Palestine Journal of Mathematics

Vol. 8(Special Issue: I, 2019) , 482–489 

# SOME RESULTS ON A GRAPH OF A POSET

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Communicated by T. Tamizh Chelvam

MSC 2010 Classifications: Primary 05C69, 06B99, 05C15; Secondary 05C25.

Keywords and phrases: Poset, atom, complete bipartite graph, chromatic number, clique number.

**Abstract** In this paper, we introduce a simple graph on a finite poset $P$ with the least element 0. Two elements $a, b, \in P$ are adjacent if and only if the set of nonzero lower bounds of $a, b$ in $P$ is nonempty and contains atoms only. We call this graph as the atom based graph of $P$ and denote it by $\Gamma_a(P)$. We obtain some properties of $\Gamma_a(P)$. We give a necessary and sufficient condition for $\Gamma_a(P)$ to be connected, We also give a characterization for a complete bipartite graph to be a $\Gamma_a(P)$. We prove characterization for $\Gamma_a(P)$ to be triangle free.

## 1 Introduction

In 1988, Beck [1] developed the notion of the zero-divisor graph of a commutative ring with identity. Since then, a number of researchers have studied zero-divisor graphs associated to rings and other algebraic structures. In 2007 Nimbhorkar et. al. [8] have studied graphs derived from a meet-semilattice with 0. This was generalized in 2009 by Halaš and Jukl [3] for a poset with 0.

We begin with the some concepts and terminology. Let $P$ be a nonempty set. A binary relation $\leq$ on $P$ is called a partial order if $\leq$ is reflexive, antisymmetric and transitive. Then $(P, \leq)$ is called a *partially ordered set* or a poset. Let $(P, \leq)$ be a poset and $a, b \in P$. If neither $a \leq b$ nor $b \leq a$ hold. Then we say that $a$ and $b$ are incomparable. We write this as $a \parallel b$. Let $(P, \leq)$ be a poset and let $S \subseteq P$. The set $S^l = \{x \in P \mid x \leq s \text{ for every } s \in S\}$ is called the *lower cone* of $S$. If $S = \{a, b\}$, then we write $(a, b)^l$ for $S^l$ and if $S = \{a\}$, then we denote $S^l$ by $a^l$. Similarly $S^u = \{x \in P \mid x \geq s \text{ for every } s \in S\}$ is called the upper cone of $S$. If $S = \{a, b\}$, then we write $(a, b)^u$ for $S^u$ and if $S = \{a\}$, then we denote $S^u$ by $a^u$. A nonempty subset $I$ of a poset $P$ is called an *ideal* if $a, b \in I$ implies $(a, b)^{ul} \subseteq I$.

The study of the zero-divisor graph of a poset can be found in the work of many researchers, e.g., Nimbhorkar and Wasdikar [7], Lu and Wu [6] and Xue and Liu [9]. In [7] Nimbhorkar and Wasdikar, have associated a graph with a poset $P$ not necessarily with 0 (the smallest element), whose vertex set consists of those elements $x \in P$, for which, there is some $y \in P$ with the property $(x, y)^{lu} = P$ and two vertices are adjacent if and only if $(x, y)^{lu} = P$. Halaš and Julk [3] defined the zero divisor graph of a poset with 0, denoted by $G(P)$, with vertex set $P$ and $x, y \in P$ are adjacent if and only if $(x, y)^l = \{0\}$.

Joshi [5] defined the zero divisor graph of a poset $P$ with respect to an ideal $I$, denoted by $G_I(P)$, as follows: The set of vertices of $G_I(P)$ is $V(G_I(P)) = \{x \in P \backslash I | (x, y)^l \subseteq I \text{ for some } y \in P \backslash I\}$ and two distinct vertices $x, y$ are adjacent if and only if $(x, y)^l \subseteq I$. If $I = \{0\}$, then the zero divisor graph is denoted by $G_{\{0\}}(P)$. Thus the Ideal based zero divisor graph with respect to the ideal $(0]$ is the zero divisor graph of the poset.

In this paper we introduce a simple graph on a finite poset $P$ with 0, called as the *atom based graph* of $P$ and denote it by $\Gamma_a(P)$.

The undefined terms related to graph theory are from West [10]. All the graphs are assumed to be simple and finite. Throughout in this paper $P$ denotes a finite poset with 0 as a consequence $P$ contains an atom. For two elements $a, b \in P$, $a < b$, we write $a \prec b$ if there is no $x \in P$ such that $a < x < b$.

