MODULES WHICH ARE INVARIANT UNDER IDEMPOTENT ENDOMORPHISMS OF THEIR INJECTIVE HULLS

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Abstract A module which is invariant under automorphisms of its injective hull is called an automorphism-invariant module. Automorphism-invariant modules, which generalize the notion of quasi-injective modules, are precisely the pseudo-injective modules. Here, a study on modules which are invariant under idempotent endomorphisms of their injective hulls is carried out and such modules will be called idempotent-invariant modules. In this paper, we discuss that idempotent-invariant modules, which also generalize the notion of quasi-injective modules and show that the classes of automorphism-invariant modules and idempotent-invariant modules are not contained one in the other. Some facts and results of this class of modules are obtained. Automorphism-invariant modules are clean but not so for idempotent-invariant modules. We investigate certain conditions under which idempotent-invariant modules can actually be clean. We also establish some relations of idempotent-invariant modules with ADS, SIP, SSP and pseudo-continuous modules.

1 Introduction

Let \( R \) be an associative ring with unity. An element \( a \in R \) is said to be clean if \( a = e + u \) where \( e \) is an idempotent and \( u \) is a unit in \( R \). If every element of \( R \) is clean, then \( R \) is called a clean ring. Clean rings were introduced by W. K. Nicholson in [17]. Nicholson proved that every clean ring is an exchange ring, and a ring with central idempotents is clean if and only if it is an exchange ring. A ring is said to be clean (almost clean) if each of its elements is the sum of a unit (regular element) and an idempotent. A module is clean (almost clean) if its endomorphism ring is clean (almost clean). A module which is invariant under automorphisms of its injective hull is called an automorphism-invariant module, i.e., \( M \) is an automorphism-invariant module if \( f(M) \subseteq M \) for all \( f \in \text{Aut}(E(M)) \). It has been proved in [9, Theorem 16] that automorphism-invariant modules are precisely the pseudo-injective modules, where a module \( M \) is called pseudo-injective, if for any submodule \( A \) of \( M \), every monomorphism \( f : A \to M \) can be extended to an endomorphism of \( M \).

A module which is invariant under idempotent endomorphisms of its injective hull will be called an idempotent-invariant module, i.e., \( M \) will be called an idempotent-invariant module if \( f(M) \subseteq M \) for all \( f^2 = f \in \text{End}(E(M)) \). It has been proved in [11, Theorem 1.1] that idempotent-invariant modules are precisely the \( \pi \)-injective (quasi-continuous) modules.

Consider the following conditions for an \( R \)-module \( M \):

\((C_1)\) Every submodule of \( M \) is essential in a direct summand of \( M \).

\((C_2)\) Every submodule of \( M \) that is isomorphic to a direct summand of \( M \) is itself a direct summand of \( M \).

\((C_3)\) If \( A \) and \( B \) are direct summands of \( M \) with \( A \cap B = 0 \) then \( A \oplus B \) is also a direct summand of \( M \).

\( M \) is called a CS (or an extending) module if it satisfies \((C_1)\); \( M \) is called continuous if it satisfies \((C_1) \) and \((C_2)\); \( M \) is called quasi-continuous if it satisfies \((C_1) \) and \((C_3)\).

Modules satisfying \((C_1)\), \((C_2)\) and \((C_3)\) are called \( C1 \), \( C2 \) and \( C3 \)-modules respectively.

It is well known that the following implications hold:
Injective $\implies$ quasi-injective $\implies$ continuous $\implies$ quasi-continuous $\implies$ CS.

But none of the converses hold in general. For background on injective, quasi-injective, continuous, quasi-continuous and CS modules, we refer to [8] and [16].

A right $R$-module $M$ is said to satisfy the exchange property if for every right $R$-module $A$ and any two direct sum decompositions $A = M_1 \oplus N = \oplus_{i \in I} A_i$ with $M_1 \cong M$ there exist submodules $B_i$ of $A_i$ such that $A = M_1 \oplus (\oplus_{i \in I} B_i)$. If this holds only for $|I| < \infty$, then $M$ is said to satisfy the finite exchange property. It is well known that a continuous module is idempotent-invariant and continuous modules satisfy the exchange property but it is not so for idempotent-invariant modules. For an idempotent-invariant module, the finite exchange property implies the full exchange property.

