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On Multiple Primitive Pythagorean Triplets

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Abstract. This paper revisits the topic of Pythagorean triples with a different perspective. While numerous methods were explored to generate Pythagorean triples, none of them is whole in terms of producing all of the triples without repetitions. Indeed, many current methods focus on producing primitive triples but there does not exist multiple triples.We explores a new formation of Primitive Pythagorean triple for an even and odd shorter leg, which helps to investigate multiple primitive triples and some conjectures based on graphical representation. We genuinely hope researchers, students, and instructors of mathematics will enjoy this article and search for new directions.

1 Introduction

The theory of primitive Pythagorean triples (PPTs) is related to other areas of number theory such as Gaussian numbers, modular forms [18], and spinors [19]. A triplet (a, b, c), with a, b, and c some natural numbers where c is the hypotenuse is said to be Pythagorean triplets if it satisfies the condition:

$$a^2 + b^2 = c^2$$

Definition 1.1. We say a triplet (a, b, c), with a, b, and c some natural numbers with satisfying the following properties is known as primitive triplets or primitive Pythagorean triplets:
(1) G.C.D. of a and $b = 1$, Symbolically $gcd(a, b) = 1$,
(2) $a < b$ and
(3) $a^2 + b^2 = c^2$.

These properties will encourage the differentiation of a primitive triplet from a triplet. Once the fundamental property existing between the integers a, b, and c is known, there has remained, as it is known from historical facts, many ways and supportive ideas to search for triplets following some mathematical recurrence for odd and even integers. Primitive Pythagorean triples (PPTs) have potential applications to cryptography [20].

One might also additionally examine programs of PPTs to cryptography springing from the fact that every triple is a set of three connected numbers as (a, b, c). If those numbers are sufficiently large, then they can be used as keys which can be dispensed with the aid of using the Certification Authority (CA), who sends the numbers a and b one by one to Alice and Bob, for use as seeds to generate the key. The CA will preserve the third number c for audit purposes. In this article, we are more interested in developing the strict triple of the generation of Multiple PPTs.

Frink [1] presented almost Pythagorean triplets, and Prince et al. [2] explored new form of Pythagorean triples. In 1982, [3], Bernstein has presented idea about Primitive Pythagorean triplets.Many researchers has worked on Pythagorean triplets and their conjectures (See [4, 5, 6, 7, 8]). Some of them which will be cleared in this article. We notice from the different articles, that different form of Pythagorean triples has been implemented, analyses with their statistics of triples by many researchers [9, 10, 11, 17, 21, 25]. Bhanotar et.al [12], discovered a new interesting algebraic approach and extended the field $Q^+ \cup \{0\}$ to explore dual of given triplets with some important sequences and inter-connectivity. We see some note, algebraic property for primitive triples and its association with the generalization form has been presented in [13, 14, 15]. The solution of the Boolean Pythagorean Triples problem has been discussed by peter et.al in [22]. Billiards on Pythagorean triples and their Minkowski functions is presented in [25]. Agrawal [16] mentioned the history of Pythagorean triples before and after Pythagoras. Many mathematicians and experts in other branches have developed and communicated different aspects to express their thoughts about the construction, characterization and geometrical representation of Primitive Pythagorean triples in [21, 23, 24]. In this article, we explore it to outmatch in the new form and brief discussion about some important notion MPPT (Multiple Primitive Pythagorean Triplets). In addition to this, we leave it to the reader's discretion to compare this result with that of Euclid's formula wherein we use any one of the results. Let's have look at Euclid's introductory notion as follows[15]:

(i) $a = 2pq, b = p^2 - q^2$, and $c = p^2 + q^2$ with $gcd(p, q) = 1$, where $p > q$. In this structure, the triple starts with a positive even integer. As a condition of triple for which $a < b < c$, we should have,
$$2pq < p^2 - q^2 \Rightarrow (p + q)^2 < 2p^2,$$
which violate for $p = 2$, and $q = 1$.

(ii) $a = p^2 - q^2, b = 2pq$, and $c = p^2 + q^2$ with $gcd(p, q) = 1$, where $p > q$. In this structure, we argue to keep p as a positive odd integer and q as an even positive integer and that can motivate a to be an odd positive integer and the term $b = 2pq > a$ with $c = p^2 + q^2$ will develop the triplet an odd Pythagorean triplet.

