $g \nsim g'$ and $h \nsim h'$. If $d = d_G(g,g')$ and $d' = d_H(h,h')$, then the following two paths exist respectively in G and H:

$$g \sim x_1 \sim x_2 \sim \ldots \sim x_d \sim g'$$

$$h \sim y_1 \sim y_2 \sim \ldots \sim y_{d'} \sim h'$$

As a consequence, $(g,h) \sim (x_1, y_{d'}) \sim (g', h')$ and $(g,h) \sim (x_1, y_{d'}) \sim (g, h')$ in Γ. Hence, $diam(\Gamma) = 2$. $\square$

Now, we are ready to prove the following theorem.

Theorem 5.2. *Let G and H be two non-complete graphs. If $\Gamma = G \odot H$ and $A(\Gamma)$ is the antipodal graph of Γ, then*

$$A(\Gamma) = \overline{G} \times \overline{H}.$$

Proof. Let (g,h) and (g',h') be two vertices of $V(\Gamma)$. By Proposition 5.1, we have $diam(\Gamma) = d((g,h),(g',h'))$ if and only if (g,h) is not adjacent to (g',h') in Γ. By the definition of the co-normal product, (g,h) is not adjacent to (g',h') in Γ if and only if $g \nsim g'$ in G and $h \nsim h'$ in H, equivalently $g \sim g'$ in $\overline{G}$ and $h \sim h'$ in $\overline{H}$. Therefore $d_\Gamma((g,h),(g',h')) = diam(\Gamma)$ if and only if $(g,h) \sim (g',h')$ in $\overline{G} \times \overline{H}$. $\square$

Lemma 5.3. *For any graphs G and H, $\overline{G \odot H} \simeq \overline{G} \boxtimes \overline{H}$.*

Proof. As we know, for each $g, g' \in V(G)$ and $h, h' \in H$, $(g,h) \sim (g',h')$ in $\overline{G \odot H}$ if and only if $(g,h) \nsim (g',h')$ in $G \odot H$. Hence, $(g,h) \sim (g',h')$ in $\overline{G \odot H}$ if and only if one of the following statements is true:

- $g = g'$ and $h \nsim h'$ in H;
- $h = h'$ and $g \nsim g'$ in G;
- $g \nsim g'$ and $h \nsim h'$ in G and H, respectively.

Hence, $(g,h) \sim (g',h')$ in $\overline{G \odot H}$ if and only if one of the following is true:

- $g = g'$ and $h \sim h'$ in $\overline{H}$;
- $h = h'$ and $g \sim g'$ in $\overline{G}$;
- $g \sim g'$ and $h \sim h'$ in $\overline{G}$ and $\overline{H}$, respectively.

Therefore, $(g,h) \sim (g',h')$ in $\overline{G \odot H}$ if and only if $(g,h) \sim (g',h')$ in $\overline{G} \boxtimes \overline{H}$. $\square$

Corollary 5.4. *For any graphs $\overline{G}$ and $\overline{H}$, $diam(\overline{G \odot H}) = \max\{diam(\overline{G}), diam(\overline{H})\}$.*

Proof. If G and H are two connected graphs, then the result follows from Proposition 3.1 and Lemma 5.3. Let $\overline{G}$ be a disconnected graph. Then there are $g, g' \in V(G)$ such that $d_{\overline{G}}(g,g') = \infty$. Hence, for each $h \in H$, $d_{\overline{G \odot H}}((g,h),(g',h)) = \infty$. Thus, $diam(\overline{G \odot H}) = \infty$. This completes the proof. $\square$

Theorem 5.5. *The graph $G \odot H$ is an antipodal graph if and only if G and H are two antipodal graphs and $diam(\overline{G}) = diam(\overline{H})$.*

Proof. Let $\Gamma = G \odot H$ and G and H be two antipodal graphs such that $diam(\overline{G}) = diam(\overline{H})$. Then, by Corollary 5.4, $daim(\overline{\Gamma}) = diam(\overline{G}) = diam(\overline{H})$. Let $(g,h) \sim (g',h')$ in Γ. There are two cases:

Case I. $g \sim g'$ in G and $h \nsim h'$ in H or $g \nsim g'$ in G and $h \sim h'$ in H. Without loss of generality suppose that $g \sim g'$ in G and $h \nsim h'$ in H. Hence, $d_{\overline{G}}(g,g') = diam(\overline{G})$. By Lemmas 1.6 and 5.3,

$$d_{\overline{\Gamma}}((g,h),(g',h')) = \max\{diam(\overline{G}), 1\} = diam(\overline{G}) = daim(\overline{\Gamma}).$$

Case II. $g \sim g'$ in G and $h \sim h'$ in H. Then $d_{\overline{G}}(g,g') = diam(\overline{G})$ and $d_{\overline{H}}(h,h') = diam(\overline{H})$. Thus

$$d_{\overline{\Gamma}}((g,h),(g',h')) = \max\{diam(\overline{G}), diam(\overline{H})\} = daim(\overline{\Gamma}).$$

Therefore, Γ is an antipodal graph.
Now, let Γ be an antipodal graph. We will show that $diam(\overline{G}) = diam(\overline{H})$ and that both factors of Γ are antipodal graphs. Let $g \sim g'$ in G. Then for each $h \in V(H)$, $(g,h) \sim (g',h)$ in Γ. Since Γ is an antipodal graph,

$$diam(\overline{\Gamma}) = d_{\overline{\Gamma}}((g,h),(g',h)) = \max\{d_{\overline{G}}(g,g')\,,\,0\} = d_{\overline{G}}(g,g').$$

With similar argument for each $h_1 \sim h_2$ in H and $u \in G$ we have

$$diam(\overline{\Gamma}) = d_{\overline{\Gamma}}((u,h_1),(u,h_2)) = \max\{0\,,\,d_{\overline{H}}(h_1,h_2)\} = d_{\overline{H}}(h_1,h_1).$$

Let $diam(\overline{\Gamma}) = diam(\overline{G})$. Then $diam(\overline{H}) \leq diam(\overline{G})$ and $diam(\overline{G}) = d_{\overline{H}}(h_1,h_1) \leq diam(\overline{H})$. Hence, $diam(\overline{H}) = diam(\overline{G}) = diam(\overline{\Gamma})$. Also, $d_{\overline{G}}(g,g') = diam(\overline{G})$ and $d_{\overline{H}}(h_1,h_2) = diam(\overline{H})$. Therefore, G and H are two antipodal graphs. □

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