It has been proved in [12] that automorphism-invariant modules are clean and they satisfy the exchange property. Similar results do not hold for idempotent-invariant modules. In this paper, we investigate certain conditions under which an idempotent-invariant module can be clean or almost clean. Every idempotent-invariant module is pseudo-continuous but the converse is not true. Here we investigate certain conditions under which a pseudo-continuous module is idempotent-invariant. We provide examples to show that the classes of automorphism-invariant modules and idempotent-invariant modules are not contained one in the other and also show that a direct sum of idempotent-invariant modules need not be idempotent-invariant although summands of idempotent-invariant modules inherit the property. Finally, we also establish some of the relations of idempotent-invariant modules with ADS modules and modules with the SIP and SSP.

Throughout, all rings $R$ are associative with unity and all modules are unitary $R$-modules, unless otherwise stated. For a module $M$, we use $E(M)$, $\text{End}(M)$ and $\text{Aut}(M)$ to denote the injective hull, the endomorphism ring and the group of automorphisms of $M$, respectively. $\ker f$ and $\text{Im}f$ denote the kernel of $f$ and the image of $f$ respectively. We write $N \subseteq M$ if $N$ is a submodule of $M$, $N \subseteq^{ess} M$ if $N$ is an essential submodule of $M$ and $N \subseteq^{=} M$ if $N$ is a direct summand of $M$.

### 2 Automorphism-invariant modules and idempotent-invariant modules

A module which is invariant under automorphisms of its injective hull is called an automorphism-invariant module, i.e., $M$ is called an automorphism-invariant module if $f(M) \subseteq M$ for all $f \in \text{Aut}(E(M))$.

Quasi-injective modules are automorphism-invariant but the converse is not true, in general.

**Theorem 2.1.** [15, Corollary 13] A module $M$ is quasi-injective if and only if it is automorphism-invariant CS.

**Theorem 2.2.** [7, Theorem 2.6] Every pseudo-injective module $M$ satisfies $(C_2)$.

**Corollary 2.3.** Every CS automorphism-invariant module is continuous.

**Proof.** Let $M$ be a CS automorphism-invariant module. Since automorphism-invariant modules are precisely the pseudo-injective modules, by Theorem 2.2 $M$ satisfies $(C_2)$. By assumption $M$ is CS and $M$ also satisfies $(C_2)$. Hence $M$ is continuous.

It is pertinent to mention that the classes of automorphism-invariant modules and idempotent-invariant modules are not contained one in the other as shown by the following examples.

**Example 2.4.** If $\mathbb{Z}$, $\mathbb{Q}$ denote the ring of integers and rational numbers respectively, $\mathbb{Z}_2$ is an idempotent-invariant module which is not automorphism-invariant because the injective hull $\mathbb{Q}_2$ of $\mathbb{Z}_2$ has the automorphism $\varphi : \mathbb{Q} \to \mathbb{Q}$ defined by $\varphi(q) = \frac{q}{2}$ but $\varphi(\mathbb{Z}) \not\subseteq \mathbb{Z}$.

**Example 2.5.** If $R$ is the ring of all eventually constant sequences $(x_n)_{n \in \mathbb{N}}$ of elements in $\mathbb{Z}_2$, then $E(R_R) = \prod_{n \in \mathbb{N}} \mathbb{Z}_2$ has only one automorphism, namely the identity automorphism. Thus $R_R$ is an automorphism-invariant module but $R_R$ cannot be CS because by Theorem 2.1 a module $M$ is quasi-injective if and only if it is automorphism-invariant CS. Hence $R_R$ is not an idempotent-invariant module.
Theorem 2.6. [15, Theorem 12] Every automorphism-invariant module satisfies \((C_3)\).

Though the classes of automorphism-invariant and idempotent-invariant modules are not contained one in the other, the following corollary, which is a consequence of the above theorem, shows that \(CS\) automorphism-invariant modules are indeed idempotent-invariant.