## 2 Some results of $\Gamma_a(P)$

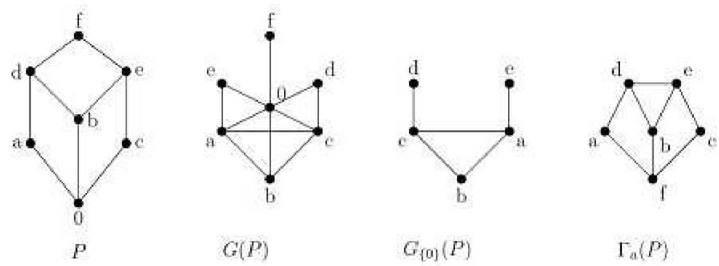
We denote the set of all atoms in $P$ by $\Omega(P)$. We define the *atom based graph* of $P$, denoted by $\Gamma_a(P)$ as follows.

**Definition 2.1.** Let $P$ be a finite poset with 0. We associate a simple graph with $P$, where the vertex set is $P \setminus \{0\}$ and two distinct elements $x, y \in P$ are adjacent if and only if the set of nonzero lower bounds of $x, y$ in $P$ is nonempty and contains atoms only. $i.e.$ $x^l \cap y^l \setminus \{0\} \neq \emptyset$ and $x^l \cap y^l \setminus \{0\} \subseteq \Omega(P)$. We denote this graph by $\Gamma_a(P)$ and call it the atom based graph on $P$.

If $a, b \in P$ are adjacent, then we write $a \leftrightarrow b$ and if $a, b \in P$ are nonadjacent, then we denote it by $a \not\leftrightarrow b$.

The following example shows that $\Gamma_a(P)$ is different from the zero divisor graph $G(P)$, defined by Halăs [3] and from the zero divisor graph $G_{\{0\}}(P)$ defined by Joshi [5].

**Example 2.2.**




Poset $P$ whose $\Gamma_a(P)$ is different from $G(P)$ and $G_{\{0\}}(P)$.
Figure 1

We note the following:

(i) If $a, b \in \Omega(P)$, then $a \not\leftrightarrow b$ in $\Gamma_a(P)$.

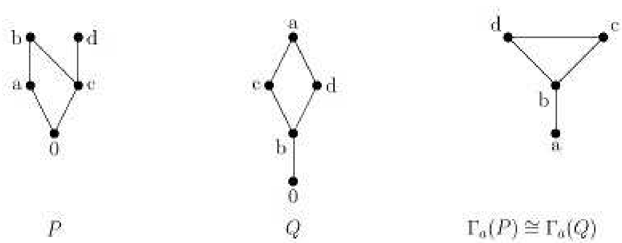
(ii) If $a \in \Omega(P)$, then for any $x \in a^u = \{x \mid x \in P \text{ and } a \leq x\}$, $a \leftrightarrow x$ and $a \not\leftrightarrow y$ for $y \notin a^u$.

(iii) For every $a \in \Omega(P)$, $deg(a) = |a^u| - 1$ in $\Gamma_a(P)$.

(iv) Two vertices $a, b$ are adjacent in $G(P)$ (Zero divisor graph considered by Halaš et. al.) or $G_{\{0\}}(P)$ implies $a, b$ are nonadjacent in $\Gamma_a(P)$.

(v) Let $P$, $P_1$ be posets with 0. If $P \cong P_1$, then $\Gamma_a(P) \cong \Gamma_a(P_1)$. The following example shows that the converse need not hold.

**Example 2.3.**




Nonisomorphic posets may have the same atom based graph.
Figure 2

Now we discuss the connectedness of $\Gamma_a(P)$.

**Lemma 2.4.** *Suppose that* $|\Omega(P)| = 1$. *Then* $\Gamma_a(P)$ *is connected and* $diam(\Gamma_a(P)) \leq 2$.

**Proof.** Let $x$ be the only atom in $P$. If $y \in P$, $y \neq x$, then $y \in x^u$. Hence $x \leftrightarrow y$ in $\Gamma_a(P)$. Thus $d(x, y) = 1$.

