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# SOME PROPERTIES OF WEAKLY PRIMARY AND WEAKLY 2-ABSORBING PRIMARY IDEALS IN LATTICES

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**Abstract** We introduce the concepts of a weakly primary and a weakly 2-absorbing primary ideal in a lattice. We show that under certain condition, an ideal $I$ of a lattice $L$ is weakly primary if and only if $I$ is primary. We define a free triple zero ideal of a lattice and give a result related to it.

## 1 Introduction

Many researchers have introduced different types of ideals such as primary, weakly primary etc. in a commutative ring. We introduce some of these concepts in a lattice. Anderson and Smith [1], defined a weakly prime ideal in a commutative ring, as a proper ideal $P$ of $R$ with the property that, if whenever $a, b \in R$, $0 \neq ab \in P$ implies either $a \in P$ or $b \in P$. Badawi [2] generalized the concept of a prime ideal to a 2-absorbing ideal. A proper ideal $I$ of a commutative ring $R$ is said to be a 2-absorbing ideal, if whenever $abc \in I$ for $a, b, c \in R$, then either $ab \in I$ or $bc \in I$ or $ac \in I$. Badawi and Darani [3] call a proper ideal $I$ of a commutative ring $R$ to be a weakly 2-absorbing ideal, if whenever $0 \neq abc \in I$ for $a, b, c \in R$, then either $ab \in I$ or $bc \in I$ or $ac \in I$. Atani and Farzalipour [6] defined a proper ideal $P$ of a commutative ring $R$ to be a weakly primary ideal, if whenever $a, b \in R$, $0 \neq ab \in P$ implies either $a \in P$ or $b \in \sqrt{P}$ (where $\sqrt{P}$ is the radical of $P$). Badawi et al. [4] defined a proper ideal $I$ of a commutative ring $R$ to be a 2-absorbing primary ideal, if whenever $abc \in I$ for $a, b, c \in R$, then either $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. Badawi et al. [5] defined a proper ideal $I$ of a commutative ring $R$ to be a weakly 2-absorbing primary ideal, if whenever $0 \neq abc \in I$ for $a, b, c \in R$, then either $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$.

Moreover, Wasadikar and Gaikwad [9] introduced a 2-absorbing and a weakly 2-absorbing ideal in a lattice. A proper ideal $I$ of a lattice $L$ is called a 2-absorbing ideal, if whenever $a \wedge b \wedge c \in I$ for $a, b, c \in L$, then either $a \wedge b \in I$ or $a \wedge c \in I$ or $b \wedge c \in I$. A proper ideal $I$ of a lattice $L$ with zero is called a weakly 2-absorbing ideal, if whenever $0 \neq a \wedge b \wedge c \in I$ for $a, b, c \in L$, then either $a \wedge b \in I$ or $a \wedge c \in I$ or $b \wedge c \in I$. Wasadikar and Gaikwad [10], [11] introduced the concepts of radical of an ideal, a primary ideal, a weakly primary ideal, a 2-absorbing primary ideal and a weakly 2-absorbing ideal in a lattice.

In this paper we introduce and study the concepts of a weakly primary and a weakly 2-absorbing primary ideal in a lattice. Throughout in this paper $L$ denotes a lattice with 0. The undefined terms are from Grätzer [7].

## 2 Preliminaries

In this paper we generalize the concepts of weakly primary, weakly 2-absorbing primary ideals studied in Wasadikar and Gaikwad [10], [11].

**Definition 2.1.** Let $I$ be an ideal of $L$. The radical of $I$ is the intersection of all prime ideals of $L$ containing $I$ and we denote it by $\sqrt{I}$.

**Remark 2.2.** If there does not exists a prime ideal containing an ideal $I$, then $\sqrt{I} = L$.

**Definition 2.3.** A proper ideal $I$ of $L$ is called weakly prime if for $a, b \in L$, $0 \neq a \wedge b \in I$ implies that either $a \in I$ or $b \in I$.

**Example 2.4.** Consider the lattice shown in Figure 1. Here the ideal $(p]$ is a weakly prime ideal.

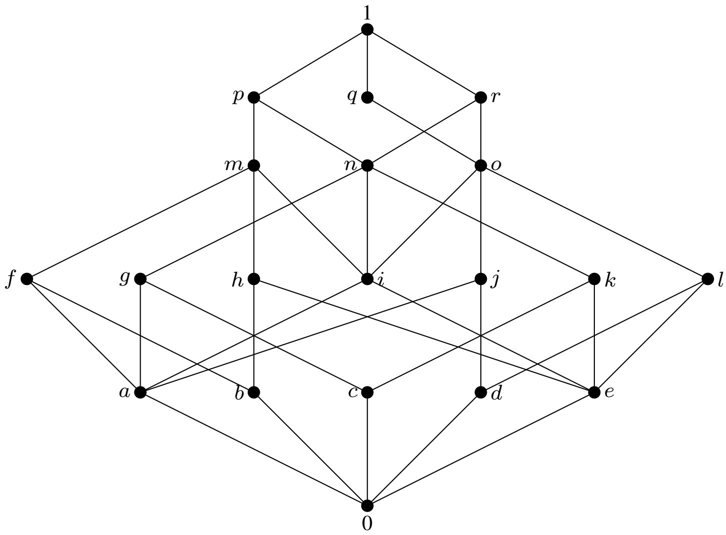


Figure 1

**Definition 2.5.** A proper ideal $I$ of $L$ is called primary if for $a, b \in L$, $a \wedge b \in I$ implies that either $a \in I$ or $b \in \sqrt{I}$.

