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SOME PROPERTIES OF SEMIGROUPS GENERATED FROM A CAYLEY FUNCTION

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Abstract Any transformation on a set S is called a Cayley function on S if there exists a semigroup operation on S such that β is an inner-translation. In this paper we describe a method to generate a semigroup with k number of idempotents, study some properties of such semigroups like greens relations and bi-ordered sets.

1 Introduction

Let β be a transformation on a set S. Following [8] we say that β is a Cayley function on S if there is a semigroup with universe S such that β is an inner translation of the semigroup S. A section of group theory has developed historically through the characterisation of inner translations as regular permutations. The problem of characterising inner translations of semigroups was raised by Schein [7] and solved by Goralcik and Hedrlin [3].In 1972 Zupnik characterised all Cayley functions algebraically(in powers of β) [8]. In 2016, Araoujo et all characterised the Cayley functions using functional digraphs[1]. In this paper we use a Cayley permutation to generate a semigroup with n elements and $k \leq n$ idempotents and discuss some of its properties.

In the sequel β will denote a function on a non-empty set S onto itself. For any positive integer n, β^n denotes the n^{th} iterate of β. By β^0 we mean the identity function on S, so $\beta^0(x) = x$. Let S be a set, then $T(S)$ denotes the set of all transformations from S to S.

2 Preliminaries

A semigroup is a non empty set S along with a binary operation $*$ on S such that $(S, *)$ is associative. An idempotent element ϵ in S is an element such that $\epsilon^2 = \epsilon * \epsilon = \epsilon$. The set of all idempotents in S is denoted by $E(S)$

If a is an element of a semigroup S, the smallest left ideal containing a is $Sa \cup \{a\}$ or S^1a the principal left ideal generated by a. The equivalence relation $\mathcal{L}$ on S is defined on S by $a \,\mathcal{L}\, b$ if and only if $S^1a = S^1b$. Similarly we say that $a \,\mathcal{R}\, b$ if and only if $aS^1 = bS^1$. The following is due to J.A. Green.

Lemma 2.1. *Let a, b be elements of a semigroup S. Then*

- *$a \,\mathcal{L}\, b$ if and only if $\exists\, x, y \in S^1$ such that $xa = b$ and $yb = a$*
- *$a \,\mathcal{R}\, b$ if and only if $\exists\, x, y \in S^1$ such that $ax = b$ and $by = a$*

The following lemma is lemma 2.1 of [2]

Lemma 2.2. *[2] The relations $\mathcal{L}$ and $\mathcal{R}$ commute and so the relation $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ is the smallest equivalence relation $\mathcal{L} \vee \mathcal{R}$ containing both $\mathcal{L}$ and $\mathcal{R}$. We define $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$*

Let S be a semigroup. For a fixed $a \in S$, the mapping $\lambda_a : S \to S$ $[\rho_a : S \to S]$ defined by $\lambda_a(x) = ax$ $[\rho_a(x) = xa]$ is called a left [right] inner translation of S.

Definition 2.3. Let β be a transformation on a set S. We say that β is a Cayley function on S if there is a semigroup with universe S such that β is an inner translation of the semigroup S.

Note that β is a left inner translation of a semigroup $(S, *)$ if and only if β is a right inner translation of the semigroup $(S, .)$, where for all $a, b \in S$, $a * b = b.a$.

By a partial algebra E we mean a set with a partial binary operation. That is a mapping from $D_E \subseteq E \times E$ into E. We call D_E the domain of the partial binary operation. Let E be a partial algebra. On E define:

$$\begin{aligned}
\omega^r &= \{(e,f) : fe = e\} \\
\omega^l &= \{(e,f) : ef = e\} \\
\mathcal{R} &= \omega^r \cap \omega^{r^{-1}} \\
\mathcal{L} &= \omega^l \cap \omega^{l^{-1}} \\
\omega &= \omega^r \cap \omega^l
\end{aligned}$$

Definition 2.4. The partial algebra E is called a biordered set if it satisfies the following axioms and their duals. For e, $f \in E$

B11 ω^r and ω^l are quasi-orders on E

B12 $D_E = (\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}$

B21 for all $e\omega^r f \implies e\mathcal{R}ef\omega f$

B22 $g\omega^r f\omega^r e \implies gf = gef$

B3 For $g, h \in \omega^r(e)$ and $g\omega^l f \implies ge\omega^l fe$ and $(fg)e = (fe)(ge)$

B4 If $f, g \in \omega^r(e)$ and $ge\omega^l fe$, then there exists $g_1 \in \omega^r(e)$ such that $g_1\omega^l f$ and $g_1 e = ge$