Let $y, z \in P$, $(y \neq x, z \neq x)$. If $y \not\leftrightarrow z$, then $y \leftrightarrow x \leftrightarrow z$ which implies that $d(y, z) = 2$. Hence $diam(\Gamma_a(P)) \leq 2$. □

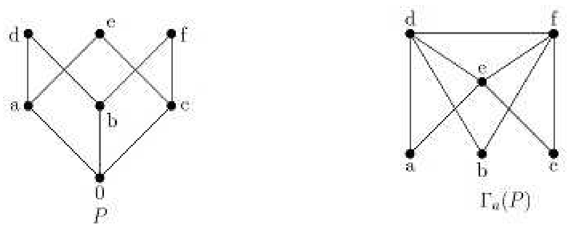
The following corollary follows from Lemma 2.4.

**Corollary 2.5.** *If a disconnected graph is realizable as $\Gamma_a(P)$ for some $P$, then $|\Omega(P)| \neq 1$.*

**Remark 2.6.** The Example 2.3 shows that the condition $a \prec x$ is necessary. The poset $Q$ shown in Example 2.3 contains only one atom. As $a \not\leftrightarrow d$, also $a \not\leftrightarrow c$. $\Gamma_a(Q)$ is not a complete graph.

**Remark 2.7.** In Lemma 2.4 we have noted that if $|\Omega(P)| = 1$, then $\Gamma_a(P)$ is connected and its diameter is at most 2. However, there exist posets with $|\Omega(P)| > 1$ whose atom based graph may be connected or may be disconnected.


A poset, whose $\Gamma_a(P)$ is disconnected.
Figure 3




A poset, whose $\Gamma_a(P)$ is connected.
Figure 4

In the following theorem we give a characterization for the connectedness of $\Gamma_a(P)$.

**Theorem 2.8.** *$\Gamma_a(P)$ is connected if and only if either (i) $(a, b)^u \neq \emptyset$ for all $a, b \in \Omega(P)$ or (ii) if $(a, b)^u = \emptyset$ for some $a, b \in \Omega(P)$, then there exist $p_1, p_2, \cdots, p_n \in \Omega(P)$ such that, (without loss of generality) $(a, p_1)^u \neq \emptyset$, $(p_i, p_j)^u \neq \emptyset$ (for $i \neq j$) and $(b, p_n)^u \neq \emptyset$.*

**Proof.** Suppose that $\Gamma_a(P)$ is connected. If $(a, b)^u \neq \emptyset$ for all $a, b \in \Omega(P)$, then nothing to prove.

Suppose that $(a, b)^u = \emptyset$ for some $a, b \in \Omega(P)$. Since $\Gamma_a(P)$ is connected, there exists a path from $a$ to $b$. Since $a, b \in \Omega(P)$, $a \leftrightarrow b$ is not possible. Let $a \leftrightarrow x_1 \leftrightarrow x_2 \leftrightarrow x_3 \leftrightarrow \cdots \leftrightarrow x_{n-1} \leftrightarrow x_n \leftrightarrow b$ be a path from $a$ to $b$. Therefore $x_i \wedge x_{i+1} = c_i$, for some $c_i \in \Omega(P)$. Clearly, $x_1 \in (a, c_1)^u$, $x_2 \in (c_1, c_2)^u$, $x_3 \in (c_2, c_3)^u$, $\cdots$, $x_n \in (c_n, b)^u$. Thus the claim.

Conversely, assume that either (i) $(a, b)^u = \emptyset$ for all $a, b \in \Omega(P)$ or (ii) if $(a, b)^u = \emptyset$ for some $a, b \in \Omega(P)$, then there exist $p_1, p_2, \cdots, p_n \in \Omega(P)$ such that $(a, p_1)^u \neq \emptyset$, $(p_i, p_j)^u \neq \emptyset$ (for $i \neq j$)and $(b, p_n)^u \neq \emptyset$.

To prove $\Gamma_a(P)$ is connected. Let $x, y \in \Gamma_a(P)$. Since $P$ is finite, there exist $p, q \in \Omega(P)$ such that $p \leq x$, $q \leq y$. Then $p \leftrightarrow x$, $q \leftrightarrow y$.

To show there is a path from $x$ to $y$, it is enough to show that there is a path between $p$ and $q$.

Suppose that $(a, b)^u \neq \emptyset$ for all $a, b \in \Omega(P)$. Then there exists $t \in (p, q)^u$. In this case $x \leftrightarrow p \leftrightarrow t \leftrightarrow q \leftrightarrow y$ and we are through.