**Example 2.6.** The ideal $(m]$ in the lattice shown in Figure 1 is a primary ideal.

**Definition 2.7.** A proper ideal $I$ of $L$ is called weakly primary if for $a, b \in L$, $0 \neq a \wedge b \in I$ implies that either $a \in I$ or $b \in \sqrt{I}$.

**Example 2.8.** The ideal $(m]$ in the lattice shown in Figure 1 is a weakly primary ideal.

**Definition 2.9.** A proper ideal $I$ of $L$ is called 2-absorbing if for every $a, b, c \in L$, $a \wedge b \wedge c \in I$ implies that either $a \wedge b \in I$ or $a \wedge c \in I$ or $b \wedge c \in I$.

**Example 2.10.** The ideal $(j]$ in the lattice shown in Figure 1 is a 2-absorbing ideal.

**Definition 2.11.** A proper ideal $I$ of $L$ is called weakly 2-absorbing if for every $a, b, c \in L$, $0 \neq a \wedge b \wedge c \in I$ implies that either $a \wedge b \in I$ or $a \wedge c \in I$ or $b \wedge c \in I$.

**Example 2.12.** The ideal $(i]$ in the lattice shown in Figure 1 is a weakly 2-absorbing ideal.

**Definition 2.13.** A proper ideal $I$ of $L$ is called 2-absorbing primary if for every $a, b, c \in L$, $a \wedge b \wedge c \in I$ implies that either $a \wedge b \in I$ or $a \wedge c \in \sqrt{I}$ or $b \wedge c \in \sqrt{I}$.

**Example 2.14.** The ideal $(m]$ in the lattice shown in Figure 1 is a 2-absorbing primary ideal.

**Definition 2.15.** A proper ideal $I$ of $L$ is called weakly 2-absorbing primary if for every $a, b, c \in L$, $0 \neq a \wedge b \wedge c \in I$ implies that either $a \wedge b \in I$ or $a \wedge c \in \sqrt{I}$ or $b \wedge c \in \sqrt{I}$.

**Example 2.16.** The ideal $(f]$ in the lattice shown in Figure 1 is a weakly 2-absorbing primary ideal.

## 3 Some Properties of Weakly Primary Ideals

We note that every primary ideal is weakly primary. However, the converse need not hold.

**Example 3.1.** Consider the ideal $I = (0]$ of the lattice shown in Figure 1. Then $\sqrt{I} = (p] \cap (q] \cap (r] = (i]$. Here $I$ is a weakly primary ideal. However, $j \wedge k = 0 \in I$, neither $j \in I$ nor $k \in I$. Also, neither $j \in \sqrt{I}$ nor $k \in \sqrt{I}$. Hence $I$ is not a primary ideal.

**Remark 3.2.** If $I$ is a prime ideal of $L$, then $\sqrt{I}$ is a primary ideal of $L$.

We have the following example which shows that the converse is not true.

**Example 3.3.** Consider the ideal $I = (m]$ of the lattice shown in Figure 1. Then $\sqrt{I} = (p]$. The ideal $\sqrt{I}$ is prime. However, $I$ is not a prime ideal since $n \wedge o = i \in I$, but neither $n \in I$ nor $o \in I$.

**Remark 3.4.** (i) If $I$ is a weakly primary ideal of $L$, then $\sqrt{I}$ need not be a weakly primary ideal of $L$.

(ii) If $I$ is a weakly prime ideal of a lattice $L$, then $\sqrt{I}$ need not be a weakly primary ideal of $L$.

We have the following example for above remarks.

**Example 3.5.** Consider the ideal $I = (0]$ of the lattice shown in Figure 1. Hence $\sqrt{I} = (p] \cap (q] \cap (r] = (i]$. Here $I$ is a weakly prime as well as a weakly primary ideal. However, $\sqrt{I}$ is not a weakly primary ideal, since for $m \wedge n = i \in \sqrt{I}$, neither $m \in \sqrt{I}$ nor $n \in \sqrt{I}$.

Let $I$ be a proper ideal of a lattice $L$. For $x \in L$, let $(I :_L x) = \{y \in L | y \wedge x \in I\}$.
The following characterization is an analogue of Theorem 2.1 from Atani et al. [6].

**Theorem 3.6.** *Let $I$ be a proper ideal of $L$. Then the following statements are equivalent:*

*(i) $I$ is a weakly primary ideal of $L$.*

*(ii) For $x \in L - \sqrt{I}$, $(I :_L x) = I \cup (0 :_L x)$.*

*(iii) For $x \in L - \sqrt{I}$, $(I :_L x) = I$ or $(I :_L x) = (0 :_L x)$.*

**Proof.** $(1) \Rightarrow (2)$. Let $x \in L - \sqrt{I}$ and $y \in (I :_L x)$. Then $x \wedge y \in I$. If $x \wedge y \neq 0$ then $y \in I$. If $x \wedge y = 0$, then $y \in (0 :_L x)$. Thus $(I :_L x) \subseteq I \cup (0 :_L x)$. It is clear that $I \cup (0 :_L x) \subseteq (I :_L x)$. Hence $(I :_L x) = I \cup (0 :_L x)$.
$(2) \Rightarrow (3)$. If an ideal is a union of two ideals, then it is equal to one of them.
$(3) \Rightarrow (1)$. Let $0 \neq x \wedge y \in I$ with $x \notin \sqrt{I}$. Then $y \in (I :_L x) = I \cup (0 :_L x)$, by (3). Since $x \wedge y \neq 0$, we have $y \in I$. Thus $I$ is a weakly primary ideal of $L$. □

**Definition 3.7.** Let $I$ be a weakly primary ideal of $L$. Let $x, y \in L$ be such that $x \wedge y = 0$, $x \notin I, y \notin \sqrt{I}$, then (x,y) is called a twin-zero of $I$.

