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δ-PRIMARY IDEALS IN LATTICES

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Abstract In this paper we define an ideal expansion on a lattice. Also we define a δ-primary ideal and a weakly δ-primary ideal with the help of an ideal expansion and investigate some properties of these ideals. We show that for an expansion function δ, an ideal is δ-primary if and only if that ideal is strongly δ-primary ideal of lattice. Also we define a δ-twin-zero and prove some results based on δ-twin-zero.

1 Introduction

The study of expansions of ideals and δ-primary ideals for commutative rings is carried out by Zaho Dongsheng [4] where δ is a mapping with some additional properties. The aim of this paper is to generalize these results to lattices. In section 2, we introduce the concept of an ideal expansion δ where δ is a mapping that assigns to each ideal I an ideal $\delta(I)$ of the same lattice. We define a δ-primary ideal with respect to such an expansion. In section 3, we investigate ideal expansions satisfying some additional conditions and some properties of δ-primary ideals with respect to homomorphisms. The study of a δ-zero-divisor, a δ-nilpotent element is done by Atani [1]. In this paper we introduce these concepts in a lattice. In section 4, we define a weakly δ-primary ideal with respect to such an expansion. Also we define a δ-twin-zero and prove some results based on δ-twin-zero.

Throughout in this paper L denotes a lattice with a least element 0. It is known that $Id(L)$ the set of all ideals of a lattice L is a lattice under set inclusion see Grätzer [2, p. 22]. The undefined concepts related to lattice theory are from Grätzer [2].

2 Expansions of ideals and δ-primary ideals

The notion of the radical of an ideal I in a lattice, denoted by $\sqrt{I}$, can be found in Wasadikar and Gaikwad [3], $\sqrt{I}=\bigcap\{P \in Id(L) | P$ is a prime ideal, $I \subseteq P\}$.

In this section, we introduce an ideal expansion δ and define a δ-primary ideals with respect to such an expansion.

Definition 2.1. An expansion of ideals, or an ideal expansion, is a function $\delta : Id(L) \to Id(L)$, satisfying $I \subseteq \delta(I)$, and $P \subseteq Q$ implies $\delta(P) \subseteq \delta(Q)$.

Example 2.2. (i) The identity function $\delta_0 : Id(L) \to Id(L)$, is an expansion of ideals.

(ii) The function $\mathbf{B}$ that assigns the biggest ideal L to each ideal is an expansion of ideals.

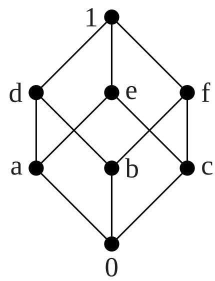
(iii) For a proper ideal P, the mapping $\mathbf{M}(P) : Id(L) \to Id(L)$, where $\mathbf{M}(P) = \cap\{I \in Id(L) | P \subseteq I, I$ is a maximal ideal other than $L\}$, and $\mathbf{M}(L) = L$. Then $\mathbf{M}$ is an expansion of ideals.

(iv) For each ideal I define $\delta_1(I) = \sqrt{I}$. Then δ_1 is an expansion of ideals.

Definition 2.3. Given an expansion δ of ideals, an ideal I of L is called δ-primary if for all $a, b \in L$, $a \wedge b \in I$ and $a \notin I$ then $b \in \delta(I)$.

The definition of a δ-primary ideal can be stated as: if $a \wedge b \in I$ and $a \notin \delta(I)$, then $b \in I$ for all $a, b \in L$.