Now assume that $(p, q)^u = \emptyset$. By assumption there exist $p_1, p_2, \cdots, p_n \in \Omega(P)$ such that $(p_2, p_j)^u \neq \emptyset$, $(p, p_1)^u \neq \emptyset$ and $(q, p_n)^u \neq \emptyset$.

Let $t_1 \in (p, p_1)^u$, $t_2 \in (p_1, p_2)^u$, $\cdots$, $t_n \in (p_{n-1}, p_n)^u$ and $t_{n+1} \in (q, p_n)^u$. Thus we have a path $x \leftrightarrow p \leftrightarrow t_1 \leftrightarrow p_1 \leftrightarrow t_2 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_{n-1} \leftrightarrow t_n \leftrightarrow p_n \leftrightarrow t_{n+1} \leftrightarrow q \leftrightarrow y$. Thus $\Gamma_a(P)$ is connected. □

**Definition 2.9.** For $x \in P$, the atom based annihilator of $x$, denoted by $Ann_{\Omega}(x)$, is defined as $Ann_{\Omega}(x) = \{y \mid x^l \cap y^l \setminus \{0\}(\neq \emptyset) \subseteq \Omega(P)\}$.

In the following lemma, we identify a situation where in $diam(\Gamma_a(P)) \leq 4$.

**Lemma 2.10.** *If for some $z \in P$, $\Omega(P) \subseteq Ann_{\Omega}(z)$. Then $\Gamma_a(P)$ is a connected graph and $diam(\Gamma_a(P)) \leq 4$.*

**Proof.** Suppose that for some $z \in P$, $\Omega(P) \subseteq Ann_{\Omega}(z)$. Let $d(x, y) > 4$ for some $x, y \in P$. Assume that $x \leftrightarrow a \leftrightarrow b \leftrightarrow c \leftrightarrow d \leftrightarrow y$ in $\Gamma_a(P)$.
Now $x \leftrightarrow a$ implies that either (1) $x$ or $a$ is an atom or (2) $x, a \notin \Omega(P)$. Also $d \leftrightarrow y$ implies that either (1) $d$ or $y$ is an atom or (2) $d, y \notin \Omega(P)$. Without loss of generality, we assume that there exist two atoms say $p_1, p_2 \in P$ with $x \in p_1^u$ and $y \in p_2^u$. As $\Omega(P) \subseteq Ann_{\Omega}(z)$ implies that $z \in p_1^u \cap p_2^u$. This implies that $x \leftrightarrow p_1 \leftrightarrow z \leftrightarrow p_2 \leftrightarrow y$, a contradiction. Therefore, $d(x, y) = 4$. Thus $\Gamma_a(P)$ is always connected and $diam(\Gamma_a(P)) \leq 4$. □

**Lemma 2.11.** *If $P$ contains a unique atom, then $\Gamma_a(P)$ is a complete graph. The converse holds if each $x \in P$ $x \neq a$ satisfies $a \prec x$.*

**Proof.** Let $P$, be a poset with a unique atom, say $a$. Let $x, y \in P$,$x \neq a$, $y \neq a$. By assumption, $a \prec x$ and $a \prec y$. Hence the set of nonzero lower bounds of $x$ and $y$ in $P$ contains $a$. This implies $x \leftrightarrow y$ in $\Gamma_a(P)$. By uniqueness of $a$, $a \leftrightarrow x$ for every $x \in P$, $x \neq a$. Hence $\Gamma_a(P)$ is a complete graph.

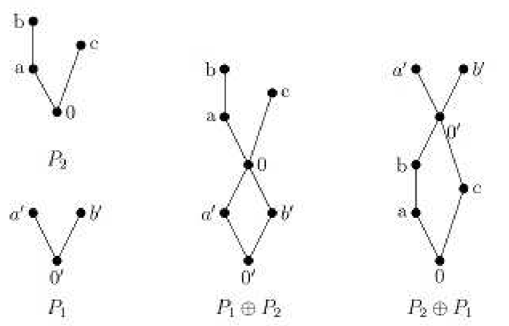
Conversely, assume that $\Gamma_a(P)$ is a complete graph. We have already noted that if $x, y \in \Omega(P)$ then $x \not\leftrightarrow y$ in $\Gamma_a(P)$ hence $|\Omega(P)| = 1$. Moreover, if $a \in \Omega(P)$, then for all $x \in P$, $x \in a^u$. If for some $y \in a^u$, $a \not\prec y$ then there exists $z \in P$ such that $a < z < y$, this implies that $a \leftrightarrow z$, $a \leftrightarrow y$ but $y \not\leftrightarrow z$ in $\Gamma_a(P)$, a contradiction to the completeness of $\Gamma_a(P)$. Hence $a \prec x$ for every $x \in P$, $x \notin \Omega(P)$. □