Corollary 2.7. Every \(CS\) automorphism-invariant module is idempotent-invariant.

Proof. Let \(M\) be a \(CS\) automorphism-invariant module. Then by Theorem 2.6, \(M\) satisfies \((C_3)\) and by assumption \(M\) being \(CS\), satisfies \((C_1)\). Thus \(M\) is an idempotent-invariant module.

A module \(M\) is called an \(ADS\) (Absolute Direct Summand) module if for every decomposition \(M = A \oplus B\) of \(M\) and every complement \(C\) of \(A\), \(M = A \oplus C\). A module \(M\) is called \(square-free\) if it contains no non-zero isomorphic submodules \(A\) and \(B\) with \(A \cap B = 0\). An \(R\)-module \(M\) satisfies \((C_4)\) if, whenever \(A\) and \(B\) are submodules of \(M\) with \(M = A \oplus B\) and \(f : A \to B\) is an \(R\)-homomorphism with \(\ker f \subseteq \oplus A\), then \(\text{Im} f \subseteq \oplus B\). An \(R\)-module \(M\) satisfying \((C_4)\) is called a \(C4\)-module. A \(C4\)-module which is also \(CS\), is called a pseudo-continuous module.

Some examples of \(C4\)-modules are \(ADS\) modules, automorphism-invariant modules and square-free modules.

We know that \((C_2) \Rightarrow (C_3)\) and also \((C_3) \Rightarrow (C_4)\). Hence we have the following implications:

\((C_2) \Rightarrow (C_3) \Rightarrow (C_4)\) which further yields the implications continuous \(\Rightarrow\) quasi-continuous \(\Rightarrow\) pseudo-continuous.

But the converses are not true in general.

Theorem 2.8. Every \(CS\) automorphism-invariant module is pseudo-continuous.

Proof. Let \(M\) be an automorphism-invariant module. Then by Theorem 2.6, \(M\) satisfies \((C_3)\) and since \((C_3)\) implies \((C_4)\), \(M\) satisfies \((C_4)\). Thus \(M\) being a \(CS\) automorphism-invariant module, is pseudo-continuous.

The following theorem by Goel and Jain is vital in establishing the equivalence between the classes of \(\pi\)-injective and idempotent-invariant modules.

Theorem 2.9. [11, Theorem 1.1] For any \(R\)-module \(M\) the following are equivalent:

(a) \(M\) is \(\pi\)-injective.

(b) For every idempotent \(e\) in \(\hom(\hat{M}, \hat{M})\), \(eM \subseteq M\).

(c) If \(\hat{M} = N_1 \oplus N_2\), then \(M = (N_1 \cap M) \oplus (N_2 \cap M)\).

(d) If \(\hat{M} = \bigoplus_{i \in A} N_i\), then \(M = \bigoplus_{i \in A} (N_i \cap M)\) for any index set \(A\).

(Note that in [11], \(\hat{M}\) denotes the injective hull of \(M\) and \(\hom(\hat{M}, \hat{M})\) denotes the endomorphism ring of \(R\)-homomorphisms of \(\hat{M}\)).

In [11], Goel and Jain call a module \(M\) \(\pi\)-injective if for every pair of submodules \(M_1\) and \(M_2\) with \(M_1 \cap M_2 = 0\), each projection \(\pi_i : M_1 \oplus M_2 \to M_i, i = 1, 2\), can be lifted to an endomorphism of \(M\). \(\pi\)-injective modules are the same as the quasi-continuous modules defined by Jeremy in [14].

We call a module which is invariant under idempotent endomorphisms of its injective hull an idempotent-invariant module, i.e., \(M\) is called an idempotent-invariant module if \(f(M) \subseteq M\) for all \(f^2 = f \in \text{End}(E(M))\). Hence Theorem 2.9 can be restated as:

Theorem 2.10. For any \(R\)-module \(M\) the following are equivalent:

(a) \(M\) is quasi-continuous.

(b) \(M\) is idempotent-invariant, i.e., for every idempotent \(f\) in \(\text{End}(E(M))\), \(f(M) \subseteq M\).

(c) If \(E(M) = N_1 \oplus N_2\), then \(M = (N_1 \cap M) \oplus (N_2 \cap M)\).