**Example 3.8.** Consider the ideal $I = (a]$ of the lattice shown in Figure 2. Then $\sqrt{I} = (d]$ as (d] is the only prime ideal containing $I$. $I$ is a weakly primary ideal. For $d \wedge e = 0 \in I$, neither $d \in I$ nor $e \in \sqrt{I}$. Thus $(d, e)$ is a twin zero of $I$.

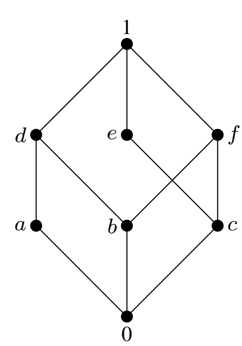


Figure 2

**Remark 3.9.** The following example shows if $(x, y)$ is a twin-zero of an ideal $I$, then $(y, x)$ need not be a twin-zero of $I$.

**Example 3.10.** In the Example 3.8 we observe that $(d, e)$ is a twin-zero of $I$. However, $e \wedge d = 0 \in I$, $e \notin I$ but $d \in \sqrt{I}$. Thus $(e, d)$ is not a twin-zero of $I$. Hence $(d, e) \neq (e, d)$.

**Remark 3.11.** If $I$ is a weakly primary ideal of $L$ such that $I$ is not a primary ideal, then $I$ has a twin-zero $(x, y)$ for some $x, y \in L$.

**Definition 3.12.** For an ideal $I$ of a lattice $L$, we define $x \wedge I = \{x \wedge i : \ i \in I\}$.

We prove some results by using the concept of twin-zero.

The following Theorem is an analogue of Theorem 3.2 from Badawi et al. [5].

**Theorem 3.13.** *Let $I$ be a weakly primary ideal of a modular lattice $L$. Suppose that $(x, y)$ is a twin-zero of $I$ for some $x, y \in L$. Then $x \wedge I = y \wedge I = 0$.*

**Proof.** Suppose that $x \wedge I \neq 0$. Then $x \wedge i \neq 0$ for some $i \in I$. Hence $0 \neq (x \wedge y) \vee (x \wedge i) \in I$. As $L$ is modular, $0 \neq x \wedge (y \vee (x \wedge i)) \in I$. As $x \notin I$ and $I$ is a weakly primary ideal, we have $y \vee (x \wedge i) \in \sqrt{I}$. Implies that $y \in \sqrt{I}$, a contradiction to $(x, y)$ is a twin-zero of $I$. Thus $x \wedge I = 0$.

Similarly, we can show that $y \wedge I = 0$. □

The following Theorem is an analogue of Theorem 3.5 from Badawi et al. [5].

**Theorem 3.14.** *Let $I$ be a weakly prime ideal of a lattice $L$ and suppose that $(x, y)$ is a twin-zero of $I$. If $x \wedge l \in I$ for some $l \in L$, then $x \wedge l = 0$.*

**Proof.** Suppose that $0 \neq x \wedge l \in I$ for some $l \in L$. As $I$ is weakly prime, $l \in I$. By Theorem 3.13 $x \wedge l = 0$, a contradiction. Thus $x \wedge l = 0$. □

In the following theorem we show that under some condition the concepts of a primary ideal and a weakly primary ideal coincide.

**Theorem 3.15.** *Suppose that $\sqrt{0}$ is a prime ideal of $L$. Let $I$ be a proper ideal of $L$. Then $I$ is a weakly primary ideal of $L$ if and only if $I$ is a primary ideal of $L$.*

**Proof.** Suppose that $I$ is a weakly primary ideal of $L$. Let $a \wedge b \in I$ for some $a, b \in L$.
Case 1: Suppose that $a \wedge b \neq 0$. As $I$ is a weakly primary ideal and $0 \neq a \wedge b \in I$, either $a \in I$ or $b \in \sqrt{I}$.
Case 2: Suppose that $a \wedge b = 0$ and $a \notin I$. Since $a \wedge b = 0$ and $\sqrt{0}$ is a prime ideal of $L$, we conclude that either $a \in \sqrt{0}$ or $b \in \sqrt{0}$. Since $\sqrt{0} \subseteq \sqrt{I}$, we have $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Thus $I$ is a primary ideal of $L$.

The converse is clear. □

We omit the obvious proof of the following lemma.

**Lemma 3.16.** *Let $I$ be an ideal of $L$. Then $\sqrt{I} = \sqrt{\sqrt{I}}$.*

**Lemma 3.17.** *Let $I$ be an ideal of $L$. If $\sqrt{I}$ is a primary ideal, then $I$ is a primary ideal.*

**Proof.** Let $a \wedge b \in I$. Then $a \wedge b \in \sqrt{I}$. As $\sqrt{I}$ is primary, either $a \in \sqrt{I}$ or $b \in \sqrt{\sqrt{I}}$. By Lemma 3.16, $\sqrt{I} = \sqrt{\sqrt{I}}$. Hence, either $a \in \sqrt{I}$ or $b \in \sqrt{I}$. □

Now we prove some results in product lattices.