Now we define the linear sum of two posets as follows,

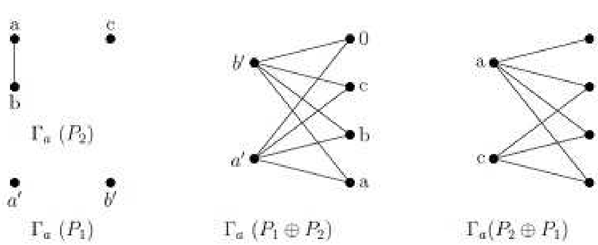
**Definition 2.12.** Let $P_1$ and $P_2$ be finite posets with 0. The linear sum of $P_1$ and $P_2$, denoted by $P_1 \oplus P_2$, is obtained by placing the diagram of $P_1$ directly below the diagram of $P_2$ and adding line segments from all the maximal elements of $P_1$ to the zero element of $P_2$.

We note that: (1) $P_1 \oplus P_2$ is a poset with 0. (2) $\Omega(P) = \Omega(P_1)$.

**Example 2.13.**




Linear sum of two Posets
Figure 5



Atom based graphs of $P_1$, $P_2$ and $P_1 \oplus P_2$ and $P_2 \oplus P_1$.
Figure 6

From the above examples, we note that $\Gamma_a(P_1 \oplus P_2)$ is a complete bipartite graph whereas $\Gamma_a(P_2 \oplus P_1)$ is a bipartite graph.

Moreover, $\Gamma_a(P_1 \oplus P_2)$ is a complete bipartite graph with six 4-cycles namely,

(1) $a' \leftrightarrow 0 \leftrightarrow b' \leftrightarrow c \leftrightarrow a'$ (2) $a' \leftrightarrow 0 \leftrightarrow b' \leftrightarrow b \leftrightarrow a'$
(3) $a' \leftrightarrow 0 \leftrightarrow b' \leftrightarrow a \leftrightarrow a'$ (4) $a' \leftrightarrow a \leftrightarrow b' \leftrightarrow c \leftrightarrow a'$
(5) $a' \leftrightarrow b \leftrightarrow b' \leftrightarrow c \leftrightarrow a'$ (6) $a' \leftrightarrow a \leftrightarrow b' \leftrightarrow b \leftrightarrow a'$

which are formed by 2 atoms $a'$, $b'$ and 4 nonatoms $a, b, c, 0$ of $P_1 \oplus P_2$ in $\binom{4}{2}\binom{2}{2}$ ways as shown in above cycles. We use this in the following characterization for a complete bipartite graph.

**Theorem 2.14.** *The complete bipartite graph $K_{m,n}$ is a $\Gamma_a(P)$ for some finite poset $P$ with 0 if and only if $P = Q_1 \oplus Q_2$, where $|Q_1| = m + 1$ and $Q_1$ is a finite poset with $m$ atoms and $Q_2$ is a finite poset with 0 and $|Q_2| = n$. Moreover, $\Gamma_a(P)$ contains $\binom{m}{2}\binom{n}{2}$ number of 4 cycles.*

**Proof.** Assume that $K_{m,n} = \Gamma_a(P)$, $m, n \geq 2$ for some finite poset $P$ with 0. Let $P_1$ and $P_2$ be the two partite sets of $K_{m,n}$. Since no two atoms are adjacent in $\Gamma_a(P)$ we may assume that $P_1 = \Omega(P)$. Hence the other partite set $P_2 \subseteq \{x | x \notin \Omega(P)\}$. Since every $x \notin \Omega(P)$ is adjacent to some $a \in \Omega(P)$ implies $x \in P_2$. Therefore $\{x | x \notin \Omega(P)\} \subseteq P_2$ and so $\{x | x \notin \Omega(P)\} = P_2$. Since $P_2$ is a partite set, no two $x, y \in P_2$ are adjacent to each other. Now we have for each $x \in P_2$ and for each $a \in P_1$, $x \leftrightarrow a$, which implies that $x \in a_i^u$, for every $a_i \in P_1$, therefore, $x \in \cap a_i^u$. Let $Q_1 = 0 \cup \Omega(P)$, and $Q_2 = P_2$.