(d) If \(E(M) = \bigoplus_{i \in I} N_i\), then \(M = \bigoplus_{i \in I} (N_i \cap M)\) for any index set \(I\).

It is to be noted that a summand of an idempotent-invariant module is also idempotent-invariant. However, a direct sum of idempotent-invariant modules need not be idempotent-invariant as shown by the following example.
Example 2.11. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where $F$ is any field and let $A = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$. It is clear that $A$ and $B$ are idempotent-invariant as $R$-modules. However, $R = A \oplus B$ is not idempotent-invariant as $R$ satisfies $(C_1)$ but does not satisfy $(C_3)$.

Recall that a ring is clean (almost clean) if each of its elements is the sum of a unit (regular element) and an idempotent. A module is clean (almost clean) if its endomorphism ring is clean (almost clean).

Theorem 2.12. [4, Theorem 3.9] Every continuous module is clean.

It is interesting to note that although continuous modules are clean, a similar result does not hold, either for $CS$ modules or for quasi-continuous modules. However, it has been proved in [12, Corollary 4] that automorphism-invariant modules are clean and in [12, Theorem 3] that automorphism-invariant modules satisfy the exchange property but similar results do not hold for idempotent-invariant modules.

For example, $\mathbb{Z}$ is an idempotent-invariant module but $\mathbb{Z}$ is not a clean ring; in fact, $\mathbb{Z}$ is not even an exchange ring. The question now arises under what conditions a quasi-continuous module would be clean. The following is one of the consequences of Theorem 2.12 for quasi-continuous modules.

Theorem 2.13. [4, Theorem 4.3] A quasi-continuous $R$-module $M$ is clean if and only if it has the finite exchange property if and only if it has the full exchange property.

It has been proved in [18] that for quasi-continuous modules, the finite exchange property implies the full exchange property.

Recall that the singular submodule $Z(M)$ of a module $M$ is defined as $Z(M) = \{m \in M : nm = 0 \text{ for some essential right ideal } I \text{ of } R\}$ and a module $M$ is called singular if $Z(M) = M$ and nonsingular if $Z(M) = 0$.

Theorem 2.14. [1, Theorem 2.6] If $M$ is a quasi-continuous and non-singular module, then $M$ is almost clean.

Because of the equivalence of the classes of quasi-continuous modules and idempotent-invariant modules, the following results are immediate consequences of the above two theorems.

Corollary 2.15. An idempotent-invariant module $M$ is clean if and only if it has the finite exchange property if and only if it has the full exchange property.

Proof. The proof follows from Theorem 2.13.

Corollary 2.16. Every non-singular idempotent-invariant module is almost clean.

Proof. The proof follows from Theorem 2.14.

For any $R$-module $M$, a monomorphism $f \in \text{End}(M)$ is called an essential monomorphism if $\text{Im}(f) \subseteq \text{ess} M$ (i.e., $\text{Im}(f)$ is an essential submodule of $M$). A module $M$ is called essentially co-Hopfian if every essential monomorphism in $\text{End}(M)$ is an isomorphism. A module $M$ is called co-Hopfian if every monomorphism in $\text{End}(M)$ is an isomorphism.

Theorem 2.17. [4, Proposition 4.5] If $M$ is a $CS$ module, then every element $f \in \text{End}(M)$ can be written as $e + v$ where $e = e^2 \in \text{End}(M)$ and $v \in \text{End}(M)$ is a monomorphism.

Theorem 2.18. [1, Proposition 2.5] If $M$ is quasi-continuous, then every endomorphism of $M$ is the sum of an idempotent and an essential monomorphism.

With the aid of the two above results, we can now prove that every co-Hopfian $CS$ module is clean and every essentially co-Hopfian idempotent-invariant module is clean.

Corollary 2.19. Every co-Hopfian $CS$ module is clean.
Proof. Let $M$ be a co-Hopfian $CS$ module. Since $M$ is $CS$, by Theorem 2.17 every $f \in \text{End}(M)$ is the sum of an idempotent and a monomorphism. But $M$ being co-Hopfian, every monomorphism is an isomorphism. Thus every $f \in \text{End}(M)$ being the sum of an idempotent and an isomorphism, $M$ is clean.

