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MODULES WHICH ARE INVARIANT UNDER IDEMPOTENT ENDOMORPHISMS OF THEIR INJECTIVE HULLS

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Abstract A module which is invariant under automorphisms of its injective hull is called an automorphism-invariant module. Automorphism-invariant modules, which generalize the notion of quasi-injective modules, are precisely the pseudo-injective modules. Here, a study on modules which are invariant under idempotent endomorphisms of their injective hulls is carried out and such modules will be called idempotent-invariant modules. In this paper, we discuss that idempotent-invariant modules, which also generalize the notion of quasi-injective modules, are precisely the quasi-continuous modules and show that the classes of automorphism-invariant modules and idempotent-invariant modules are not contained one in the other. Some facts and results of this class of modules are obtained. Automorphism-invariant modules are clean but not so for idempotent-invariant modules. We investigate certain conditions under which idempotent-invariant modules can actually be clean. We also establish some relations of idempotent-invariant modules with ADS, SIP, SSP and pseudo-continuous modules.

1 Introduction

Let R be an associative ring with unity. An element $a \in R$ is said to be *clean* if $a = e + u$ where e is an idempotent and u is a unit in R. If every element of R is clean, then R is called a clean ring. Clean rings were introduced by W. K. Nicholson in [17]. Nicholson proved that every clean ring is an exchange ring, and a ring with central idempotents is clean if and only if it is an exchange ring . A ring is said to be clean (almost clean) if each of its elements is the sum of a unit (regular element) and an idempotent. A module is clean (almost clean) if its endomorphism ring is clean (almost clean). A module which is invariant under automorphisms of its injective hull is called an *automorphism-invariant module*, i.e., M is called an automorphism-invariant module if $f(M) \subseteq M$ for all $f \in Aut(E(M))$. It has been proved in [9, Theorem 16] that automorphism-invariant modules are precisely the pseudo-injective modules, where a module M is called pseudo-injective, if for any submodule A of M, every monomorphism $f : A \to M$ can be extended to an endomorphism of M.

A module which is invariant under idempotent endomorphisms of its injective hull will be called an *idempotent-invariant module*, i.e., M will be called an idempotent-invariant module if $f(M) \subseteq M$ for all $f^2 = f \in End(E(M))$. It has been proved in [11, Theorem 1.1] that idempotent-invariant modules are precisely the π-injective (quasi-continuous) modules.

Consider the following conditions for an R-module M:

(C_1)Every submodule of M is essential in a direct summand of M.

(C_2)Every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M.

(C_3)If A and B are direct summands of M with $A \cap B = 0$ then $A \oplus B$ is also a direct summand of M.

M is called a CS (or an *extending*) module if it satisfies (C_1); M is called *continuous* if it satisfies (C_1) and (C_2); M is called *quasi-continuous* if it satisfies (C_1) and (C_3).

Modules satisfying (C_1), (C_2) and (C_3) are called $C1$-, $C2$- and $C3$- modules respectively.

It is well known that the following implications hold:

Injective $\Longrightarrow$ quasi-injective $\Longrightarrow$ continuous $\Longrightarrow$ quasi-continuous $\Longrightarrow$ CS.
But none of the converses hold in general. For background on injective, quasi-injective, continuous, quasi-continuous and CS modules, we refer to [8] and [16].
A right R-module M is said to satisfy the *exchange property* if for every right R-module A and any two direct sum decompositions $A = M_1 \oplus N = \oplus_{i\in I} A_i$ with $M_1 \simeq M$ there exist submodules B_i of A_i such that $A = M_1 \oplus (\oplus_{i\in I} B_i)$. If this holds only for $|I| < \infty$, then M is said to satisfy the finite exchange property. It is well known that a continuous module is idempotent-invariant and continuous modules satisfy the exchange property but it is not so for idempotent-invariant modules. For an idempotent-invariant module, the finite exchange property implies the full exchange property.
It has been proved in [12] that automorphism-invariant modules are clean and they satisfy the exchange property. Similar results do not hold for idempotent-invariant modules. In this paper, we investigate certain conditions under which an idempotent-invariant module can be clean or almost clean. Every idempotent-invariant module is pseudo-continuous but the converse is not true. Here we investigate certain conditions under which a pseudo-continuous module is idempotent-invariant. We provide examples to show that the classes of automorphism-invariant modules and idempotent-invariant modules are not contained one in the other and also show that a direct sum of idempotent-invariant modules need not be idempotent-invariant although summands of idempotent-invariant modules inherit the property. Finally, we also establish some of the relations of idempotent-invariant modules with ADS modules and modules with the SIP and SSP.
Throughout, all rings R are associative with unity and all modules are unitary R-modules, unless otherwise stated. For a module M, we use $E(M)$, $End(M)$ and $Aut(M)$ to denote the injective hull, the endomorphism ring and the group of automorphisms of M, respectively. $ker f$ and $Im f$ denote the kernel of f and the image of f respectively. We write $N \subseteq M$ if N is a submodule of M, $N \subseteq^{ess} M$ if N is an essential submodule of M and $N \subseteq^{\oplus} M$ if N is a direct summand of M.

