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Identities related to the Stirling numbers and modified Apostol-type numbers on Umbral Calculus

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Abstract By using umbral calculus and umbral algebra methods, we derive several interesting identities and relations related to the modified and unification of the Bernoulli, Euler and Genocchi polynomials and numbers and the generalized (β-) Stirling numbers of the second kind. Finally, we give some applications and remarks related to these numbers and polynomials.

Introduction, definitions and preliminaries

Throughout this paper, we use the following standard notations: Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{C}$ denote the sets of positive integers, integers, rational numbers, real numbers, positive real numbers and complex numbers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We also assume that $\log z$ denotes the principal branch of the multi-valued function $\log z$ with the imaginary part $\Im(\log z)$ constrained by $-\pi < \Im(\log z) < \pi$. For all $0 \leq k \leq n$, let $(n)_k = k!\binom{n}{k}$ (*cf.* [17]).

The unification of the Bernoulli, Euler and Genocchi polynomials is defined by Ozden [6]:

$$g_\beta(x,t;k,a,l) := \frac{2^{1-k}t^k e^{tx}}{\beta^l e^t - a^l} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}(x;k,a,l)\frac{t^n}{n!}, \tag{0.1}$$

where if $\beta = a$, then $|t| < 2\pi$ and if $\beta \neq a$, $k \in \mathbb{N}_0$, $a, l \in \mathbb{C}\backslash\{0\}$, then $|t| < l \log\left(\frac{\beta}{a}\right)$.

Remark 0.1. Note that Equation (0.1) with $x = 1$ reduces to the generating functions for the unification of the Bernoulli, Euler and Genocchi numbers.

Remark 0.2. Using the special values of a, l, k and β in (0.1), the polynomials $\mathcal{Y}_{n,\beta}(x;k,a,l)$ provide us with a generalization and unification of the Apostol-Bernoulli polynomials, Apostol-Euler polynomials and Apostol-Genocchi polynomials, respectively:

$$\mathcal{B}_n(x,\beta) = \mathcal{Y}_{n,\beta}(x;1,1,1),$$

$$\mathcal{E}_n(x,\beta) = \mathcal{Y}_{n,\beta}(x;0,-1,1)$$

and

$$\mathcal{G}_n(x,\beta) = 2\mathcal{Y}_{n,\beta}(x;1,-1,1)$$

(*cf.* [1]-[19] and the references cited in each of these earlier works). Moreover, for the classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$, one easily has

$$B_n(x) = \mathcal{B}_n(x,1),$$

$$E_n(x) = \mathcal{E}_n(x,1)$$

and

$$G_n(x) = \mathcal{G}_n(x, 1).$$

Substituting $x = 0$, one also has the classical Bernoulli numbers B_n, the classical Euler numbers E_n and the classical Genocchi numbers G_n:

$$B_n = B_n(0),$$

$$E_n = 2^n E_n(\frac{1}{2}),$$

and

$$G_n = G_n(0)$$

(*cf.* [1]-[19] and the references cited in each of these earlier works).

In [9], Ozden and Simsek modified the polynomials $\mathcal{Y}_{n,\beta}(x; k, a, l)$ as follows:

$$f(t; k, a, b, \beta) = \left(\frac{t^k 2^{1-k}}{\beta b^t - a^t}\right)^v b^{xt} = \sum_{n=0}^{\infty} Y_{n,\beta}^{(v)}(x, k, a, b)\frac{t^n}{n!}, \tag{0.2}$$

where the polynomials $Y_{n,\beta}^{(v)}(x, k, a, b)$ are called *modification and unification of the Apostol-type polynomials of order* v. One easily sees that

$$Y_{n,\beta}^{(v)}(0, k, a, b) = Y_{n,\beta}^{(v)}(k, a, b),$$

which denotes *modification and unification of the Apostol-type numbers* of order v, and

$$\mathcal{Y}_{n,\beta}^{(v)}(x; k, 1, 1) = Y_{n,\beta}^{(v)}(x, k, 1, e)$$

which denotes Apostol-type polynomials (*cf.* [6], [8]).