Example 2.4. Consider the lattice shown in Figure 1. From Example 2.2, for the ideal $P = \{0, a, c, e\}$, $\delta_0(P) = P$, $\delta_1(P) = P$, $\mathbf{M}(P) = P$ and $\mathbf{B}(P) = L$. Thus P is a δ_0-primary, δ_1-primary, $\mathbf{M}$-primary, $\mathbf{B}$-primary ideal. However the ideal $I = \{0, a\}$, satisfies $\delta_0(I) = I$, $\delta_1(I) = I$, $\mathbf{M}(I) = I$ and $\mathbf{B}(I) = L$, $f \wedge e = a \in I$ and $f \notin I$ but $e \notin \delta_0(I)$, $e \notin \delta_1(I)$, $e \notin \mathbf{M}(I)$. Thus I is not a δ_0-primary, δ_1-primary, $\mathbf{M}$-primary ideal but it is a $\mathbf{B}$-primary ideal. Also the ideal $\{0\}$ is not a δ_0-primary, δ_1-primary, $\mathbf{M}$-primary ideal but it is a $\mathbf{B}$-primary ideal.



Lattice showing primary ideals

Figure 1

In general, the intersection of two δ-primary ideals is not δ-primary. The ideals $N = \{0, a, b, d\}$,$P = \{0, a, c, e\}$ in Figure 1 are δ-primary but $N \cap P = \{0, a\}$ is not δ-primary.

Lemma 2.5. *An ideal I is δ_0-primary if and only if it is if prime.*

Proof. First assume that I is δ_0-primary then $a \wedge b \in I$ and $a \notin I$ imply $b \in \delta_0(I) = I$. Hence we get $a \wedge b \in I$ implies $a \in I$ or $b \in I$, for all $a, b \in L$. Therefore I is a prime ideal.

Conversely, assume that I is a prime ideal. Let $a \wedge b \in I$ and $a \notin I$ as I is prime we get $b \in I = \delta_0(I)$ thus I is δ_0- primary. □

Lemma 2.6. *An ideal I of L is δ_1-primary if and only if it is primary.*

Proof. If I is δ_1-primary, then $a \wedge b \in I$ and $a \notin I$ imply $b \in \delta_1(I) = \sqrt{I}$. Therefore I is a primary ideal.

Conversely, assume that I is a primary ideal. Let $a \wedge b \in I$ and $a \notin I$. Then as I is primary, we get $b \in \sqrt{I} = \delta_1(I)$. Thus I is δ_1- primary. □

Example 2.7. Every ideal is $\mathbf{B}$-primary.

Lemma 2.8. *If δ and γ are two ideal expansions and $\delta(I) \subseteq \gamma(I)$ for each ideal I, then every δ-primary ideal is also γ-primary. Thus, in particular, a prime ideal is δ-primary for every ideal expansion δ.*

Proof. Let P be a δ-primary ideal. Suppose that $a \wedge b \in P$ for some $a, b \in L$ and $a \notin P$. If P is δ-primary, then $b \in \delta(P) \subseteq \gamma(P)$. So P is γ-primary.
Next, suppose that P is a prime ideal then by Lemma 2.5, P is δ_0-primary ideal.
For any ideal expansion δ, $P \subseteq \delta(P)$, so $\delta_0(P) = P \subseteq \delta(P)$. Thus we get $\delta_0(P) \subseteq \delta(P)$ and P is δ_0-primary. Therefore P is δ-primary. □

In the following Lemma we prove that the intersection of two ideal expansions is an ideal expansion. Generally intersection of any collection of ideal expansions is an ideal expansion.

Lemma 2.9. *Given two ideal expansions δ_1 and δ_2, define $\delta(I) = \delta_1(I) \cap \delta_2(I)$. Then δ is also an ideal expansion.*

Proof. Let $\delta : Id(L) \to Id(L)$ be a function, defined by $\delta(I) = \delta_1(I) \cap \delta_2(I)$.
As δ_1 and δ_2 are expansions of ideals, we get $I \subseteq \delta_1(I) \cap \delta_2(I)$
and if $P \subseteq Q$ then $\delta_1(P) \cap \delta_2(P) \subseteq \delta_1(Q) \cap \delta_2(Q)$ which implies $\delta(P) \subseteq \delta(Q)$.
Thus δ is an ideal expansion. □

Remark 2.10. Let δ be an ideal expansion.
Define $E_\delta(P) = \cap\{J \in Id(L) | P \subseteq J, J$ is a $\delta -$ primary ideal$\}$. Then E_δ is an ideal expansion. For example, $E_{\delta_0} = \delta_1$, $E_{\mathbf{M}} = \mathbf{M}$ and $E_{\delta_1} = \delta_1$.