We claim that $P_2$ is a finite poset containing 0. For instance, if $Q_2$ does not contain 0, then for $x, y \in P_2, x, y \in a_i^u, a_i \in P_1$, implies that $x^l \cap y^l \setminus \{0\} (\neq \emptyset) \subseteq \Omega(P)$. Therefore $x \leftrightarrow y \in \Gamma_a(P)$, a contradiction, as $P_2$ is partite set. Hence we write $P = Q_1 \oplus Q_2$.

Conversely, assume that $P$ is a finite poset and $P = Q_1 \oplus Q_2$, where $Q_1$ and $Q_2$ are finite posets with 0, $|Q_2| = n$, $|Q_1| = m + 1$ and $|\Omega(Q_1)| = m$.

We note that $x \in Q_2$ implies $x \in a_i^u$ for every $a_i \in \Omega(P)$ as $x \leftrightarrow a_i$. Since no two atoms are adjacent in $\Gamma_a(P)$, we take $\Omega(P) = \Omega(Q_1)$ as one partite set say $P_1$ and considering other the partite set $P_2 = \{x \mid x \notin \Omega(P)\} = Q_2$.

We claim that $x \not\leftrightarrow y$ if $x, y \in P_1$ *or* $x, y \in P_2$. If $x \leftrightarrow y$ for some $x, y \in P$ then clearly $x, y \notin P_1$ as no two atoms can be adjacent. For $x, y \in P_2$, $x^l \cap y^l \setminus \{0\} \nsubseteq \Omega(P)$ hence $x \not\leftrightarrow y$ therefore $P_2 = Q_2$ will be the other partite set.

Hence $\Gamma_a(P)$ is a complete bipartite graph $K_{m,n}$ where $m = |\Omega(P)|$ and $|Q| = n$.

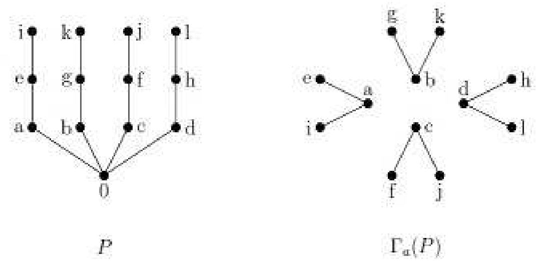
As $\Gamma_a(P)$ is a complete bipartite graph with $|P_1| = m$, $|P_2| = n$, a cycle of length 4 that is $K_{2,2}$ can be formed from $K_{m,n}$ in $\binom{m}{2}\binom{n}{2}$ many ways. Hence $\Gamma_a(P)$ contains $\binom{m}{2}\binom{n}{2}$ number of 4 cycles. □

Since a star graph is a particular case of a complete bipartite graph, we have the following theorem.

**Definition 2.15.** The lattice $M_n$ is defined as a set $\{a_1, ...., a_n, 0, 1\}$, with $0 < a_i < 1$ and $a_i \parallel a_j$ for $i \neq j$.

**Theorem 2.16.** $\Gamma_a(P)$ *is a star graph if and only if either (1) poset $P$ is $M_n$ or (2) $P$ is a poset of the type $C_2 \oplus P_1$, where $C_2$ is the two element chain and $P_1$ is a poset.*

**Example 2.17.**




Poset whose $\Gamma_a(P)$ is disconnected.
Figure 7

What happens if each atom is covered by unique and distinct element? See the following Example 2.17. Hence we have following lemma.

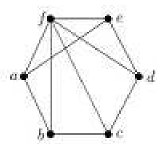
**Lemma 2.18.** *If $P$ is a poset consisting of the chains $C_i$, $1 \leq i \leq n$, such that $C_i \cap C_j = \{0\}$. Then $\Gamma_a(P)$ is a disconnected graph with $n$ number of star components.*