By Theorem 2.12, we know that every continuous module is clean and by Theorem 2.19 every co-Hopfian $CS$ module is clean. So it seems quite natural to raise the following question: Is every clean module continuous? The following example negates the possibility of every clean module being continuous.

Let $R = \left( \begin{array}{cc} F & F \\ 0 & F \end{array} \right)$ where $F$ is any field. Then $R_R$ is an Artinian (and thus a co-Hopfian) $CS$ module and so $R_R$ is clean but it is not quasi-continuous since $R_R$ satisfies $(C_1)$ but does not satisfy $(C_3)$. Hence $R_R$ does not satisfy $(C_2)$ either and so $R_R$ is not continuous.

Theorem 2.20. Every essentially co-Hopfian idempotent-invariant module is clean.

Proof. Let $M$ be an essentially co-Hopfian idempotent-invariant module. Since $M$ is idempotent-invariant, by Theorem 2.18 every $f \in \text{End}(M)$ is the sum of an idempotent and an essential monomorphism. Since $M$ is essentially co-Hopfian, every essential monomorphism is an isomorphism. Thus every $f \in \text{End}(M)$ being the sum of an idempotent and an isomorphism, $M$ is clean.

(Alternative Proof). Let $M$ be an essentially co-Hopfian idempotent-invariant module. Then $M$ is continuous (see [16, Lemma 3.14]) and so by Theorem 2.12, $M$ is clean.

Lemma 2.21. [2, Lemma 3.1] An $R$-module $M$ is $ADS$ if and only if for each decomposition $M = A \oplus B$, $A$ and $B$ are mutually injective.

Lemma 2.22. [8, Lemma 7.5] Let $A$ and $B$ be $R$-modules and let $M = A \oplus B$. Then the following are equivalent:

(a) $B$ is $A$-injective.

(b) For every submodule $N$ of $M$ such that $N \cap B = 0$, there exists a submodule $M_1$ of $M$ such that $M = M_1 \oplus B$ and $N \leq M_1$.

Theorem 2.23. [19, Theorem 2.7] Every $ADS$ module is a $C3$-module.

Proof. Let $M$ be an $ADS$ module and let $A \subseteq M$, $B \subseteq M$ such that $A \cap B = 0$. We have to show that $A \oplus B$ is a direct summand of $M$. Let $M = A \oplus A_1$ and $M = B \oplus B_1$ for some submodules $A_1, B_1$ of $M$. Then by Lemma 2.21, $A$ is $A_1$-injective. Hence by Lemma 2.22, there exists $N \leq M$ such that $M = N \oplus A$ and $B \leq N$. It follows that $N = B \oplus (N \cap B_1)$ and so $M = (A \oplus B) \oplus (N \cap B_1)$, thereby implying that $M = A \oplus B$ is a direct summand of $M$.

Every idempotent-invariant module is $ADS$ as proved in [16, Theorem 2.8] but the converse is not true. As a consequence of Theorem 2.23, we have the following result.

Corollary 2.24. Every $ADS$ module which is also $CS$ is idempotent-invariant.

Proof. Let $M$ be an $ADS$ module. Then by Theorem 2.23, $M$ is a $C3$-module and $M$ also being $CS$, is idempotent-invariant.

Theorem 2.25. Every essentially co-Hopfian $ADS$ module which is $CS$ is clean.

Proof. Let $M$ be an essentially co-Hopfian $ADS$ module which is $CS$. Then by Corollary 2.24, $M$ is idempotent-invariant. By assumption, $M$ is essentially co-Hopfian. Thus by Theorem 2.20, $M$ is clean.

Theorem 2.26. Every $ADS$ module which is also $CS$ is pseudo-continuous.