Ozden and Simsek [9] gave an explicit formula for the polynomials $Y_{n,\beta}^{(v)}(x, k, a, b)$ as follows:

$$Y_{n,\beta}^{(v)}(x, k, a, b) = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} (x \ln b)^{n-j} Y_{j,\beta}^{(v)}(k, a, b).$$

Ozden and Simsek [9] also gave the following recurrence relation for the numbers $Y_\beta(k, a, b)$ as follows:

$$\beta \left(Y_\beta(k, a, b) + \ln b\right)^n - \left(Y_\beta(k, a, b) + \ln a\right)^n = \begin{cases} 2^{1-k} k! & n = k, \\ 0 & n \neq k. \end{cases}$$

where $(Y_\beta(k, a, b))^m$ is replaced by $Y_{m,\beta}(k, a, b)$.

Remark 0.3. If we substitute $k = a = \beta = v = 1$ and $b = e$ into (0.2), we have

$$Y_{n,1}^{(1)}(1, 1, e) = B_n,$$

where B_n denotes the classical Bernoulli numbers. If we substitute $k = 0$, $a = v = 1$, $\beta = -1$ and $b = e$ into (0.2), we have

$$Y_{n,-1}^{(1)}(0, 1, e) = -E_n,$$

where E_n denotes the classical Euler numbers. If we substitute $k = a = v = 1$, $\beta = -1$ and $b = e$ into (0.2), we have

$$Y_{n,-1}^{(1)}(1, 1, e) = -\frac{1}{2} G_n,$$

where G_n denotes the classical Genocchi numbers.

The generalized β-Stirling type numbers of the second kind are given by the following definition:

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Definition 0.4. (see [13]) Let a, $b \in \mathbb{R}^+$ (
type numbers of the second kind $\mathcal{S}(n, v; a,$
function:

$$f_{S,v}(t; a, b; \beta) = \frac{(\beta b}{}$$

$n=0$

Substituting $a = 1$ and $b = e$ into (0.3), we have the β-Stirling numbers of the second kind

$$\mathcal{S}(n, v; 1, e; \beta) = S(n, v; \beta)$$

(*cf.* [5], [16], [17]). If $\beta = 1$, then we get the classical Stirling numbers of the second kind as follows:

$$S(n, v; 1) = S(n, v)$$

(*cf.* [1]-[19]).

Proof of the following theorem was given by Simsek [15].

Theorem 0.5. *We have*

$$\mathcal{S}(n, v; a, b; \beta) = \frac{1}{v!} \sum_{j=0}^{v} (-1)^j \binom{v}{j} \beta^{v-j} \left(j \ln a + (v - j) \ln b \right)^n \tag{0.4}$$

and

$$\mathcal{S}(n, v; a, b; \beta) = \frac{1}{v!} \sum_{j=0}^{v} (-1)^{v-j} \binom{v}{j} \beta^{j} \left(j \ln b + (v - j) \ln a \right)^n . \tag{0.5}$$

Remark 0.6. Note that by setting $a = 1$ and $b = e$ in the assertions (0.4) of Theorem 0.5, we have the following result:

$$S(n, v; \beta) = \frac{1}{v!} \sum_{j=0}^{v} \binom{v}{j} \beta^{v-j} (-1)^j (v - j)^n .$$

The above relation has been studied by Srivastava [16] and Luo [5]. For $\beta = 1$, we have

$$S(n, v) = \frac{1}{v!} \sum_{j=0}^{v} \binom{v}{j} (-1)^j (v - j)^n$$

(*cf.* [1]-[19]).

Definition 0.7. ([15]) Let a, $b \in \mathbb{R}^+$ $(a \neq b)$, $x \in \mathbb{R}$, $\beta \in \mathbb{C}$ and $v \in \mathbb{N}_0$. The generalized array type polynomials $\mathcal{S}_v^n(x; a, b; \beta)$ are defined by means of the following generating function:

$$g_v(x, t; a, b; \beta) = \frac{1}{v!} \left(\beta b^t - a^t \right)^v b^{xt} = \sum_{n=0}^{\infty} \mathcal{S}_v^n(x; a, b; \beta) \frac{t^n}{n!}. \tag{0.6}$$

By using (0.6), we have

$$\mathcal{S}_v^n(x; a, b; \beta) = \frac{1}{v!} \sum_{j=0}^{v} (-1)^{v-j} \binom{v}{j} \beta^j \left(\ln \left(a^{v-j} b^{x+j} \right) \right)^n \tag{0.7}$$

(*cf.* [15]).