**Proof.** If $P$ is a poset consisting of the chains $C_i$, $1 \leq i \leq n$, such that $C_i \cap C_j = \{0\}$. Clearly for each $a \in \Omega(P)$, there is only one chain $C_i$ of $P$ such that $a \in C_i$. Clearly $\Gamma_a(C_i)$ is a star graphs with $a$ as the centre.
There does not exist any path between $x_i, x_j \in P$ if $x \in C_i$ and $y \in C_j$, $i \neq j$. As $C_i \cap C_j = \{0\}$ and $a \nleftrightarrow b$, if $a, b \in \Omega(P)$, also no element $x \in C_i$ can be adjacent to any $y \in C_j$, $i \neq j$. Hence $\Gamma_a(P)$, $1 \leq i \leq n$ is a disconnected graph containing $n$ number of star components. □

Beck[1] conjectured that for a commutative ring $\chi(R) = clique(R)$. For the graph in Figure 2.19, we note that $\chi(G) = 4$ and $clique(G) = 3$. In the following theorem we show that this graph is not realizable as $\Gamma_a(P)$ for a finite poset $P$ with 0.

We know that a subset $C$ of a graph $G$ is a clique if any two distinct vertices of $C$ are adjacent. If $G$ contains a clique with $n$ elements and every clique has at most $n$ elements then the clique number of $G$ is $\omega(G) = n$. If the size of the clique are not bounded, then $\omega(G) = \infty$. The minimum number of colors needed to color the vertices of a graph such that no two adjacent vertices have the same color is called the chromatic number of $G$. It is denoted by $\chi(G)$.

**Example 2.19.** In the following example $\chi(G) = 4$ and $clique(G) = 3$, $i.e.$ $\chi(G) \neq clique(G)$.




Graph with $\chi(G) \neq clique(G)$.
Figure 8

**Theorem 2.20.** *The graph shown in Example 2.19 cannot be a component of the atom based graph of any finite poset with* 0.

**Proof.** Suppose that the graph shown in Example 2.19 is a component of the atom based graph of a finite poset $P$ with 0.

As no two atoms are adjacent in $\Gamma_a(P)$. Hence $|\Omega(P)| < 3$. Therefore, either $(i)$ $|\Omega(P)| = 1$ or $(ii)$ $|\Omega(P)| = 2$
$(i)$ Let $|\Omega(P)| = 1$. Let $x \in \Omega(P)$. Clearly if $y \in P, y \neq x$, then $y \in x^u$ and so $x \leftrightarrow y$. From the graph $G$ in Example 2.19, we note that $f$ is adjacent to all other elements, hence $f \in \Omega(P)$.

As $a \leftrightarrow b$, $b \leftrightarrow c$, $c \leftrightarrow d$, $d \leftrightarrow e$, $a \leftrightarrow e$ in the graph $G$ shown in Example 2.19, so by the definition of the atom based graph each of $a^l \cap b^l \setminus \{0\}$, $b^l \cap c^l \setminus \{0\}$, $c^l \cap d^l \setminus \{0\}$, $d^l \cap e^l \setminus \{0\}$, $a^l \cap e^l \setminus \{0\}$ contains $f$ only. Since $a \nleftrightarrow c$ ,$a \nleftrightarrow d$, $b \nleftrightarrow d$, $b \nleftrightarrow e$, $c \nleftrightarrow e$ in $G$. So by the definition of the atom based graph each of $a^l \cap c^l \setminus \{0\}$, $a^l \cap d^l \setminus \{0\}$, $b^l \cap d^l \setminus \{0\}$ $b^l \cap e^l \setminus \{0\}$, $c^l \cap e^l \setminus \{0\}$ contain $f$ and some other nonzero element.
As $a \nleftrightarrow d$ hence there exists some $x \in P$ with $x \neq f, x \neq 0$ such that $x \in a^l \cap d^l \setminus \{0\}$. We have the following possibilities:-
(1) $x = a$ or (2) $x = b$ or (3) $x = c$ or (4) $x = d$ or (5) $x = e$
(1) If $x = a$, then $a^l \subseteq d^l$. Hence $a^l \cap c^l \subseteq c^l \cap d^l = \{0, f\}$, a contradiction to $a \nleftrightarrow c$.
(2) If $x = b$, then $b \in a^l \cap d^l$. Hence $b^l \subseteq a^l \Rightarrow b \in a^l \cap b^l = \{0, f\}$, a contradiction.
(3) If $x = c$, then $c \in a^l \cap d^l$. Hence $c^l \subseteq d^l \Rightarrow c \in c^l \cap d^l = \{0, f\}$, a contradiction.
(4) If $x = d$, then $d^l \subseteq a^l$. Hence $d^l \cap b^l \subseteq a^l \cap b^l = \{0, f\}$, a contradiction to $b \nleftrightarrow d$.
(5) If $x = e$, then $e \in a^l \cap d^l$. Hence $e^l \subseteq a^l$. As $a \leftrightarrow e$, therefore $e^l \cap a^l = \{0, f\}$. Together imply that $e = 0$ or $e = f$.