The following Theorem is from Wasadikar and Gaikwad [10].

**Theorem 3.18.** *Let $L = L_1 \times L_2$, where each $L_i$, $(i = 1, 2)$ is a lattice with 1. Then the following hold:*

*(i) If $I_1$ is an ideal of $L_1$, then $\sqrt{I_1 \times L_2} = \sqrt{I_1} \times L_2$.*

*(ii) If $I_2$ is an ideal of $L_2$, then $\sqrt{L_1 \times I_2} = L_1 \times \sqrt{I_2}$.*

The following Theorem is an analogue of Lemma 2.5 from Atani et al. [6].

**Theorem 3.19.** *Let $L = L_1 \times L_2$ where $L_1$ and $L_2$ are bounded lattices. Then the following statements hold:*

*(i) $I_1$ is a primary ideal of $L_1$ if and only if $I_1 \times L_2$ is a primary ideal of $L$.*

*(ii) $I_2$ is a primary ideal of $L_2$ if and only if $L_1 \times I_2$ is a primary ideal of $L$.*

**Proof.** (1) Let $(a_1, a_2) \wedge (b_1, b_2) \in I_1 \times L_2$ where $(a_1, a_2), (b_1, b_2) \in L$. That is $(a_1 \wedge b_1, a_2 \wedge b_2) \in I_1 \times L_2$. Then $a_1 \wedge b_1 \in I_1$. As $I_1$ is a primary ideal of $L_1$, either $a_1 \in I_1$ or $b_1 \in \sqrt{I_1}$. Hence either $(a_1, a_2) \in I_1 \times L_2$ or $(b_1, b_2) \in \sqrt{I_1} \times L_2$. By Theorem 3.18, either $(a_1, a_2) \in I_1 \times L_2$ or $(b_1, b_2) \in \sqrt{I_1 \times L_2}$.
Conversely, let $a \wedge b \in I_1$. Then $(a \wedge b, c) \in I_1 \times L_2$. That is $(a, c) \wedge (b, c) \in I_1 \times L_2$. As $I_1 \times L_2$ is a primary ideal of $L$, either $(a, c) \in I_1 \times L_2$ or $(b, c) \in \sqrt{I_1 \times L_2}$. By Theorem 3.18, either $(a, c) \in I_1 \times L_2$ or $(b, c) \in \sqrt{I_1} \times L_2$. Thus, either $a \in I_1$ or $b \in \sqrt{I_1}$. Hence $I_1$ is a primary ideal of $L_1$.
(2) Proof is similar as in statement (1). □

**Remark 3.20.** The following example shows that, if $I_1$ is a weakly primary ideal of $L_1$, then $I_1 \times L_2$ need not be a weakly primary ideal of $L = L_1 \times L_2$.

**Example 3.21.** Consider the ideal $I_1 = (0]$ of the lattice $L_1$ shown in Figure 3. Thus $I_1 \times L_2 = \{(0,0), (0,c), (0,1)\}$. Clearly $\sqrt{I_1} = (0]$ and $\sqrt{I_1 \times L_2} = \{(0,0), (0,c), (0,1)\}$. Here $I_1$ is a weakly primary ideal of $L_1$. However, $I_1 \times L_2$ is not a weakly primary ideal of $L = L_1 \times L_2$ since $(a, 0) \wedge (b, 0) = (0, 0) \in I_1 \times L_2$ but neither $(a, 0) \in \sqrt{I_1 \times L_2}$ nor $(b, 0) \in \sqrt{I_1 \times L_2}$ and hence neither $(a, 0) \in I_1 \times L_2$ nor $(b, 0) \in I_1 \times L_2$.

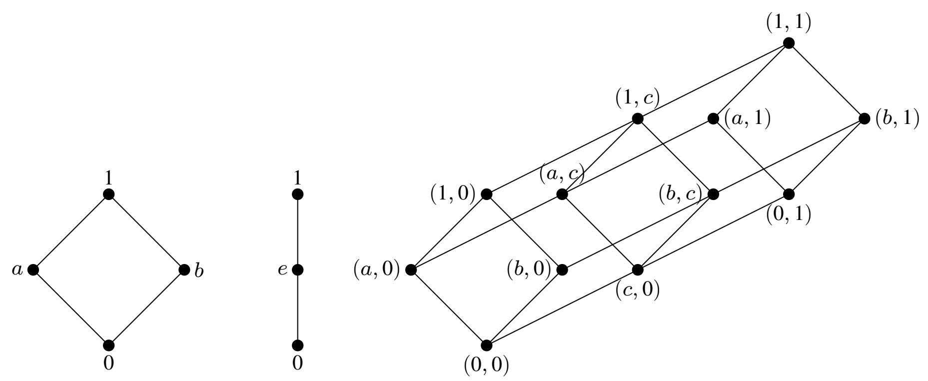


Figure 3

## 4 Some properties of weakly 2-absorbing primary ideals

We start with the definition of a triple zero from Wasadikar and Gaikwad [9].