We here note that the polynomials $\mathcal{S}_v^n(x; a, b; \beta)$ are called the *generalized λ-array type polynomials*.

Substituting $x = 0$ into (0.7), we arrive at (0.5):

$$\mathcal{S}_v^n(0; a, b; \beta) = \mathcal{S}(n, v; a, b; \beta).$$

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Setting $a = \beta = 1$ and $b = e$ in (0

$$S_v^n(x) =$$

a result due to Chang and Ha [2
$S_n^n(x) = 1$, $S_0^n(x) = x^n$, and for

Theorem 0.8. ([15]) *We have*

$$\mathcal{S}_v^n(x; a, b; \lambda) = \sum_{j=0} \binom{n}{j} \mathcal{S}(j, v; a, b; \lambda) (\ln b^x)^{n-j}. \tag{0.8}$$

In this paper, by using umbral calculus and umbral algebra methods, we derive many interesting identities and relations related to the modified and unification of the Bernoulli, Euler and Genocchi polynomials and numbers and the generalized (β-) Stirling numbers of the second kind. We also give some applications and remarks related to these numbers and polynomials.

1 Identities on Umbral Calculus and Umbral Algebra

In this section, we give relation between the modification and unification of the Apostol-type polynomials of order v and the Stirling numbers of the second kind on the umbral calculus and umbral algebra.

We need some identities of the umbral algebra and calculus. Here we note that the following formulas and notations are given in work of Roman [11]:

Let P be the algebra of polynomials in the single variable x over the complex number field. Let P^* be the vector space of all linear functionals on P. Let $\langle L \mid p(x) \rangle$ be the action of a linear functional L on a polynomial $p(x)$. Let F denote the algebra of formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k. \tag{1.1}$$

Let $f \in F$ define a linear functional on P and for all $k \in \mathbb{N}_0$,

$$a_k = \langle f(t) \mid x^k \rangle. \tag{1.2}$$

The order $o(f(t))$ of a power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. A series $f(t)$ for which $o(f(t)) = 1$ is called a delta series. And a series $f(t)$ for which $o(f(t)) = 0$ is called a invertible series.

Let $f(t)$, $g(t)$ be in F. Then we have

$$\langle f(t)g(t) \mid p(x) \rangle = \langle f(t) \mid g(t)p(x) \rangle. \tag{1.3}$$

For all $p(x)$ in P, we have

$$\langle e^{yt} \mid p(x) \rangle = p(y) \tag{1.4}$$

and

$$e^{yt} p(x) = p(x+y). \tag{1.5}$$

The Appell polynomials are defined by means of the following generating function

$$\sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k = \frac{1}{g(t)} e^{xt}. \tag{1.6}$$

(*cf.* [11]).

Theorem 1.1. ([11, p. 20, Theorem 2.3.6]) *Let $f(t)$ be a delta series and let $g(t)$ be an invertible series. Then there exist a unique sequence $s_n(x)$ of polynomials satisfying the orthogonality conditions*

$$\langle g(t) f(t)^k \mid s_n(x) \rangle = n! \delta_{n,k} \tag{1.7}$$

for all $n, k \in \mathbb{N}_0$.

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Let

$$s_n(x)$$

derivative formula

$$ts_n(x) = s_n'$$

Proofs of (1.8)-(1.9) were given by Roma

By (1.8), we easily obtain the following l

Lemma 1.2. *Let* $n \in \mathbb{N}_0$. *The following relat*

$$Y_{n,\beta}^{(v)}(x;k,a,b) =$$

Lemma 1.3. *Let* $n \in \mathbb{N}_0$. *The following relationship holds true:*

$$\left\langle (\beta b^t - a^t)^j \mid Y_{n,\beta}(x;k,a,b) \right\rangle = \sum_{m=0}^{j} \binom{j}{m} (-1)^{(j-m)} \beta^m Y_{n,\beta}(m\ln b + (j-m)\ln a, k, a, b).$$

Proof.