Proof. Let $M$ be an $ADS$ module which is also $CS$. Then, by Corollary 2.24, $M$ is idempotent-invariant. Since every idempotent-invariant module is pseudo-continuous, $M$ is pseudo-continuous.
Recall that a module $M$ is said to have the $SIP$ (Summand Intersection Property), if the intersection of every pair of direct summands of $M$ is a direct summand of $M$. A module $M$ is said to have the $SSP$ (Summand Sum Property), if the sum of every pair of direct summands of $M$ is a direct summand of $M$. A module $M$ is said to be polyform if for every $N \subseteq M$ and homomorphism $f : N \to M$, $\ker f$ is closed in $N$.

**Lemma 2.27.** [13, Theorem 2.3] Let $M$ be an $R$-module. Then

(a) $M$ has the $SIP$ if and only if for every decomposition $M = A \oplus B$ and every $R$-homomorphism $f : A \to B$, the kernel of $f$ is a direct summand of $A$, i.e., $\ker f \subseteq \oplus A$.

(b) $M$ has the $SSP$ if and only if for every decomposition $M = A \oplus B$ and every $R$-homomorphism $f : A \to B$, the image of $f$ is a direct summand of $B$, i.e., $\text{Im} f \subseteq \oplus B$.

It has been proved in [3, Lemma 19] that if $M$ is a $C3$-module with the $SIP$, then $M$ has the $SSP$. As a consequence of this, we can have the following result.

**Corollary 2.28.** Every idempotent-invariant module with the $SIP$ has the $SSP$.

The next theorem provides a sufficient condition for the endomorphism ring of an idempotent-invariant module to have the summand sum property.

**Theorem 2.29.** If $M$ is a nonsingular idempotent-invariant module, then $\text{End}(M)$ has the $SSP$.

**Proof.** Let $M$ be a nonsingular idempotent-invariant module and let $A, B \subseteq M$ such that $M = A \oplus B$ and let $f : A \to B$ be an $R$-homomorphism. Since $M$ is nonsingular, $A/\ker f$ is nonsingular and $\ker f$ is a closed submodule of $M$. Therefore, $\ker f \subseteq \oplus A$ and so by Lemma 2.27 (a), $M$ has the $SIP$. By Corollary 2.28, $M$ also has the $SSP$. Since $M$ has both the $SIP$ and the $SSP$, $\text{End}(M)$ has the $SSP$ (see [10, Theorem 2.3]).

We know that every module with the $SSP$ is a $C3$-module and also every $C3$-module is a $C4$-module. Hence every module with the $SSP$ is a $C4$-module.

The following result is analogous to a result on $C3$-modules discussed earlier.

**Theorem 2.30.** If $M$ is a $C4$-module with the $SIP$, then $M$ has the $SSP$.

**Proof.** Let $A, B \subseteq M$ such that $M = A \oplus B$ and let $f : A \to B$ be an $R$-homomorphism. Since $M$ is an $SIP$-module, by Lemma 2.27 (a), $\ker f \subseteq \oplus A$. Hence, by definition of $C4$-module, $\text{Im} f \subseteq \oplus B$. Thus by Lemma 2.27 (b), $M$ has the $SSP$.

The following corollary is an immediate consequence of the above theorem.

**Corollary 2.31.** Every pseudo-continuous module with the $SIP$ has the $SSP$.

**Theorem 2.32.** If $M$ is a nonsingular $CS$ module, then $M$ has the $SIP$.

**Proof.** Let $M$ be a nonsingular $CS$ module. We use Lemma 2.27 (a) to show that $M$ has the $SIP$, i.e., we show that if $A, B \subseteq M$ such that $M = A \oplus B$ and $f : A \to B$ is an $R$-homomorphism, then $\ker f \subseteq \oplus A$. Since $M$ is a $CS$ module and $A \subseteq \oplus M$, $A$ is also a $CS$ module and so $\ker f \subseteq \text{ess} L \subseteq \oplus A$ for a submodule $L$ of $M$. Since $M$ is nonsingular, $A/\ker f$ is nonsingular and $\ker f$ is a closed submodule of $M$. Therefore $\ker f \subseteq \oplus A$ and so by Lemma 2.27 (a), $M$ has the $SIP$.

**Corollary 2.33.** If $M$ is a nonsingular pseudo-continuous module, then $M$ has the $SSP$. In particular, $M$ is an idempotent-invariant module.