2 Automorphism-invariant modules and idempotent-invariant modules

A module which is invariant under automorphisms of its injective hull is called an automorphism-invariant module, i.e., M is called an automorphism-invariant module if $f(M) \subseteq M$ for all $f \in Aut(E(M))$.
Quasi-injective modules are automorphism-invariant but the converse is not true, in general.

Theorem 2.1. *[15, Corollary 13] A module M is quasi-injective if and only if it is automorphism-invariant CS .*

Theorem 2.2. *[7, Theorem 2.6] Every pseudo-injective module M satisfies (C_2) .*

Corollary 2.3. *Every CS automorphism-invariant module is continuous.*

Proof. Let M be a CS automorphism-invariant module. Since automorphism-invariant modules are precisely the pseudo-injective modules, by Theorem 2.2 M satisfies (C_2). By assumption M is CS and M also satisfies (C_2). Hence M is continuous.

It is pertinent to mention that the classes of automorphism-invariant modules and idempotent-invariant modules are not contained one in the other as shown by the following examples.

Example 2.4. If $\mathbb{Z}$, $\mathbb{Q}$ denote the ring of integers and rational numbers respectively, $\mathbb{Z}_{\mathbb{Z}}$ is an idempotent-invariant module which is not automorphism-invariant because the injective hull $\mathbb{Q}_{\mathbb{Z}}$ of $\mathbb{Z}_{\mathbb{Z}}$ has the automorphism $\varphi : \mathbb{Q} \to \mathbb{Q}$ defined by $\varphi(q) = \dfrac{q}{2}$ but $\varphi(\mathbb{Z}) \nsubseteq \mathbb{Z}$.

Example 2.5. If R is the ring of all eventually constant sequences $(x_n)_{n\in\mathbb{N}}$ of elements in $\mathbb{Z}_2$, then $E(R_R) = \prod_{n\in\mathbb{N}} \mathbb{Z}_2$ has only one automorphism, namely the identity automorphism. Thus R_R is an automorphism-invariant module but R_R cannot be CS because by Theorem 2.1 a module M is quasi-injective if and only if it is automorphism-invariant CS. Hence R_R is not an idempotent-invariant module.

Theorem 2.6. *[15, Theorem 12] Every automorphism-invariant module satisfies* (C_3).

Though the classes of automorphism-invariant and idempotent-invariant modules are not contained one in the other, the following corollary, which is a consequence of the above theorem, shows that CS automorphism-invariant modules are indeed idempotent-invariant.

Corollary 2.7. *Every CS automorphism-invariant module is idempotent-invariant.*

Proof. Let M be a CS automorphism-invariant module. Then by Theorem 2.6, M satisfies (C_3) and by assumption M being CS, satisfies (C_1). Thus M is an idempotent-invariant module.

A module M is called an *ADS (Absolute Direct Summand) module* if for every decomposition $M = A \oplus B$ of M and every complement C of A, $M = A \oplus C$. A module M is called *square-free* if it contains no non-zero isomorphic submodules A and B with $A \cap B = 0$. An R-module M satisfies (C_4) if, whenever A and B are submodules of M with $M = A \oplus B$ and $f : A \to B$ is an R-homomorphism with $kerf \subseteq^{\oplus} A$, then $Imf \subseteq^{\oplus} B$. An R-module M satisfying (C_4) is called a *$C4$-module*. A $C4$-module which is also CS, is called a *pseudo-continuous* module.
Some examples of $C4$-modules are ADS modules, automorphism-invariant modules and square-free modules.
We know that $(C_2) \Rightarrow (C_3)$ and also $(C_3) \Rightarrow (C_4)$. Hence we have the following implications: $(C_2) \Rightarrow (C_3) \Rightarrow (C_4)$ which further yields the implications
continuous $\Rightarrow$ quasi-continuous $\Rightarrow$ pseudo-continuous.
But the converses are not true in general.