Definition 2.5. Let E be a biordered set. For $e, f \in E$, let

$$M(e,f) = (\omega^l(e) \cap \omega^r(f), <)$$

where $<$ is the relation defined by

$$g, h \iff g, h \in \omega^l(e) \cap \omega^r(f) \text{ and } eg\omega^r eh, gf\omega^l hf$$

The **Sandwich set** of $e$and f is defined as

$$S(e,f) = \{h \in M(e,f) : g, h \forall g \in M(e,f)\}$$

The biordered set of a semigroup S to be the set $E = E(S)$ with a partial multiplication inherited from S with domain

$$D_E = \{(e,f) \in /E \times E : e\omega^r f \text{ or } e\omega^l f \text{ or } f\omega^r e \text{ or } f\omega^l e\}$$

Products of idempotents e and f such that $ef = e$ or $ef = f$ or $fe = e$ or $fe = f$ are called basic products of idempotents.

Definition 2.6. The stabilizer of $\beta \in T(S)$ is the smallest integer $s \geq 0$ such that $img(\beta^s) = img(\beta^{s+1})$. If such an s does not exist, we say that β has no stabilizer.

The following definition originally defined by Zupnik was modified by Araujo [1] and the same is recalled here.

Definition 2.7. Suppose $\beta \in T(S)$ has the stabilizer s. If $s > 0$, we define the subset Ω_β of S by:

$$\Omega_\beta = \{a \in S : \beta^n(a) \in Ran(\beta^s) \text{ if and only if } n \geq s-1\}$$

If $s = 0$, we define Ω_β to be S.

Theorem 2.8. *[8] Let $\beta \in T(S)$. Then β is a Cayley function if and only if exactly one of the following conditions holds:*

a *has no stabilizer and there exists $a \in S$ such that $\beta^n(a) \notin img(\beta^{n+1})$ for every $n \geq 0$;*

b *has the stabilizer s such that $\beta|img(\beta^s)$ is one-to-one and there exists $a \in \Omega_\beta$ such that $\beta^m(a) = \beta^n(a)$ implies $\beta^m = \beta^n$ for all $m, n \geq 0$; or*

c *has the stabilizer s such that $\beta|img(\beta^s)$ is not one-to-one and there exists $a \in \Omega_\beta$ such that:*

(i) $\beta^m(a) = \beta^n(a)$ implies $m = n$ for all $m, n \geq 0$;and

(ii) For every $n > s$, there are pairwise distinct elements $y_1, y_2, ...$of S such that $\beta(y_1) = \beta^n(a)$, $\beta(y_k) = y_{k-1}$ for every $k \geq 2$, and if $n > 0$ then $y_1 \neq \beta^{n-1}(a)$.

A Cayley function that is also a permutation is called a Cayley permutation. Similarly a Cayley function that also an idempotent is called a Cayley Idempotent.

3 A Semigroup from Cayley functions

In this section we construct a semigroup S_β from a Cayley permutation β on a finite set and study some of its properties.

Let S be a set with n elements, and $a \in S$ be a fixed element. For any a_1, in S we consider set $\{r : \beta^r(a) = a_1\}$ of all non-negative integers such that $\beta^r(a) = a_1$ in case the set is non empty we define $\delta_{a_i} = min\{r : \beta^r(a) = a_i\}$.

Theorem 3.1. *Let S be a set with n elements for any K $(0 < k \leq n)$ there exists a semigroup with k idempotents using a Cayley function a Cayley permutation.*

Proof.

Let $0 < k < n$, and a_1 be a fixed element of S. Let β be the permutation that such that $\beta = (a_1 a_2 a_3 ....a_{n-k} a_{n-k+1})$ an $n-k+1$ cycle in S_n that permutes $n-k+1$ terms and fixes the rest of the $k-1$ terms, it is a Cayley function by theorem 1 above. Now consider the binary operation on S given by

$$a_i * a_j = \begin{cases} \beta^{\delta_{a_i}+1}(a_j) & \text{if } a_i \text{ is not a fixed element of } \beta \\ a_i & \text{if } a_i \text{ is a fixed element of } \beta \end{cases} .$$

where $\delta_{a_i} = min\{r : \beta^r(a_1) = a_i\}$. For $k = n$ consider the identity permutation with the same construction. We can see that $*$ is well defined binary operation and that $*$ is associative. So $(S, *)$ is a semigroup. By the choice of the permutation β and the definition of $*$, we can see that a_{n-k+1} is an idempotent as

$$a_{n-k+1} * a_{n-k+1} = \beta^{n-k+1}(a_{n-k+1}) = a_{n-k+1}$$

and all the $k-1$ fixed elements of β are also idempotents. □

Example 3.2. Let $S = \{a, b, c, d, e\}$ and let $k = 3$ choosing $\beta = \begin{pmatrix} a & b & c & d & e \\ b & c & a & d & e \end{pmatrix}$ and following the construction as in the above theorem, we get the following Cayley table on S.