$$\begin{aligned}
&\left\langle (\beta b^t - a^t)^j \mid Y_{n,\beta}(x;k,a,b) \right\rangle \\
=& \left\langle \sum_{m=0}^{j} (-1)^{j-m} \binom{j}{m} \beta^m e^{t(m\ln b + (j-m)\ln a)} \mid Y_{n,\beta}(x;k,a,b) \right\rangle \\
=& \sum_{m=0}^{j} (-1)^{j-m} \binom{j}{m} \beta^m \left\langle e^{t(m\ln b + (j-m)\ln a)} \mid Y_{n,\beta}(x;k,a,b) \right\rangle .
\end{aligned}$$

Substituting Equation (1.4) into the above equation, we arrive at the desired result. □

Remark 1.4.

$$\left\langle (\beta b^t - 1)^j \mid Y_{n,\beta}(x;k,1,e) \right\rangle = \left\langle (\beta e^t - 1)^j \mid \mathcal{Y}_{n,\beta}(x,k,1,1) \right\rangle$$

(*cf.* [4, Lemm 2. Eq- (3.1)]).

Lemma 1.5. *Then the following identity holds:*

$$v^n S(n,l) = \frac{1}{l!} \left\langle \left(e^{vt} - 1\right)^l \mid x^n \right\rangle$$

where $S(n,l)$ *is the Stirling numbers of the second kind.*

Proof. We set

$$\frac{1}{l!}\left(e^{vt} - 1\right)^l = \sum_{j=0}^{\infty} v^n S(n,l) \frac{t^n}{n!}.$$

By using (1.1) and (1.2), we get the desired result. □

Remark 1.6. Substituting $v = 1$ into Lemma 1.5, we have

$$S(n,l) = \frac{1}{l!} \left\langle \left(e^t - 1\right)^l \mid x^n \right\rangle$$

(*cf.* [11]).

Theorem 1.7. *The following identity holds true:*

$$\begin{aligned}
&\sum_{m=0}^{j} \binom{j}{m} (-1)^{(j-m)} \beta^m Y_{n,\beta}(m\ln b + (j-m)\ln a, k, a, b) \\
=\ & 2^{1-k} (\ln b)^n \beta^{j-1} k! \binom{n}{k} \sum_{l=0}^{j-1} l! \binom{j-1}{l} \left(1 - \frac{1}{\beta}\right)^{j-l-1} \\
&\times \sum_{v=0}^{n-k} \binom{n-k}{v} ((j-1)\ln a)^{n-k-v} \left(\ln \frac{b}{a}\right)^v S(v,l).
\end{aligned}$$

where $S(u,v)$ *denote the Stirling numbers of the second kind.*

Proof. Using Lemma 1.2 , we get

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$$\left\langle (\beta b^t - a^t)^j \mid Y_{n,\beta}(x;k,a\right.$$

Substituting (1.3) and (1.9) into the abc

$$\left\langle (\beta b^t - a^t)^j \mid Y_{n,\beta}(x;k,a,b)\right\rangle$$

After some elementary calculations in the above equation, we obtain

$$\begin{aligned}&\left\langle (\beta b^t - a^t)^j \mid Y_{n,\beta}(x;k,a,b)\right\rangle \\ = \; & 2^{1-k}(\ln b)^n k!\binom{n}{k}\beta^{j-1}\sum_{l=0}^{j-1}\binom{j-1}{l}\left(1-\frac{1}{\beta}\right)^{j-l-1} e^{t((j-1)\ln a)} \\ & \times\left\langle \left(e^{t\left(\ln\frac{b}{a}\right)}-1\right)^l \mid x^{n-k}\right\rangle .\end{aligned}$$

By applying Lemma 1.5 with (1.3) in the above equation, after some calculation, we obtain the desired result. □

Remark 1.8. Substituting $a = 1$ and $b = e$ into Theorem 1.7, we arrive at the work of Dere et al. [4, Thorem 3, p. 3253 and Corollary 2, p. 3254]

$$\begin{aligned}&\sum_{m=0}^{j}\binom{j}{m}(-1)^{(j-m)}\beta^m\mathcal{Y}_{n,\beta}(m;k,1,1) \\ = \; & \frac{\beta^{j-1}}{2^{k-1}}k!\binom{n}{k}\sum_{l=0}^{j-1}\frac{(j-1)!}{(j-l-1)!}\left(1-\frac{1}{\beta}\right)^{j-l-1} S(n-k,l) .\end{aligned}$$