Theorem 2.8. *Every CS automorphism-invariant module is pseudo-continuous.*

Proof. Let M be an automorphism-invariant module. Then by Theorem 2.6, M satisfies (C_3) and since (C_3) implies (C_4), M satisfies (C_4). Thus M being a CS automorphism-invariant module, is pseudo-continuous.

The following theorem by Goel and Jain is vital in establishing the equivalence between the classes of π-injective and idempotent-invariant modules.

Theorem 2.9. *[11, Theorem 1.1] For any R-module M the following are equivalent:*
(a) M is π-injective.
(b) For every idempotent e in $hom(\widehat{M}, \widehat{M})$, $eM \subset M$.
(c) If $\widehat{M} = N_1 \oplus N_2$, then $M = (N_1 \cap M) \oplus (N_2 \cap M)$.
(d) If $\widehat{M} = \bigoplus_{i \in \Lambda} N_i$, then $M = \bigoplus_{i \in \Lambda}(N_i \cap M)$ for any index set Λ.

(Note that in [11], $\widehat{M}$ denotes the injective hull of M and $hom(\widehat{M}, \widehat{M})$ denotes the endomorphism ring of R-homomorphisms of $\widehat{M}$).
In [11], Goel and Jain call a module M π-injective if for every pair of submodules M_1 and M_2 with $M_1 \cap M_2 = 0$, each projection $\pi_i : M_1 \oplus M_2 \to M_i, i = 1, 2$, can be lifted to an endomorphism of M . π-injective modules are the same as the quasi-continuous modules defined by Jeremy in [14].
We call a module which is invariant under idempotent endomorphisms of its injective hull an idempotent-invariant module, i.e., M is called an idempotent-invariant module if $f(M) \subseteq M$ for all $f^2 = f \in End(E(M))$. Hence Theorem 2.9 can be restated as:

Theorem 2.10. *For any R-module M the following are equivalent:*
(a) M is quasi-continuous.
(b) M is idempotent-invariant, i.e., for every idempotent f in $End(E(M))$, $f(M) \subseteq M$.
(c) If $E(M) = N_1 \oplus N_2$, then $M = (N_1 \cap M) \oplus (N_2 \cap M)$.
(d) If $E(M) = \bigoplus_{i \in I} N_i$, then $M = \bigoplus_{i \in I}(N_i \cap M)$ for any index set I.

It is to be noted that a summand of an idempotent-invariant module is also idempotent-invariant. However, a direct sum of idempotent-invariant modules need not be idempotent-invariant as shown by the following example.

Example 2.11. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where F is any field and let $A = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$. It is clear that A and B are idempotent-invariant as R-modules. However, $R = A \oplus B$ is not idempotent-invariant as R_R satisfies (C_1) but does not satisfy (C_3).

Recall that a ring is *clean (almost clean)* if each of its elements is the sum of a unit (regular element) and an idempotent. A module is clean (almost clean) if its endomorphism ring is clean (almost clean).

Theorem 2.12. *[4, Theorem 3.9] Every continuous module is clean.*

It is interesting to note that although continuous modules are clean, a similar result does not hold, either for CS modules or for quasi-continuous modules. However, it has been proved in [12, Corollary 4] that automorphism-invariant modules are clean and in [12, Theorem 3] that automorphism-invariant modules satisfy the exchange property but similar results do not hold for idempotent-invariant modules.
For example, $\mathbb{Z}_{\mathbb{Z}}$ is an idempotent-invariant module but $\mathbb{Z}$ is not a clean ring; in fact, $\mathbb{Z}$ is not even an exchange ring. The question now arises under what conditions a quasi-continuous module would be clean. The following is one of the consequences of Theorem 2.12 for quasi-continuous modules.