*	a	b	c	d	e
a	b	c	a	d	e
b	c	a	b	d	e
c	a	b	c	d	e
d	d	d	d	d	d
e	e	e	e	e	e

For the rest of the paper we denote the semigroup generated in the above theorem as S_β.

In this section we study the greens relation of the semigroup S_β generated by the permutation β

Lemma 3.3. *Let $a,\, b \in S_\beta$. Then*

(i) If a, b are both non-fixed elements of β , then $a\ \mathcal{R}\ b$. If a is a fixed element of β, then a is $\mathcal{R}$ related only to itself.

(ii) If a, b are both non-fixed elements of β, then $a\ \mathcal{L}\ b$. If a, b are both fixed elements of β then $a\ \mathcal{L}\ b$

(iii) S_β contains two $\mathcal{D}$classes.

(iv) S_β contains k number of $\mathcal{H}$classes.

Proof. (i) If a is not a fixed element of β then $aS_\beta = S_\beta$, if a is a fixed element of β then $aS_\beta = \{a\}$. Hence (1). In correspondence to lemma1 we have for non-fixed elements a_i , a_j of S_β $a_q * a_i = a_j$ where $q + i = j mod(n - k + 1)$ for fixed elements $a_i * a_i = a_i$

(ii) If a is not a fixed element of β then $S_\beta a = S_\beta$, if a is a fixed element of β then $S_\beta a = \{b :$ b is a fixed element of $\}$. And in correspondence to lemma1

(iii) from lemma 2 $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. Hence from 1 and 2 we get two $\mathcal{D}$classes

(iv) from lemma 2 $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and hence k $\mathcal{H}$classes

□

Generally the egg box picture of the semigroup S_β is as follows.

$a_1, a_2, \ldots, a_{n-k+1}$
a_{n-k+2}
a_{n-k+3}
a_{n-k+4}
.
.
.
.
a_n

Remark 3.4. Some Properties of S_β

- $\{a_1, a_2, ...a_{n-k+1}\}$ forms a subgroup of S_β.
- a_{n-k+1} is the identity element of S_β.
- k-1 idempotents act as left zeros (absorbing elements).
- Idempotents do not commute.
- S_β is a regular semigroup.
- In fact S_β is a completely regular semigroup
- S_β is a not an inverse semigroup.
- S_β is a union of a group and a band.
- $E(S_\beta)$ forms a subsemigroup of S_β(i. e, $S_\beta$is an orthodox semigroup.)

Let us now consider the Biordered set of S_β. Since $E(S_\beta)$ forms a sub-semigroup of S_β, $E(S_\beta)$ with its semigroup operation forms a biordered set where for every $e, f \in E(S_\beta)$

$$\begin{aligned} \omega^r &= \{(e,e)\} \cup \{(e,f) : f = a_{n-k+1}\} \\ \omega^l &= \{(e,f) : e \neq a_{n-k+1}\} \cup \{(a_{n-k+1}, a_{n-k+1})\} \\ \mathcal{R} &= \{(e,e)\} \\ \mathcal{L} &= \{(e,e)\} \cup \{(e,f) : e, f \neq a_{n-k+1}\} \\ \omega &= \{e,e\} \cup \{(e,f) : f = a_{n-k+1}\}r \end{aligned}$$

$$M(e,f) = \begin{cases} E(S_\beta) & \text{if } e, f = a_{n-k+1} \\ f & \text{if } e = a_{n-k+1}, f \neq a_{n-k+1} \\ E(S_\beta) - a_{n-k+1} & \text{if } e \neq a_{n-k+1}, f = a_{n-k+1} \\ f & \text{if } e, f \neq a_{n-k+1} \end{cases}$$

and the sandwich set will be

$$S(e,f) = \begin{cases} a_{n-k+1} & \text{if } e, f = a_{n-k+1} \\ f & \text{if } e = a_{n-k+1}, f \neq a_{n-k+1} \\ E(S_\beta) - a_{n-k+1} & \text{if } e \neq a_{n-k+1}, f = a_{n-k+1} \\ f & \text{if } e, f \neq a_{n-k+1} \end{cases}$$

To conclude we have constructed a semigroup with k-idempotents and n elements and also studied some basic properties on the biordered set of the generated semigroup.

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