**Definition 4.1.** Let $I$ be a weakly 2-absorbing ideal of $L$. Let $a, b, c \in L$ be such that $a \wedge b \wedge c = 0$, $a \wedge b \notin I$, $a \wedge c \notin I$ and $b \wedge c \notin I$, then $(a, b, c)$ is called a triple zero of $I$.

**Definition 4.2.** Let $I$ be a weakly 2-absorbing primary ideal of $L$ and suppose that $I_1 \cap I_2 \cap I_3 \subseteq I$ for some ideals $I_1$, $I_2$ and $I_3$ of $L$. We say that $I$ is a free triple-zero with respect to $I_1 \cap I_2 \cap I_3$ if $(a, b, c)$ is not a triple-zero of $I$ for any $a \in I_1$, $b \in I_2$ and $c \in I_3$.

**Example 4.3.** Consider the lattice shown in Figure 1. Here the ideal $I = (l]$ is a free triple zero with respect to $(a] \cap (d] \cap (e]$.

**Conjecture 4.4.** Let $I$ be a weakly 2-absorbing primary ideal of a lattice $L$. Suppose that $(0] \neq I_1 \cap I_2 \cap I_3 \subseteq I$ for some ideals $I_1$, $I_2$ and $I_3$ of $L$. Then $I$ is a free triple-zero with respect to $I_1 \cap I_2 \cap I_3$.

The following Theorem is an analogue of Lemma 2.29 from Badawi et al. [5].

**Theorem 4.5.** *Let $I$ be a weakly 2-absorbing primary ideal of a lattice $L$ and suppose that $x \wedge y \wedge J \subseteq I$ for some $x, y \in L$ and some ideal $J$ of $L$ such that $(a, b, c)$ is not a triple-zero of $I$ for every $z \in J$. If $x \wedge y \notin I$, then $x \wedge J \subseteq \sqrt{I}$ or $y \wedge J \subseteq \sqrt{I}$.*

**Proof.** Let $x \wedge y \notin I$. Suppose that $x \wedge J \nsubseteq \sqrt{I}$ and $y \wedge J \nsubseteq \sqrt{I}$. Then there exist $j_1, j_2 \in J$ such that $x \wedge j_1 \notin \sqrt{I}$ and $y \wedge j_2 \notin \sqrt{I}$. As $(x, y, j_1)$ is not a triple-zero of $I$ and $x \wedge y \wedge j_1 \in I$, we have $y \wedge j_1 \in \sqrt{I}$. As $(x, y, j_2)$ is not a triple-zero of $I$ and $x \wedge y \wedge j_2 \in I$ implies $x \wedge j_2 \in \sqrt{I}$.

Since $(x, y, j_1 \vee j_2)$ is not a triple-zero of $I$ and $x \wedge y \wedge (j_1 \vee j_2) \in I$ and $x \wedge y \notin I$, we have either $x \wedge (j_1 \vee j_2) \in \sqrt{I}$ or $y \wedge (j_1 \vee j_2) \in \sqrt{I}$. Suppose that $x \wedge (j_1 \vee j_2) \in \sqrt{I}$. Therefore, $(x \wedge j_1) \vee (x \wedge j_2) \leq x \wedge (j_1 \vee j_2) \in \sqrt{I}$ and so $(x \wedge j_1) \vee (x \wedge j_2) \in \sqrt{I}$. Hence $x \wedge j_2 \in \sqrt{I}$ and $x \wedge j_1 \in \sqrt{I}$, which is a contradiction.

Similarly, if $y \wedge (j_1 \vee j_2) \in \sqrt{I}$ then $(y \wedge j_1) \vee (y \wedge j_2) \in \sqrt{I}$. Hence $y \wedge j_1 \in \sqrt{I}$ and $y \wedge j_2 \in \sqrt{I}$, which is a contradiction. Hence $x \wedge J \subseteq \sqrt{I}$ or $y \wedge J \subseteq \sqrt{I}$. □

**Remark 4.6.** Let $I$ be a weakly 2-absorbing primary ideal of $L$ and suppose that $I_1 \cap I_2 \cap I_3 \subseteq I$ for some ideals $I_1, I_2, I_3$ of $L$ such that $I$ is free triple-zero with respect to $I_1 \cap I_2 \cap I_3$. Then if $a \in I_1, b \in I_2$ and $c \in I_3$, then either $a \wedge b \in I$ or $a \wedge c \in \sqrt{I}$ or $b \wedge c \in \sqrt{I}$.

Let $I$ be a weakly 2-absorbing primary ideal of $L$. In view of the below result, one can see that Conjecture 1 is valid if and only if whenever $0 \neq I_1 \cap I_2 \cap I_3 \subseteq I$ for some ideals $I_1, I_2, I_3$ of $L$, then either $I_1 \cap I_2 \subseteq I$ or $I_1 \cap I_3 \subseteq \sqrt{I}$ or $I_2 \cap I_3 \subseteq \sqrt{I}$.

The following Theorem is an analogue of Lemma 2.30 from Badawi et al. [5].