Theorem 2.13. *[4, Theorem 4.3] A quasi-continuous R-module M is clean if and only if it has the finite exchange property if and only if it has the full exchange property.*

It has been proved in [18] that for quasi-continuous modules, the finite exchange property implies the full exchange property.
Recall that the *singular submodule* $Z(M)$ of a module M is defined as $Z(M) = \{m \in M : mI = 0$ for some essential right ideal I of $R\}$ and a module M is called *singular* if $Z(M) = M$ and *nonsingular* if $Z(M) = 0$.

Theorem 2.14. *[1, Theorem 2.6] If M is a quasi-continuous and non-singular module, then M is almost clean.*

Because of the equivalence of the classes of quasi-continuous modules and idempotent-invariant modules, the following results are immediate consquences of the above two theorems.

Corollary 2.15. *An idempotent-invariant module M is clean if and only if it has the finite exchange property if and only if it has the full exchange property.*

Proof. The proof follows from Theorem 2.13.

Corollary 2.16. *Every non-singular idempotent-invariant module is almost clean.*

Proof. The proof follows from Theorem 2.14.

For any R-module M, a monomorphism $f \in End(M)$ is called an *essential monomorphism* if $Im(f) \subseteq^{ess} M$ (i.e., $Im(f)$ is an essential submodule of M). A module M is called *essentially co-Hopfian* if every essential monomorphism in $End(M)$ is an isomorphism. A module M is called *co-Hopfian* if every monomorphism in $End(M)$ is an isomorphism.

Theorem 2.17. *[4, Proposition 4.5] If M is a CS module, then every element $f \in End(M)$ can be written as $e + v$ where $e = e^2 \in End(M)$ and $v \in End(M)$ is a monomorphism.*

Theorem 2.18. *[1, Proposition 2.5] If M is quasi-continuous, then every endomorphism of M is the sum of an idempotent and an essential monomorphism.*

With the aid of the two above results, we can now prove that every co-Hopfian CS module is clean and every essentially co-Hopfian idempotent-invariant module is clean.

Corollary 2.19. *Every co-Hopfian CS module is clean.*

Proof. Let M be a co-Hopfian CS module. Since M is CS, by Theorem 2.17 every $f \in End(M)$ is the sum of an idempotent and a monomorphism. But M being co-Hopfian, every monomorphism is an isomorphism. Thus every $f \in End(M)$ being the sum of an idempotent and an isomorphism, M is clean.

By Theorem 2.12, we know that every continuous module is clean and by Theorem 2.19 every co-Hopfian CS module is clean. So it seems quite natural to raise the following question: *Is every clean module continuous?*
The following example negates the possibility of every clean module being continuous.
Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where F is any field. Then R_R is an Artinian (and thus a co-Hopfian) CS module and so R_R is clean but it is not quasi-continuous since R_R satisfies (C_1) but does not satisfy (C_3). Hence R_R does not satisfy (C_2) either and so R_R is not continuous.

Theorem 2.20. *Every essentially co-Hopfian idempotent-invariant module is clean.*

Proof. Let M be an essentially co-Hopfian idempotent-invariant module. Since M is idempotent-invariant, by Theorem 2.18 every $f \in End(M)$ is the sum of an idempotent and an essential monomorphism. Since M is essentially co-Hopfian, every essential monomorphism is an isomorphism. Thus every $f \in End(M)$ being the sum of an idempotent and an isomorphism, M is clean.
(**Alternative Proof**). Let M be an essentially co-Hopfian idempotent-invariant module. Then M is continuous (see [16, Lemma 3.14]) and so by Theorem 2.12, M is clean.

Lemma 2.21. *[2, Lemma 3.1] An R-module M is ADS if and only if for each decomposition $M = A \oplus B$, A and B are mutually injective.*

Lemma 2.22. *[8, Lemma 7.5] Let A and B be R-modules and let $M = A \oplus B$. Then the following are equivalent:*

(a) B is A-injective.

(b) For every submodule N of M such that $N \cap B = 0$, there exists a submodule M_1 of M such that $M = M_1 \oplus B$ and $N \leq M_1$.

Theorem 2.23. *[19, Theorem 2.7] Every ADS module is a $C3$-module.*

Proof. Let M be an ADS module and let $A \subseteq^{\oplus} M, B \subseteq^{\oplus} M$ such that $A \cap B = 0$. We have to show that $A \oplus B$ is a direct summand of M. Let $M = A \oplus A_1$ and $M = B \oplus B_1$ for some submodules A_1, B_1 of M. Then by Lemma 2.21, A is A_1-injective. Hence by Lemma 2.22, there exists $N \leq M$ such that $M = N \oplus A$ and $B \leq N$. It follows that $N = B \oplus (N \cap B_1)$ and so $M = (A \oplus B) \oplus (N \cap B_1)$, thereby implying that $M = A \oplus B$ is a direct summand of M.