Proposition 2.11. *Let $\{J_i | i \in D\}$ be a directed collections of δ-primary ideals of L, then $J = \cup_{i\in D} J_i$ is δ-primary.*

Proof. Let $a \wedge b \in J$ and $a \notin J$, then there exists an ideal J_i with $a \wedge b \in J_i$ and $a \notin J_i$, as J_i δ-primary, we get $b \in \delta(J_i)$, and also we have $\delta(J_i) \subseteq \delta(J)$ which implies $b \in \delta(J)$. Hence J is δ-primary. □

Definition 2.12. Let δ be an expansion of ideal. A proper ideal P of L is called strongly δ-primary if $I \wedge J \subseteq P$ and $I \nsubseteq P$ then $J \subseteq \delta(P)$. whenever $I, J \in Id(L)$.

The definition of strongly δ-primary ideals can be also stated as: if $I \wedge J \subseteq P$ and $I \nsubseteq \delta(P)$, then $J \subseteq P$.

In the following result, we show that an ideal P is δ-primary if and only if P is strongly δ-primary ideal of L.

Lemma 2.13. *An ideal P is δ-primary if and only if P is a strongly δ-primary ideal of L.*

Proof. Let P be a δ-primary ideal. Suppose that $I \wedge J \subseteq P$ and $I \nsubseteq P$, now suppose on contrary $J \nsubseteq \delta(P)$ then we can choose $a \in I - P$ and $b \in J - \delta(P)$, then we get $a \wedge b \in I \wedge J \subseteq P$ but $a \notin P$ and $b \notin \delta(P)$, which contradicts P is δ-primary. Thus $J \subseteq \delta(P)$. Hence P is a strongly δ-primary ideal of L.
Conversely, suppose that P is a strongly δ-primary ideal of L, for any two elements a and b. Suppose $a \wedge b \in P$ and $a \notin P$. Then $(a] \wedge (b] \subseteq P$ and $(a] \nsubseteq P$ which implies $(b] \subseteq \delta(P)$. Hence $b \in (b] \subseteq \delta(P)$ implies $b \in \delta(P)$. Thus P is a δ-primary ideal of L. □

Definition 2.14. Let P and Q be ideals of a lattice L, the residual division of P by Q is defined to be the ideal $P : Q = \{x \in L | x \wedge y \in P$ for all $y \in Q\}$.

Theorem 2.15. *Let δ be an ideal expansion. Then*
(1) if P is a δ-primary ideal and I is an ideal with $I \nsubseteq \delta(P)$, then $(P : I) = P$;
(2) for any δ-primary ideal P and non-empty subset N of L, $P : N$ is also a δ-primary ideal.

Proof. (1) Clearly $P \subseteq P : I$. By the definition of $P : I$, we have $I \wedge (P : I) \subseteq P$, since $I \nsubseteq \delta(P)$, we get $(P : I) \subseteq P$. Therefore $(P : I) = P$.
(2) Let $a \wedge b \in P : N$ and $a \notin P : N$ then there is $n \in N$ such that $a \wedge n \notin P$, but $a \wedge n \wedge b = a \wedge b \wedge n \in P$ implies $(a \wedge n) \wedge b = (a \wedge b) \wedge n \in P$ and $a \wedge n \notin P$ and P is δ-primary then we get $b \in \delta(P) \subseteq \delta(P : N)$ implies $b \in \delta(P : N)$. Thus $P : N$ is a δ-primary ideal. □

3 Expansions with extra properties

In this section we investigate δ-primary ideals where δ satisfies additional conditions and prove more results with respect to such expansions.

Definition 3.1. An ideal expansion δ is intersection preserving if it satisfies $\delta(I \cap J) = \delta(I) \cap \delta(J)$ for any $I, J \in Id(L)$.