**Theorem 4.7.** *Let $I$ be a weakly 2-absorbing primary ideal of a lattice $L$ and suppose that $(0] \neq I_1 \cap I_2 \cap I_3 \subseteq I$ for some ideals $I_1, I_2, I_3$ of $L$ such that $I$ is a free triple-zero with respect to $I_1 \cap I_2 \cap I_3$. Then $I_1 \cap I_2 \subseteq I$ or $I_1 \cap I_3 \subseteq \sqrt{I}$ or $I_2 \cap I_3 \subseteq \sqrt{I}$.*

**Proof.** Suppose that $I$ is a weakly 2-absorbing primary ideal. Let $(0] \neq I_1 \cap I_2 \cap I_3 \subseteq I$ for some ideals $I_1, I_2, I_3$ of $L$ such that $I$ is free triple-zero with respect to $I_1 \cap I_2 \cap I_3$. Suppose that $I_1 \cap I_2 \nsubseteq I$. We show that $I_1 \cap I_3 \subseteq \sqrt{I}$ or $I_2 \cap I_3 \subseteq \sqrt{I}$. Suppose that $I_1 \cap I_3 \nsubseteq \sqrt{I}$ and $I_2 \cap I_3 \nsubseteq \sqrt{I}$. Then there exist $q_1 \in I_1$ and $q_2 \in I_2$ such that $q_1 \wedge I_3 \nsubseteq \sqrt{I}$ and $q_2 \wedge I_3 \nsubseteq \sqrt{I}$. As $q_1 \wedge q_2 \wedge I_3 \subseteq I$, we have $q_1 \wedge q_2 \in I$ by Theorem 4.5.

Since $I_1 \cap I_2 \nsubseteq I$, we have $a \wedge b \notin I$ for some $a \in I_1$, $b \in I_2$. Since $a \wedge b \wedge I_3 \subseteq I$ and $a \wedge b \notin I$, we have $a \wedge I_3 \subseteq \sqrt{I}$ or $b \wedge I_3 \subseteq \sqrt{I}$ by Theorem 4.5. We consider three cases.
Case 1: Suppose that $a \wedge I_3 \subseteq \sqrt{I}$ but $b \wedge I_3 \nsubseteq \sqrt{I}$.
Since $q_1 \wedge b \wedge I_3 \subseteq I$ and $b \wedge I_3 \nsubseteq \sqrt{I}$ and $q_1 \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $q_1 \wedge b \in I$ by Theorem 4.5. Since $(a \vee q_1) \wedge b \wedge I_3 \subseteq I$ and $a \wedge I_3 \subseteq \sqrt{I}$, but $q_1 \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $(a \vee q_1) \wedge I_3 \nsubseteq \sqrt{I}$. Since $b \wedge I_3 \nsubseteq \sqrt{I}$ and $(a \vee q_1) \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $(a \vee q_1) \wedge b \in I$ by Theorem 4.5. Since $(a \wedge b) \vee (q_1 \wedge b) \leq (a \vee q_1) \wedge b \in I$, we have $(a \wedge b) \vee (q_1 \wedge b) \in I$. Thus $q_1 \wedge b \in I$ and $a \wedge b \in I$, a contradiction.
Case 2: Suppose that $b \wedge I_3 \subseteq \sqrt{I}$, but $a \wedge I_3 \nsubseteq \sqrt{I}$.
Since $a \wedge q_2 \wedge I_3 \subseteq I$ and $a \wedge I_3 \nsubseteq \sqrt{I}$ and $q_2 \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $a \wedge q_2 \in I$ by Theorem 4.5. Since $a \wedge (b \vee q_2) \wedge I_3 \subseteq I$ and $b \wedge I_3 \subseteq \sqrt{I}$, but $q_2 \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $(b \vee q_2) \wedge I_3 \nsubseteq \sqrt{I}$. Since $a \wedge I_3 \nsubseteq \sqrt{I}$ and $(b \vee q_2) \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $a \wedge (b \vee q_2) \in I$ by Theorem 4.5. Since $(a \wedge b) \vee (a \wedge q_2) \leq a \wedge (b \vee q_2) \in I$, we have $(a \wedge b) \vee (a \wedge q_2) \in I$. Thus $a \wedge q_2 \in I$ and $a \wedge b \in I$, a contradiction.
Case 3: $a \wedge I_3 \subseteq \sqrt{I}$ and $b \wedge I_3 \subseteq \sqrt{I}$.
Since $b \wedge I_3 \subseteq \sqrt{I}$ and $q_2 \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $(b \vee q_2) \wedge I_3 \nsubseteq \sqrt{I}$. Since $q_1 \wedge (b \vee q_2) \wedge I_3 \subseteq I$ and $q_1 \wedge I_3 \nsubseteq \sqrt{I}$ and $(b \vee q_2) \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $q_1 \wedge (b \vee q_2) \in I$ by Theorem 4.5. As $(q_1 \wedge b) \vee (q_1 \wedge q_2) \leq q_1 \wedge (b \vee q_2) \in I$, we have $(q_1 \wedge b) \vee (q_1 \wedge q_2) \in I$. Hence $b \wedge q_1 \in I$. Since $a \wedge I_3 \subseteq \sqrt{I}$ and $q_1 \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $(a \vee q_1) \wedge I_3 \nsubseteq \sqrt{I}$. Since $(a \vee q_1) \wedge q_2 \wedge I_3 \subseteq I$ and $q_2 \wedge I_3 \nsubseteq \sqrt{I}$ and $(a \vee q_1) \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $(a \vee q_1) \wedge q_2 \in I$ by Theorem 4.5. As $(a \wedge q_2) \vee (q_1 \wedge q_2) \leq (a \vee q_1) \wedge q_2 \in I$, we have $(a \wedge q_2) \vee (q_1 \wedge q_2) \in I$. Hence $a \wedge q_2 \in I$. Now, since $(a \vee q_1) \wedge (b \vee q_2) \wedge I_3 \subseteq I$ and $(a \vee q_1) \wedge I_3 \nsubseteq \sqrt{I}$ and $(b \vee q_2) \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $(a \vee q_1) \wedge (b \vee q_2) \in I$ by Theorem 4.5. We conclude that $a \wedge b \in I$, a contradiction. Hence $I_1 \cap I_3 \subseteq \sqrt{I}$ or $I_2 \cap I_3 \subseteq \sqrt{I}$. □