Thus, no such $x$ exists. Hence the graph shown in Example 2.19 cannot be realizable as $\Gamma_a(P)$ for some finite poset $P$ with $|\Omega(P)| = 1$.
$(ii)$ Suppose that $|\Omega(P)| = 2$ since no two atoms can be adjacent then either (1) $a, d \in \Omega(P)$

or (2) $b, e \in \Omega(P)$ or (3) $a, c \in \Omega(P)$ or (4) $e, c \in \Omega(P)$ or (5) $b, d \in \Omega(P)$.
(1) If $a, d \in \Omega(P)$, then using $a \leftrightarrow b \leftrightarrow c \leftrightarrow d$ we conclude that $b \in a^u$, $c \in d^u$ and $b \notin d^u, c \notin a^u$. But $b \leftrightarrow c$ implies $b^l \cap c^l \setminus \{0\}$ is nonempty and contains atoms only. From $b \leftrightarrow f \leftrightarrow c \leftrightarrow b$ we conclude that $f \in \Omega(P)$, a contradiction.
Similarly, we get a contradiction in the other cases also Thus the graph shown in Example 2.19 cannot be realizable as $\Gamma_a(P)$ for some finite poset $P$ with $|\Omega(P)| = 2$. □
Thus we state the following conjecture.
**Conjecture**: If $P$ is a finite poset with 0, is $\chi(\Gamma_a(P)) = clique(\Gamma_a(P))$?(i.e. wether Becks conjecture holds for the atom based graph of a poset.)

Now, we give a necessary and sufficient condition for $\Gamma_a(P)$ to be triangle free.

**Theorem 2.21.** *For a poset $P$, the following statements are equivalent*
(1) *If $x \in P - \Omega(P)$ there does not exist $y \in P - \Omega(P)$ such a that $x \leftrightarrow y$.*
(2) *$\Gamma_a(P)$ is a triangle free graph.*
(3) *$\Gamma_a(P)$ is a bipartite graph with $\Omega(P)$ as one partite set and and the set of non atoms of $P$ as the other partite set.*

*Proof.* (1) $\Rightarrow$ (2): Suppose that $\Gamma_a(P)$ contains a triangle. We note that if $a, b \in \Omega(P)$, then $a \not\leftrightarrow b$ in $\Gamma_a(P)$. Therefore, a triangle contains only one atom. Hence the remaining two elements are non atoms which are adjacent, a contradiction to assumption.
(2) $\Rightarrow$ (3) We know that a graph $G$ is bipartite if and only if it does not contain a cycle of odd length. ( West [10] Theorem 1.2.18 Page.25 ). If $\Gamma_a(P)$ is not a bipartite graph, then $\Gamma_a(P)$ contains an odd cycle. Let $x_0 \leftrightarrow x_1 \leftrightarrow ... \leftrightarrow x_{2n} \leftrightarrow x_0$ be a cycle of length $2n+1$ for some $n \geq 1$. Since no two atoms are adjacent in $\Gamma_a(P)$, hence any cycle of length $2n+1$ contains at most $n$ atoms. Thus there exists an edge between vertices which are non atoms say $x_0 \leftrightarrow x_1$. Then $x_0, x_1 \in a^u$ for some $a \in \Omega(P)$. Then $x_0 \leftrightarrow a \leftrightarrow x_1 \leftrightarrow x_0$ form a cycle of length 3, a contradiction. Thus $\Gamma_a(P)$ is a bipartite graph, with $\Omega(P)$ as one partite set and the set of non atoms of $P$ as the other partite set.
(3) $\Rightarrow$ (1) If $\Gamma_a(P)$ is a bipartite graph with $\Omega(P)$ as one partite set and the set non atoms of $P$ as the other partite set, then for $x \in P - \Omega(P)$ there does not exist $y \in P - \Omega(P)$ such a that $x \leftrightarrow y$. □

**Acknowledgemetns:** The authors are thankful to Prof. Vinayak Joshi for his help in simplifying the proofs of some results. The authors are grateful to the referee for helpful suggestions, which improved the presentation.

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