Definition 3.2. An expansion is said to be global if for any lattice homomorphism $f : L \to K$, $\delta(f^{-1}(I)) = f^{-1}(\delta(I))$ for all $I \in Id(K)$.

The expansions δ_0 and $\mathbf{B}$ are both intersection preserving and global.
The following lemma shows that if L is a distributive lattice then, the expansion $\mathbf{M}$ has the intersection preserving property.

Lemma 3.3. *Let L be a distributive lattice. For each ideal I, let $\mathbf{M}(I) = \cap\{H \in Id(L) | I \subseteq H$, H is a maximal ideal $\}$. Then $\mathbf{M}(I \cap J) = \mathbf{M}(I) \cap \mathbf{M}(J)$ holds for any two ideals I and J.*

Proof. Let $\Re_1 = \{H | I \cap J \subseteq H$, H is a maximal ideal $\}$,
$\Re_2 = \{H | I \subseteq H$ or $J \subseteq H$, H is a maximal ideal $\}$.
Then $\cap\Re_1 = \mathbf{M}(I \cap J)$ and $\cap\Re_2 = \mathbf{M}(I) \cap \mathbf{M}(J)$.
Let $H \in \Re_1$ then $I \cap J \subseteq H$, and H is a maximal ideal so it is prime. Hence $I \subseteq H$ or $J \subseteq H$ implies $H \in \Re_2$. Thus $\Re_1 \subseteq \Re_2$.
Let $H \in \Re_2$ then $I \subseteq H$ or $J \subseteq H$ and H is maximal ideal, so we have $I \cap J \subseteq I \subseteq H$ or $I \cap J \subseteq J \subseteq H$ which implies $I \cap J \subseteq H$ implies $H \in \Re_1$ thus we get $\Re_2 \subseteq \Re_1$. Therefore $\Re_1 = \Re_2$, so $\mathbf{M}(I \cap J) = \mathbf{M}(I) \cap \mathbf{M}(J)$. □

Lemma 3.4. *Let δ be an intersection preserving ideal expansion. If $Q_1, Q_2, \ldots, Q_n$ are δ-primary ideals of L, and $P = \delta(Q_i)$ for all i, then $Q = \bigcap_{i=1}^{n} Q_i$ is δ-primary.*

Proof. Let $x \wedge y \in Q$ and $x \notin Q$. Then, for some $i = k$, $x \notin Q_k$. But $x \wedge y \in Q \subseteq Q_k$, so we get $y \in \delta(Q_k)$ but $\delta(Q) = \delta(\bigcap_{i=1}^{n} Q_i) = \bigcap_{i=1}^{n}(\delta(Q_i)) = P = \delta(Q_k)$. Thus $y \in \delta(Q)$, so Q is δ-primary. □
In the following lemma, we prove that the inverse image of a δ-primary ideal of L under a homomorphism is again a δ-primary ideal.

Lemma 3.5. *If δ is global and $f : L \to K$ is a lattice homomorphism, then for any δ-primary ideal I of K, $f^{-1}(I)$ is a δ-primary ideal of L.*

Proof. Let $a, b \in L$ with $a \wedge b \in f^{-1}(I)$ and $a \notin f^{-1}(I)$. Then $f(a) \wedge (b) \in I$ and $f(a) \notin I$ but I is δ-primary then we get $f(b) \in \delta(I)$, so $b \in f^{-1}(\delta(I)) = \delta(f^{-1}(I))$. Hence $f^{-1}(I)$ is δ-primary. □
Next result gives a characterization for a δ-primary ideal.