Remark 1.9. By setting $\beta = k = a = 1$ and $b = e$ in Theorem 1.7, we arrive at the following well-known results which was proved by Roman [11, P. 94]:

$$\left\langle (e^t-1)^j \mid B_n(x)\right\rangle = n(j-1)!S(n-1,j-1)$$

or

$$\sum_{m=0}^{j}\binom{j}{m}(-1)^{j-m}B_n(m) = n(j-1)!S(n-1,j-1) .$$

Substituting $a = 1$, $k = 0$, $\beta = -1$ and $b = e$ into Theorem 1.7, we arrive at the following corollary:

Corollary 1.10.

$$\sum_{m=0}^{j}\binom{j}{m}E_n(m) = \sum_{l=0}^{j-1}\binom{j-l}{l}2^{j-l}l!S(n,l) .$$

Substituting $a = 1$, $k = 1$, $\beta = -1$ and $b = e$ into Theorem 1.7, we arrive at the following corollary:

Corollary 1.11.

$$\sum_{m=0}^{j}\binom{j}{m}G_n(m) = n\sum_{l=0}^{j-1}\binom{j-l}{l}2^{j-l-2}l!S(n-1,l) .$$

Corollary 1.12.

$$\left\langle (e^t+1)^j \mid G_n(x)\right\rangle = \sum_{m=0}^{j}\binom{j}{m}G_n(m).$$

Proof. In the work of De

$$\langle (e^t - 1)^j \mid$$

Combining this equation

Theorem 1.13. (Recurrence relation) *Let $v \geq 2$. Then we have*

$$\beta Y_{n,\beta}^{(v)}(x + \ln b; k, a, b) = Y_{n,\beta}^{(v)}(x + \ln a; k, a, b) + 2^{1-k}(n)_k(\ln b)^k Y_{n-k,\beta}^{(v-1)}(x; k, a, b).$$

Proof. By using Lemma 1.2, we get

$$(\beta b^t - a^t)Y_{n,\beta}^{(v)}(x; k, a, b) = 2^{1-k}(n)_k(\ln b)^k Y_{n-k,\beta}^{(v-1)}(x; k, a, b). \tag{1.10}$$

We also use (1.5), we obtain

$$(\beta b^t - a^t)Y_{n,\beta}^{(v)}(x; k, a, b) = \beta Y_{n,\beta}^{(v)}(x + \ln b; k, a, b) - Y_{n,\beta}^{(v)}(x + \ln a; k, a, b). \tag{1.11}$$

By combining (1.10) and (1.11), we get the desired result. □

Remark 1.14. If we set $a = 1$ and $b = e$ in Theorem 1.13, we obtain [4, p. 3256, Theorem 6]

$$\beta \mathcal{Y}_{n,\beta}^{(v)}(x + 1; k, 1, 1) = \mathcal{Y}_{n,\beta}^{(v)}(x; k, 1, 1) + 2^{1-k}(n)_k \mathcal{Y}_{n-k,\beta}^{(v-1)}(x; k, 1, 1)$$

Remark 1.15. By substituting $a = 1$, $k = 0$, $b = e$ and $\beta = -1$ into Theorem 1.13, we arrive at the recurrence relations for the Euler polynomials of higher-order as follows: let $v \geq 2$ and $n \in \mathbb{N}$. Then we have

$$E_n^{(v)}(x + 1) = -E_n^{(v)}(x) + 2E_n^{(v-1)}(x)$$

(*cf.* [4], [11, p.103]).

Remark 1.16. By substituting $a = k = 1$, $b = e$ and $\beta = -1$ into Theorem 1.13, we have recurrence relations for the Genocchi polynomials of higher-order as follows: let $v \geq 2$ and $n \in \mathbb{N}$.

$$(e^t + 1)G_n^{(v)}(x) = 2nG_{n-1}^{(v-1)}(x)$$

(*cf.* [3, p. 760, Theorem7]).