**Proof.** Let $M$ be a nonsingular pseudo-continuous module and let $A, B \subseteq M$ such that $M = A \oplus B$ and $f : A \to B$ is an $R$-homomorphism. Since every pseudo-continuous module is obviously $CS$, by Theorem 2.32 $M$ has the $SIP$ and so $\ker f \subseteq \oplus A$. $M$ being a $C4$-module, by definition, if $A$ and $B$ are submodules of $M$ such that $M = A \oplus B$ and $f : A \to B$ is an $R$-homomorphism with $\ker f \subseteq \oplus A$, then $\text{Im} f \subseteq \oplus B$. Therefore using Lemma 2.27 (b), $M$ has the $SSP$. As every module with the $SSP$ satisfies $(C3)$ and $M$ also being $CS$, $M$ is an idempotent-invariant module.
**Theorem 2.34.** If \( M \) is a polyform CS module, then \( M \) has the SIP.

**Proof.** Let \( M \) be a polyform CS module and let \( M = A \oplus B \) and \( f : A \rightarrow B \) be an \( R \)-homomorphism. Since \( M \) is polyform, \( \ker f \) is closed in \( A \). Since \( M \) is a CS module and \( A \subseteq M, A \) is also a CS module. Hence \( \ker f \subseteq A \) and so by Lemma 2.27 (a), \( M \) has the SIP.

**Corollary 2.35.** If \( M \) is a polyform pseudo-continuous module, then \( M \) has the SSP. In particular, \( M \) is an idempotent-invariant module.

**Proof.** Let \( M \) be a polyform pseudo-continuous module and let \( M = A \oplus B \) and \( f : A \rightarrow B \) be an \( R \)-homomorphism. Since every pseudo-continuous module is obviously CS, by Theorem 2.34, \( M \) has the SIP and so \( \ker f \subseteq A \). \( M \) being a 4-module, by definition, \( \text{Im} f \subseteq B \) and so using Lemma 2.27 (b), \( M \) has the SSP. As every SSP module satisfies \((C_3)\) and \( M \) also being CS, \( M \) is an idempotent-invariant module.

**Theorem 2.36.** Let \( M \) be an idempotent-invariant polyform module. Then \( M \) is an ADS module with the SIP.

**Proof.** Let \( M = A \oplus B \) and \( f : A \rightarrow B \) be a homomorphism. \( M \) being an idempotent-invariant module is also an ADS module. Since \( M \) is polyform, \( \ker f \) is closed in \( A \) and so \( \ker f \subseteq A \). Thus by Lemma 2.27 (a), \( M \) has the SIP.

We know that every idempotent-invariant module is pseudo-continuous but the converse is not true in general. The following results can now be established as consequences of Corollary 2.33 and Corollary 2.35.

**Theorem 2.37.** A module \( M \) is idempotent-invariant if and only if \( M \) is a nonsingular pseudo-continuous module.

**Theorem 2.38.** A module \( M \) is idempotent-invariant if and only if \( M \) is a polyform pseudo-continuous module.

Recall that a module \( M \) is said to satisfy the (full) internal exchange property if for every internal direct sum decomposition \( M = \oplus_{i \in I} M_i \) and every summand \( N \subseteq M \), there exist submodules \( N_i \subseteq M_i, i \in I \), such that \( M = \oplus_{i \in I} N_i \oplus N \). If this holds only for \( |I| < \infty \), then \( M \) is said to satisfy the finite internal exchange property. We have discussed earlier that \( \mathbb{Z}_2 \) is an idempotent-invariant module which does not satisfy the finite exchange property. However, it is to be noted that \( \mathbb{Z}_2 \) satisfies the full internal exchange property. It has been proved in [6, Proposition 2.22] that if \( M \) is a C4-module with the finite internal exchange property, then \( M \) is a C3-module. Also, it has been proved in [18, Proposition 1.1] that every idempotent-invariant-module satisfies the internal exchange property. The next theorem follows as an immediate consequence of both the results.

**Theorem 2.39.** A module \( M \) is idempotent-invariant if and only if \( M \) is a pseudo-continuous module with the internal exchange property.

**References**


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