Lemma 3.6. *Let $f : L \to K$ be a surjective lattice homomorphism. Let I be an ideal of L such that $ker(f) \subseteq I$. Then I is a δ-primary ideal if and only if $f(I)$ is a δ-primary ideal of K.*

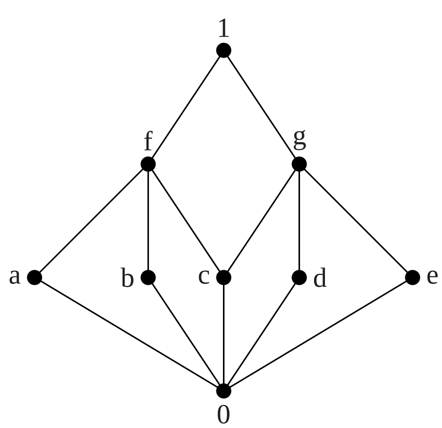
Proof. First suppose that $f(I)$ is a δ-primary and I contains $ker(f)$ we have
$f^{-1}(f(I)) = I$. Then by Lemma 3.5, I is δ-primary.
Conversely, suppose that I is δ-primary. If $a, b \in k$ and $a \wedge b \in f(I)$ and $a \notin f(I)$, then there exist $x, y \in L$ such that $f(x) = a$ and $f(y) = b$, then $f(x \wedge y) = f(x) \wedge f(y) = a \wedge b \in f(I)$ implies $x \wedge y \in f^{-1}(f(I)) = I$ and $f(x) = a \notin f(I)$ implies $x \notin I$, so $y \in \delta(I)$ and hence $b = f(y) \in f(\delta(I))$. Now $\delta(I) = \delta(f^{-1}(f(I))) = f^{-1}(\delta(f(I)))$ which implies $f(\delta(I)) = \delta(f(I))$. Thus $f(I)$ is δ-primary. □

Definition 3.7. An element of a lattice L is called δ-nilpotent if $a \in \delta(\{0\})$.

The set of all δ-nilpotent elements of L is denoted by $nil_\delta(L)$.

Definition 3.8. Let L be a lattice with an ideal expansion δ. An element $x \in L$ is called δ-zero-divisor if there exists $y \in L$ with $y \notin \delta(\{0\})$ such that $x \wedge y \in \delta(\{0\})$.

The set of all δ-zero-divisors elements of L is denoted by $Z_\delta(L)$.



Lattice showing showing sets of nilpotent elements
Figure 2

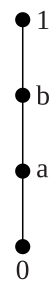
Example 3.9. In the lattice L shown in Figure 2, for ideal expansion δ_0 in Example 2.2, $\delta_0(\{0\}) = \{0\}$, set of all nilpotent element of L with ideal expansion δ_0 is $nil_{\delta_0}(L) = \{0\}$, and set of all δ_0-zero-divisors of L is $Z_{\delta_0}(L) = \{0, a, b, c, d, e, f, g\}$.
For ideal expansion $\mathbf{M}$ in Example 2.2, $\mathbf{M}(\{0\}) = \{0, c\}$, the set of all nilpotent element of L with ideal expansion $\mathbf{M}$ is $nil_{\mathbf{M}} = \{0, c\}$ and set of all $\mathbf{M}$-zero-divisors of L is $Z_{\mathbf{M}}(L) = \{0, a, b, c, d, e, f, g\}$.
Next, for ideal expansion δ_1 in Example 2.2 see Figure 4, $\delta_1(\{0\}) = \{0\}$, $nil_{\delta_1}(L) = \{0\}$, and set of all δ_1-zero-divisors of L is $Z_{\delta_1}(L) = \{0, a, b, c, d, e, f, g\}$.

Proposition 3.10. *Let δ be an ideal expansion such that $\delta(\{0\}) \neq L$. Then the following hold:*
(i) $nil_\delta(L)$ is an ideal of L with $nil_\delta(L) \subseteq Z_\delta(L)$.
(ii) If $Z_\delta(L)$ is an ideal of L, then $Z_\delta(L)$ is δ-primary.