Every idempotent-invariant module is ADS as proved in [16, Theorem 2.8] but the converse is not true. As a consequence of Theorem 2.23, we have the following result.

Corollary 2.24. *Every ADS module which is also CS is idempotent-invariant.*

Proof. Let M be an ADS module. Then by Theorem 2.23, M is a $C3$-module and M also being CS, is idempotent-invariant.

Theorem 2.25. *Every essentially co-Hopfian ADS module which is CS is clean.*

Proof. Let M be an essentially co-Hopfian ADS module which is CS. Then by Corollary 2.24, M is idempotent-invariant. By assumption, M is essentially co-Hopfian. Thus by Theorem 2.20, M is clean.

Theorem 2.26. *Every ADS module which is also CS is pseudo-continuous.*

Proof. Let M be an ADS module which is also CS. Then, by Corollary 2.24, M is idempotent-invariant. Since every idempotent-invariant module is pseudo-continuous, M is pseudo-continuous.

Recall that a module M is said to have the SIP *(Summand Intersection Property)*, if the intersection of every pair of direct summands of M is a direct summand of M. A module M is said to have the SSP *(Summand Sum Property)*, if the sum of every pair of direct summands of M is a direct summand of M. A module M is said to be *polyform* if for every $N \subseteq M$ and homomorphism $f : N \to M$, $kerf$ is closed in N.

Lemma 2.27. *[13, Theorem 2.3] Let M be an R-module. Then*

(a) M has the SIP if and only if for every decomposition $M = A \oplus B$ and every R-homomorphism $f : A \to B$, the kernel of f is a direct summand of A, i.e., $kerf \subseteq^{\oplus} A$.

(b) M has the SSP if and only if for every decomposition $M = A \oplus B$ and every R-homomorphism $f : A \to B$, the image of f is a direct summand of B, i.e., $Imf \subseteq^{\oplus} B$.

It has been proved in [3, Lemma 19] that if M is a $C3$-module with the SIP, then M has the SSP. As a consequence of this, we can have the following result.

Corollary 2.28. *Every idempotent-invariant module with the SIP has the SSP.*

The next theorem provides a sufficient condition for the endomorphism ring of an idempotent-invariant module to have the summand sum property.

Theorem 2.29. *If M is a nonsingular idempotent-invariant module, then $End(M)$ has the SSP.*

Proof. Let M be a nonsingular idempotent-invariant module and let $A, B \subseteq M$ such that $M = A \oplus B$ and let $f : A \to B$ be an R-homomorphism. Since M is nonsingular, $A/kerf$ is nonsingular and $kerf$ is a closed submodule of M. Therefore, $kerf \subseteq^{\oplus} A$ and so by Lemma 2.27 (a), M has the SIP. By Corollary 2.28, M also has the SSP. Since M has both the SIP and the SSP, $End(M)$ has the SSP (see [10, Theorem 2.3]).

We know that every module with the SSP is a $C3$-module and also every $C3$-module is a $C4$-module. Hence every module with the SSP is a $C4$-module.
The following result is analogous to a result on $C3$-modules discussed earlier.

Theorem 2.30. *If M is a $C4$-module with the SIP, then M has the SSP.*

Proof. Let $A, B \subseteq M$ such that $M = A \oplus B$ and let $f : A \to B$ be an R-homomorphism. Since M is an SIP-module, by Lemma 2.27 (a), $kerf \subseteq^{\oplus} A$. Hence, by definition of $C4$-module, $Imf \subseteq^{\oplus} B$. Thus by Lemma 2.27 (b), M has the SSP.

The following corollary is an immediate consequence of the above theorem.