**Theorem 4.8.** *Let $I$ be an ideal of $L$. If $\sqrt{I}$ is a weakly prime ideal of $L$, then $I$ is a weakly 2-absorbing primary ideal of $L$.*

**Proof.** Suppose that $0 \neq a \wedge b \wedge c \in I$ and $a \wedge b \notin I$. Since $(a \wedge c) \wedge (b \wedge c) = a \wedge b \wedge c \in I \subseteq \sqrt{I}$ and $\sqrt{I}$ is a weakly prime ideal of $L$, we have either $b \wedge c \in \sqrt{I}$ or $a \wedge c \in \sqrt{I}$. Hence $I$ is a weakly 2-absorbing primary ideal of $L$. □

The following Theorem is an analogue of Theorem 2.1 from Mostafanasab and Darani [8], which is a characterization of 2-absorbing primary ideals.

**Theorem 4.9.** *Let $I$ be a proper ideal of $L$. Then the following conditions are equivalent:*

*(i) $I$ is a 2-absorbing primary ideal of $L$.*

*(ii) For $x, y \in L$ such that $x \wedge y \notin \sqrt{I}$, either $(I :_L x \wedge y) \subseteq (I :_L x)$ or $(I :_L x \wedge y) \subseteq (\sqrt{I} :_L y)$.*

*(iii) For any $x \in L$ and any ideal $I_1$ of $L$ such that $x \wedge I_1 \nsubseteq \sqrt{I}$, either $(I :_L x \wedge I_1) \subseteq (I :_L I_1)$ or $(I :_L x \wedge I_1) \subseteq (\sqrt{I} :_L x)$.*

*(iv) For ideals $I_1, I_2$ of $L$ such that $I_1 \cap I_2 \nsubseteq I$, either $(I :_L I_1 \cap I_2) \subseteq (\sqrt{I} :_L I_1)$ or $(I :_L I_1 \cap I_2) \subseteq (\sqrt{I} :_L I_2)$.*

*(v) For ideals $I_1, I_2, I_3$ of $L$ with $I_1 \cap I_2 \cap I_3 \subseteq I$, either $I_1 \cap I_2 \subseteq I$ or $I_1 \cap I_3 \subseteq \sqrt{I}$ or $I_2 \cap I_3 \subseteq \sqrt{I}$.*