Proof. (i) Let $x, y \in nil_\delta(L)$ and $r \in L$ then $x, y \in \delta(\{0\})$, so $x \vee y \in \delta(\{0\})$ and $x \wedge r \in \delta(\{0\})$, since $\delta(\{0\})$ is an ideal of L. Hence $x \vee y \in nil_\delta(L)$ and $x \wedge y \in nil_\delta(L)$. Thus $nil_\delta(L)$ is an ideal of L.
Let $x \in nil_\delta(L)$. Since $x = x \wedge 1 \in \delta(\{0\})$ and $1 \notin \delta(\{0\}$. Thus we have $x \in nil_\delta(L)$. Therefore $nil_\delta(L) \subseteq Z_\delta(L)$.
(ii) Let $x, y \in L$ be such that $x \wedge y \in Z_\delta(L)$. Then there exists $z \in L$ such that $z \notin \delta(\{0\})$ and $x \wedge y \wedge z \in \delta(\{0\})$. Therefore if $y \wedge z \in \delta(\{0\})$, then $y \in Z_\delta(L)$ and if$y \wedge z \notin \delta(\{0\})$ then $x \in Z_\delta(L)$. Thus $Z_\delta(L)$ is a δ-primary ideal of L. □

Definition 3.11. A lattice L with an ideal expansion δ is called a δ-domainlike lattice if $Z_\delta(L) \subseteq nil_\delta(L)$.

Example 3.12. In the lattice L shown in Figure 3, $\delta_0(\{0\}) = \{0\}$, the set of all nilpotent element of L with ideal expansion δ_0 is $nil_{\delta_0}(L) = \{0\}$, and set of all δ_0-zero-divisors of L with ideal expansion δ_0 is $Z_{\delta_0}(L) = \{0, a\}$. $Z_{\delta_0}(L) \nsubseteq nil_{\delta_0}(L)$. Thus L is not δ_0-domainlike lattice.



Lattice which is not δ_0 like domain
Figure 3

Next, $\mathbf{M}(\{0\}) = \{0, a, b\}$. The set of all nilpotent element of L with ideal expansion $\mathbf{M}$ is $nil_{\mathbf{M}} = \{0, a, b\}$ and set of all $\mathbf{M}$-zero-divisors of L with ideal expansion $\mathbf{M}$ is $Z_{\mathbf{M}}(L) = \{0\}$. Thus $Z_{\mathbf{M}}(L) \subseteq nil_{\mathbf{M}}$. Also $\delta_1(\{0\}) = \{0, a\}$, the set of all nilpotent element of L with ideal expansion δ_1 is $nil_{\delta_1}(L) = \{0, a\}$, and set of all δ_1-zero-divisors of L with ideal expansion δ_1 is $Z_{\delta_1}(L) = \{0\}$. $Z_{\delta_1}(L) \subseteq nil_{\delta_1}(L)$. Hence L is a $\mathbf{M}$-domainlike lattice and δ_1-domainlike lattice.

Theorem 3.13. *A lattice L with an ideal expansion δ such that $\delta(\delta(I)) = \delta(I)$, for every ideal I of L. Then the following hold:*
(1) $\delta(\{0\})$ is a δ-primary ideal of L if and only if $Z_\delta(L) = nil_\delta(L)$. In particular,if $\delta(\{0\})$ is a δ-primary then $Z_\delta(L)$ is a δ-primary ideal of L.
(2) $\delta(\{0\})$ is δ-primary if and only if L is δ-domainlike lattice.
(3) If L is δ-domainlike lattice then $Z_\delta(L)$ is the unique minimal δ-primary ideal of L.

Proof. (1) $\delta(\{0\})$ is a δ-primary ideal of L. Let $x \in Z_\delta(L)$ then there is some $y \notin \delta(\{0\})$ such that $x \wedge y \in \delta(\{0\})$. Since $\delta(\{0\})$ is δ-primary then