Corollary 2.31. *Every pseudo-continuous module with the SIP has the SSP.*

Theorem 2.32. *If M is a nonsingular CS module, then M has the SIP.*

Proof. Let M be a nonsingular CS module. We use Lemma 2.27 (a) to show that M has the SIP, i.e., we show that if $A, B \subseteq M$ such that $M = A \oplus B$ and $f : A \to B$ is an R-homomorphism, then $kerf \subseteq^{\oplus} A$. Since M is a CS module and $A \subseteq^{\oplus} M$, A is also a CS module and so $kerf \subseteq^{ess} L \subseteq^{\oplus} A$ for a submodule L of M. Since M is nonsingular, $A/kerf$ is nonsingular and $kerf$ is a closed submodule of M. Therefore $kerf \subseteq^{\oplus} A$ and so by Lemma 2.27 (a), M has the SIP.

Corollary 2.33. *If M is a nonsingular pseudo-continuous module, then M has the SSP. In particular, M is an idempotent-invariant module.*

Proof. Let M be a nonsingular pseudo-continuous module and let $A, B \subseteq M$ such that $M = A \oplus B$ and $f : A \to B$ is an R-homomorphism. Since every pseudo-continuous module is obviously CS, by Theorem 2.32 M has the SIP and so $kerf \subseteq^{\oplus} A$. M being a $C4$-module, by definition, if A and B are submodules of M such that $M = A \oplus B$ and $f : A \to B$ is an R-homomorphism with $kerf \subseteq^{\oplus} A$, then $Imf \subseteq^{\oplus} B$. Therefore using Lemma 2.27 (b), M has the SSP. As every module with the SSP satisfies $(C3)$ and M also being CS, M is an idempotent-invariant module.

Theorem 2.34. *If M is a polyform CS module, then M has the SIP.*

Proof. Let M be a polyform CS module and let $M = A \oplus B$ and $f : A \to B$ be an R-homomorphism. Since M is polyform, $ker f$ is closed in A. Since M is a CS module and $A \subseteq^{\oplus} M$, A is also a CS module. Hence $ker f \subseteq^{\oplus} A$ and so by Lemma 2.27 (a), M has the SIP.

Corollary 2.35. *If M is a polyform pseudo-continuous module, then M has the SSP. In particular, M is an idempotent-invariant module.*

Proof. Let M be a polyform pseudo-continuous module and let $M = A \oplus B$ and $f : A \to B$ be an R-homomorphism. Since every pseudo-continuous module is obviously CS, by Theorem 2.34, M has the SIP and so $ker f \subseteq^{\oplus} A$. M being a $C4$-module, by definition, $Im f \subseteq^{\oplus} B$ and so using Lemma 2.27 (b), M has the SSP. As every SSP module satisfies (C_3) and M also being CS, M is an idempotent-invariant module.

Theorem 2.36. *Let M be an idempotent-invariant polyform module. Then M is an ADS module with the SIP.*

Proof. Let $M = A \oplus B$ and $f : A \to B$ be a homomorphism. M being an idempotent-invariant module is also an ADS module. Since M is polyform, $ker f$ is closed in A and so $ker f \subseteq^{\oplus} A$. Thus by Lemma 2.27 (a), M has the SIP.

We know that every idempotent-invariant module is pseudo-continuous but the converse is not true in general. The following results can now be established as consequences of Corollary 2.33 and Corollary 2.35.

Theorem 2.37. *A module M is idempotent-invariant if and only if M is a nonsingular pseudo-continuous module.*

Theorem 2.38. *A module M is idempotent-invariant if and only if M is a polyform pseudo-continuous module.*

Recall that a module M is said to satisfy the *(full) internal exchange property* if for every internal direct sum decomposition $M = \oplus_{i \in I} M_i$ and every summand $N \subseteq^{\oplus} M$, there exist submodules $N_i \subseteq M_i$, $i \in I$, such that $M = \oplus_{i \in I} N_i \oplus N$. If this holds only for $|I| < \infty$, then M is said to satisfy the finite internal exchange property. We have discussed earlier that $\mathbb{Z}_{\mathbb{Z}}$ is an idempotent-invariant module which does not satisfy the finite exchange property. However, it is to be noted that $\mathbb{Z}_{\mathbb{Z}}$ satisfies the full internal exchange property.
It has been proved in [6, Proposition 2.22] that if M is a $C4$-module with the finite internal exchange property, then M is a $C3$-module. Also, it has been proved in [18, Proposition 1.1] that every idempotent-invariant-module satisfies the internal exchange property. The next theorem follows as an immediate consequence of both the results.

Theorem 2.39. *A module M is idempotent-invariant if and only if M is a pseudo-continuous module with the internal exchange property.*

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