Remark 1.17. By substituting $a = k = 1$, $b = e$ and $\beta = 1$ into Theorem 1.13, Dere et al [4] and Roman [11, p. 95, Eq. (4.2.6)] gave recurrence relations for the Bernoulli polynomials of higher-order as follows: Let $v \geq 2$ and $n \in \mathbb{N}$. Then we have

$$B_n^{(v)}(x + 1) = B_n^{(v)}(x) + nB_{n-1}^{(v-1)}(x).$$

By using Lemma 1.2 with (1.10), for $v = 1$, we get the following theorem, which is very useful in the theory of the Diophantine equation:

Theorem 1.18. *Let $n, k \in \mathbb{N}_0$ with $n \geq k$. Then we have*

$$\left(\beta b^t - a^t\right) Y_{n,\beta}(x; k, a, b) = 2^{1-k}(n)_k(\ln b)^n x^{n-k}. \tag{1.12}$$

Remark 1.19. Substituting $a = 1$ and $b = e$ into (1.12), we have

$$\beta \mathcal{Y}_{n,\beta}(x + 1; k, 1, 1) - \mathcal{Y}_{n,\beta}(x; k, 1, 1) = 2^{1-k}(n)_k x^{n-k}$$

(*cf.* [10]). By substituting $a = k = 1$, $b = e$ and $\beta = 1$ into (1.12), we have

$$B_n(x + 1) - B_n(x) = nx^{n-1}$$

(*cf.* [10], [11, p. 95], [17]). By substituting $a = 1$, $k = 0$, $b = e$ and $\beta = -1$ into (1.12), we have

$$E_n(x + 1) + E_n(x) = 2x^n$$

(*cf.* [10], [11, p. 95], [17]). By substituting $a = k = 1$, $b = e$ and $\beta = -1$ into (1.12), we have

$$G_n(x + 1) + G_n(x) = 2nx^{n-1}$$

(*cf.* [3, p. 760, Corollary 1], [10]).

Retracted upon the authors request

References

[1] T. M. Apostol, On t

[2] C.-H. Chang and C.
tions of the Bernoul

[3] R. Dere and Y. Sims
218(3) (2011), 756-

[4] R. Dere, Y. Simsek
type polynomials ba
133 (2013), 3245-3263.

[5] Q.-M. Luo and H. M. Srivastava, Some generalizations of the Apostol-Genocchi polynomials and the Stirling numbers of the second kind, Appl. Math. Comput. 217 (2011), 5702-5728.

[6] H. Ozden, Unification of generating function of the Bernoulli, Euler and Genocchi numbers and polynomials, Amer. Inst. Phys. Conf. Proc. 1281 (2010), 1125-1128.

[7] H. Ozden and Y. Simsek, A new extension of q-Euler numbers and polynomials related to their interpolation functions, Appl. Math. Lett. 21 (2008), 934-939.

[8] H. Ozden, Y. Simsek and H. M. Srivastava, A unified presentation of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials, Comput. Math. Appl. 60 (2010) 2779-2787.

[9] H. Ozden and Y. Simsek, Modification and unification of the Apostol-type numbers and polynomials, Appl. Math. Comput. 235 (2014), 338-351.

[10] Unified presentation of p-adic L-functions associated with unification of the special numbers, preprint.

[11] S. Roman, *The Umbral Calculus*, Dover Publ. Inc. New York, 2005.

[12] Y. Simsek, Twisted (h, q)-Bernoulli numbers and polynomials related to twisted (h, q)-zeta function and L-function, J. Math. Anal. Appl. 324 (2006), 790-804.

[13] Y. Simsek, Interpolation function of generalized q-Bernstein type polynomials and their application, Curve and Surface, Springer Verlag Berlin Heidelberg 2011, LNCS 6920, (2011), 647-662.

[14] Y. Simsek, On q-deformed Stirling numbers, Int. J. Math. Comput. 15 (2012), 70-80.

[15] Y. Simsek, Generating functions for generalized Stirling type numbers, Array type polynomials, Eulerian type polynomials and their applications, Fixed Point Theory Appl. 2013, 2013:87.

[16] H. M. Srivastava, Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials, Appl. Math. Inform. Sci. 5 (2011), 390-444.

[17] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London, NewYork, 2012.

[18] H.M. Srivastava, M. Garg and S. Choudhary, A new generalization of the Bernoulli and related polynomials. Russian J. Math. Phys. **17**, 251-261 (2010).

[19] H.M. Srivastava, M. Garg and S. Choudhary, Some new families of the generalized Euler and Genocchi polynomials. Taiwanese J. Math. **15**, 283-305 (2011).

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