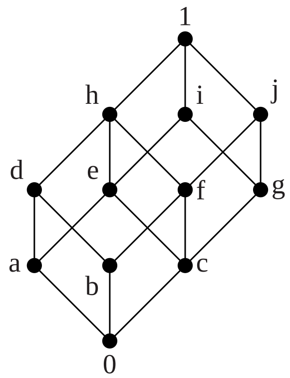
$x \in \delta(\delta(\{0\})) = \delta(\{0\})$. Thus $Z_\delta(L) \subseteq nil_\delta(L)$. By Proposition 3.10, we have equality. Conversely, let $a, b \in L$ be such that $a \wedge b \in \delta(\{0\})$ but $b \notin \delta(\{0\})$ then $a \in Z_\delta(L) = nil_\delta(L)$, so $a \in \delta(\delta(\{0\})) = \delta(\{0\})$ and hence $\delta(\{0\})$ is a δ-primary.
(2) Follows from (1).
(3) To prove (3), as $Z_\delta(L) = nil_\delta(L)$ by (1). We have that $Z_\delta(L)$ is δ-primary ideal of L. Since $\delta(\{0\})$ is δ-primary. Now if J is δ-primary ideal, then $Z_\delta(L) = nil_\delta(L)$, as required. □

4 Weakly δ-primary ideals

Definition 4.1. Given an expansion δ of ideals, an ideal I of L is called weakly δ-primary if for all $a, b \in L$, $0 \neq a \wedge b \in I$ and $a \notin I$ then $b \in \delta(I)$.

The definition of weakly $\delta-$primary ideals can be also stated as: if $0 \neq a \wedge b \in I$ and $a \notin \delta(I)$, then $b \in I$ for all $a, b \in L$.

Example 4.2. Consider the lattice shown in Figure 4. From Example 2.2, for the ideal $I = \{0, a, c, e, g, i\}$, $\delta_0(I) = I$, $\delta_1(I) = I$, $\mathbf{M}(I) = I$ and $\mathbf{B}(I) = L$. Thus I is a weakly δ_0-primary, weakly δ_1-primary, weakly $\mathbf{M}$-primary, weakly $\mathbf{B}$-primary ideal. However the ideal $J = \{0, c\}$, satisfies $\delta_0(J) = J$, $\delta_1(J) = J$, $\mathbf{M}(J) = J$ and $\mathbf{B}(J) = L$, $e \wedge g = c \in J$ and $g \notin J$ but $e \notin \delta_0(J)$, $e \notin \delta_1(J)$, $e \notin \mathbf{M}(J)$. Thus J is not a weakly δ_0-primary, weakly δ_1-primary, weakly $\mathbf{M}$-primary ideal but it is a weakly $\mathbf{B}$-primary ideal.



Lattice with a weakly primary ideal
Figure 4

Lemma 4.3. *Every δ-primary ideal of L is a weakly δ-primary ideal of L.*

Proof. Suppose that I is δ-primary ideal of L. Let for $a, b \in L$, $0 \neq a \wedge b \in I$ and $a \notin I$. As I is δ-primary ideal of L, then we have $b \in \delta(I)$. Thus I is a weakly δ-primary ideal of L. □

The following example shows that the converse of Lemma 4.3 does not hold.

Example 4.4. Consider the lattice shown in Figure 4. Here for the ideal $\{0\}$, $\delta_0(\{0\}) = \{0\}$, $\delta_1(\{0\}) = \{0\}$, $\mathbf{M}(\{0\}) = \{0, c\}$. $\{0\}$ is a weakly δ_0-primary, weakly δ_1-primary, weakly $\mathbf{M}$-primary ideal but it is not a δ_0-primary, δ_1-primary, $\mathbf{M}$-primary ideal, as $a \wedge f = 0 \in \{0\}$ and $a \notin \{0\}$ but $f \notin \delta_0(\{0\})$, $f \notin \delta_1(\{0\})$, $f \notin \mathbf{M}(\{0\})$.

Theorem 4.5. *Let L be a lattice. If I is a weakly δ-primary ideal of L then either $(I : x) = I$ or $(I : x) = (\{0\} : x)$ for every $x \notin \delta(I)$.*