**Proof.** $(1) \Rightarrow (2)$. Suppose that $x, y \in L$ such that $x \wedge y \notin \sqrt{I}$. Let $a \in (I :_L x \wedge y)$. Then $a \wedge x \wedge y \in I$. As $I$ is 2-absorbing primary, either $a \wedge x \in I$ or $a \wedge y \in \sqrt{I}$. That is either $a \in (I :_L x)$ or $a \in (\sqrt{I} :_L y)$. Hence $(I :_L x \wedge y) \subseteq (I :_L x) \cup (I :_L y)$. We know that if an ideal is a subset of the union of two ideals, then it is subset of one of them. Hence $(I :_L x \wedge y) \subseteq (I :_L x)$ or $(I :_L x \wedge y) \subseteq (\sqrt{I} :_L y)$.
$(2) \Rightarrow (3)$. Suppose that $x \in L$ and $I_1$ is an ideal of $L$ such that $x \wedge I_1 \nsubseteq \sqrt{I}$. Let $a \in (I :_L x \wedge I_1)$. Then $a \wedge x \wedge I_1 \subseteq I$. Thus $I_1 \subseteq (I :_L a \wedge x)$. If $a \wedge x \in \sqrt{I}$, then $a \in (\sqrt{I} :_L x)$. If $a \wedge x \notin \sqrt{I}$, then $(I :_L a \wedge x) \subseteq (I :_L a)$ or $(I :_L a \wedge x) \subseteq (\sqrt{I} :_L x)$, by (2). Therefore $I_1 \subseteq (I :_L a)$ or $I_1 \subseteq (\sqrt{I} :_L x)$. If $I_1 \subseteq (\sqrt{I} :_L x)$, then $x \wedge I_1 \subseteq \sqrt{I}$ which is a contradiction to our hypothesis. Therefore the only possibility is that $I_1 \subseteq (I :_L a)$. Then $a \in (I :_L I_1)$. Hence $(I :_L x \wedge I_1) \subseteq (I :_L I_1) \cup (\sqrt{I} :_L x)$. We conclude that $(I :_L x \wedge I_1) \subseteq (I :_L I_1)$ or $(I :_L x \wedge I_1) \subseteq (\sqrt{I} :_L x)$.
$(3) \Rightarrow (4)$. Suppose that $I_1, I_2$ are ideals of $L$ such that $I_1 \cap I_2 \nsubseteq I$. Let $a \in (I :_L I_1 \cap I_2)$. Then $a \wedge I_1 \wedge I_2 \subseteq I$. Thus $I_2 \subseteq (I :_L a \wedge I_1)$. If $a \wedge I_1 \subseteq \sqrt{I}$, then $a \in (\sqrt{I} :_L I_1)$. Hence we assume that $a \wedge I_1 \nsubseteq \sqrt{I}$. Which implies that $(I :_L a \wedge I_1) \subseteq (I :_L I_1)$ or $(I :_L a \wedge I_1) \subseteq (\sqrt{I} :_L a)$. If $(I :_L a \wedge I_1) \subseteq (I :_L I_1)$, then $I_1 \cap I_2 \subseteq I$, which is a contradiction. Thus $(I :_L a \wedge I_1) \subseteq (\sqrt{I} :_L a)$ which implies that $a \in (\sqrt{I} :_L I_2)$. Hence $(I :_L I_1 \cap I_2) \subseteq (\sqrt{I} :_L I_1) \cup (\sqrt{I} :_L I_2)$. Thus $(I :_L I_1 \cap I_2) \subseteq (\sqrt{I} :_L I_1)$ or $(I :_L I_1 \cap I_2) \subseteq (\sqrt{I} :_L I_2)$.
$(4) \Rightarrow (5)$. Suppose that $I_1, I_2, I_3$ are ideals of $L$. Let $I_1 \cap I_2 \cap I_3 \subseteq I$ such that $I_1 \cap I_2 \nsubseteq I$. Then we show that either $I_1 \cap I_3 \subseteq \sqrt{I}$ or $I_2 \cap I_3 \subseteq \sqrt{I}$. As $I_1 \cap I_2 \nsubseteq I$, either $(I :_L I_1 \cap I_2) \subseteq (\sqrt{I} :_L I_1)$ or $(I :_L I_1 \cap I_2) \subseteq (\sqrt{I} :_L I_2)$. As $I_1 \cap I_2 \cap I_3 \subseteq I$, $I_3 \subseteq (I :_L I_1 \cap I_2)$. Which implies that either $I_3 \subseteq (\sqrt{I} :_L I_1)$ or $I_3 \subseteq (\sqrt{I} :_L I_2)$. That is either $I_1 \cap I_3 \subseteq \sqrt{I}$ or $I_2 \cap I_3 \subseteq \sqrt{I}$.
$(5) \Rightarrow (1)$. It is clear. □

The following Theorem is an analogue of Theorem 2.8 from Mostafanasab and Darani [8].

**Theorem 4.10.** *Suppose that $P_1$ is a weakly primary ideal of $L$ such that $\sqrt{P_1} = I_1$ is a weakly primary ideal of $L$, and suppose that $P_2$ is a weakly primary ideal of $L$ such that $\sqrt{P_2} = I_2$ is a weakly prime ideal of $L$. Then $P_1 \cap P_2$ is a weakly 2-absorbing primary ideal of $L$.*

**Proof.** Let $P = P_1 \cap P_2$. Thus $\sqrt{P} = \sqrt{P_1 \cap P_2} = \sqrt{P_1} \cap \sqrt{P_2}$. As $\sqrt{P_1} = I_1$ and $\sqrt{P_2} = I_2$, $\sqrt{P} = I_1 \cap I_2$. Suppose that $0 \neq = x \wedge y \wedge z \in P$ for some $x, y, z \in L$, $x \wedge z \notin \sqrt{P}$ and $y \wedge z \notin \sqrt{P}$. Then $x, y, z \notin \sqrt{P}$. We show that $x \wedge y \in P$.

As the intersection of two distinct weakly prime ideals is weakly 2-absorbing, we know that $\sqrt{P} = I_1 \cap I_2$ is a weakly 2-absorbing ideal of $L$. Since $0 \neq x \wedge y \wedge z \in \sqrt{P}$, $x \wedge z, y \wedge z \notin \sqrt{P}$ and as $\sqrt{P}$ is weakly 2-absorbing, we have $x \wedge y \in \sqrt{P}$.

Now we claim that $x \wedge y \in P$. Since $0 \neq x \wedge y \in I_1$, we may assume that $x \in I_1$. Since $x \notin \sqrt{P} = I_1 \cap I_2$ and $0 \neq x \wedge y \in I_2$, we conclude that $x \notin I_2$ and so $y \in I_2$. If $x \in P_1$ and $y \in P_2$, then $x \wedge y \in P$ and we are done.

Therefore assume that $x \notin P_1$. Since $P_1$ is a weakly primary ideal of $L$ and $x \notin P_1$, we have $y \wedge z \in \sqrt{P_1} = I_1$. Since $y \in I_2$ and $y \wedge z \in I_1$, we have $y \wedge z \in \sqrt{P}$, which is a contradiction. Thus $x \in P_1$.

Similarly, assume that $y \notin P_2$. Since $P_2$ is a weakly primary ideal of $L$ and $y \notin P_2$, we have $x \wedge z \in \sqrt{P_2} = I_2$. Since $x \wedge z \in I_2$ and $x \in I_1$, we have $x \wedge z \in \sqrt{P}$, which is a contradiction. Thus $y \in P_1$. Hence $x \wedge y \in P_1 \cap P_2 = P$.

Thus $P_1 \cap P_2$ is a weakly 2-absorbing ideal of $L$. □

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