Proof. Let $y \in (I : x)$ for some $x \notin \delta(I)$. Then $x \wedge y \in I$. If $x \wedge y = 0$, then $y \in (\{0\} : x)$. If $x \wedge y \neq 0$, $x \notin \delta(I)$, then $y \in I$, as I is a weakly δ-primary ideal of L. Thus $(I : x) \subseteq I \cup (\{0\} : x)$. Now let $y \in I \cup (\{0\} : x)$. If $y \in I$, then $x \wedge y \in x \wedge I \subseteq I$, So we get $x \wedge y \in I$ implies that $y \in (I : x)$. If $y \in (\{0\} : x)$ then $x \wedge y = 0$, which is contradiction to I is a weakly δ-primary ideal of L. Thus $I \cup (\{0\} : x) \subseteq (I : x)$. So we get $(I : x) = I \cup (\{0\} : x)$. As $(I : x)$ is an ideal of L, so we conclude that either $(I : x) = I$ or $(I : x) = (\{0\} : x)$ for every $x \notin \delta(I)$. □

Definition 4.6. Let I be a weakly δ-primary ideal of L. We say that (a,b) is a δ-twin-zero of I, if $a \wedge b = 0$, $a \notin I$ and $b \notin \delta(I)$.

Remark 4.7. If I is a weakly δ-primary ideal of L that is not δ-primary ideal of L, then I has a δ-twin-zero of (a,b), for some $a,b \in L$.

Theorem 4.8. *Let I be a weakly δ-primary ideal of a distributive lattice L. Suppose that (a,b) is a δ-twin-zero of I, for some $a,b \in L$. Then $a \wedge I = b \wedge I = 0$, where $x \wedge I = \{x \wedge y | y \in I\}$.*

Proof. Suppose that $a \wedge I \neq 0$ then there exist an element $i \in I$ such that $a \wedge i \neq 0$. $(a \wedge b) \vee (a \wedge i) \neq 0$ implies that $a \wedge (b \vee i) \neq 0$ and $0 \neq a \wedge (b \vee i) \in I$ and $a \notin I$, as I is a weakly δ-primary ideal of L, we conclude that $b \vee i \in \delta(I)$ which implies that $b \in \delta(I)$, which is a contradiction to (a,b) is a δ-twin-zero of I. Thus $a \wedge I = 0$.
Similarly, we show that $a \wedge I = 0$. □

Theorem 4.9. *Let L be a distributive lattice. Let I be a weakly δ-primary ideal of L. If I is not δ-primary ideal of L, then $I^2 = 0$, where $I^2 = \{a \wedge b : a \neq b; a,b \in I\}$.*

Proof. Let I be a weakly δ-primary ideal of L that is not δ-primary ideal of L. Then I has a δ-twin-zero (a,b), for some $a,b \in L$. Suppose that $i_1 \wedge i_2 \neq 0$, for some $i_1, i_2 \in I$. Consider $(a \vee i_1) \wedge (b \vee i_2) = (a \wedge b) \vee (a \wedge i_2) \vee (b \wedge i_1) \vee (i_1 \wedge i_2) = i_1 \wedge i_2 \neq 0$, since (a,b) is δ-twin-zero and by Theorem 4.8. Thus $(a \vee i_1) \wedge (b \vee i_2) = i_1 \wedge i_2 \in I$ and I be a weakly δ-primary ideal of L and $a \vee i_1 \notin I$, as $a \notin I$ and $i_1 \in I$. Thus we get $b \vee i_2 \in \delta(I)$ implies that $b \in \delta(I)$, which is a contradiction to (a,b) is a δ-twin-zero of I. Thus $I^2 = 0$. □

Theorem 4.10. *Let L be a distributive lattice. Let I be a weakly δ-primary ideal of L and suppose that (a,b) is a δ-twin-zero of I, for some $a,b \in L$. If $a \wedge r \in I$ for some $r \in L$ then $a \wedge r = 0$.*

Proof. Suppose that $0 \neq a \wedge r \in I$ for some $r \in L$. If $(a \wedge b) \vee (a \wedge r) \neq 0$, then $0 \neq [a \wedge (b \vee r)] \in I$. As $a \notin I$ and I be a weakly δ-primary ideal of L, we get $b \vee r \in \delta(I)$. Hence $b \in \delta(I)$, which is a contradiction to (a,b) is a δ-twin-zero of I. Thus $a \wedge r = 0$. □
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