With Euclid's formula, to find triplets we need to find p and q. In addition to this, we are unable to say, how many PPTs (Pythagorean Primitive Triplets) are there for the same positive integer 'a'?. In this article, We propose a new method to express multiple Primitive Pythagorean triples for an odd and an even series.

The main contribution of this article is as follows. In Section 2 , We begin by extending the results to present the generation of primitive Pythagorean triples in such a way one can obtain as odd and even Pythagorean triplets. In Section 3 , The different forms of Pythagorean triples according to the position of the shorter leg are presented. We investigate MPTs (multiple primitive triples) and their graphical representation based on its statistical have been mentioned in section-4. Section 5, is devoted to the conclusion with some open questions.

2 The Generation of PPTs (Primitive Pythagorean Triplets)

In this section, we develop a new generalized form of odd and even PPTs along with some important results and examples. We implement some notations to distinguish triplets as follows: The set of natural numbers $\mathbb{N}$ is divided into two sets $\mathbb{N}_1$ and $\mathbb{N}_2$ in such a way, $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2$ and $\mathbb{N}_1 \cap \mathbb{N}_2 = \emptyset$. where $\mathbb{N}_1$ is the set of odd positive integers, and $\mathbb{N}_2$ is the set of even positive integers.

2.1 For $\mathbb{N}_1$ - the set of odd positive integers

Let a belong to $\mathbb{N}_1 - \{1\}$ be a given positive integer, then for some $j \in \mathbb{N}_1$. We have integers a, $b = \frac{a^2-j^2}{2j}, c = \frac{a^2+j^2}{2j}$, which satisfies $a^2 + b^2 = c^2$. The generated triple (a, b, c) is called a Pythagorean triplet. One can notice, $c = b + j$, We call such a triple (a, b, c) as an odd triplet for which the smallest of all three integers is an odd integer. To establish the condition for j. We have the basic condition for strictly Pythagorean triple $c > b > a$. And hence we can write,

$$\frac{a^2 + j^2}{2j} > \frac{a^2 - j^2}{2j} > a. \tag{2.1}$$

From the above-said relation (2.1), we have, $\frac{a^2-j^2}{2j} > a$; which conveys that,

$$(a - j)^2 > 2(j)^2 \Leftrightarrow a - j > (\sqrt{2})j \text{ and } a - j < -(\sqrt{2})j.$$

The only possible relation is,

$$a - j > (\sqrt{2})j \Leftrightarrow a > (\sqrt{2} + 1)j \Leftrightarrow j < a(\sqrt{2} - 1). \tag{2.2}$$

This gives feasible values for j and according to j we can have additional Pythagorean triplets if triple satisfies the condition, $a^2 + b^2 = c^2$.

Definition 2.1. OPPT (Odd primitive Pythagorean triplet): For some natural numbers a, b, and c such that a triple (a, b, c), is said to be an OPPT if it satisfies the conditions of Definition 1.1 for positive odd integer a.

The examples of Odd PPTs are $(3, 4, 5), (5, 12, 17), (7, 24, 25), \ldots$

The prime condition for the OPPT is, $j < a(\sqrt{2} - 1)$ must be an odd integer, to get evenness of $b = \frac{a^2 - j^2}{2j}$.

For instance, for $a = 5$, we have a feasible value of odd positive integer $j = 1$, the integer $b = \frac{a^2 - j^2}{2j} = 12$, and the next hypotenuse is, $c = \frac{a^2 + j^2}{2j} = b + j = 13$. The generated triple $(5, 12, 13)$ satisfy the conditions of Definition 1.1. That is, $5^2 + 12^2 = 13^2$, which convey the triple $(5, 12, 13)$ is an OPPT.

Theorem 2.2. *For some choice of odd integer a and $j \in \mathbb{N}_1$, satisfying, $j < a(\sqrt{2} - 1)$. The integer $b = \frac{a^2 - j^2}{2j}$ is always an even integer.*

Proof. Let a be of the form, $2k + 1$, for some $k \in \mathbb{N}$ and j is of the form, $2m + 1$, for some $m \in N$, which satisfy the condition, $j < a(\sqrt{2} - 1)$. Then,

$$\begin{aligned} b = \frac{a^2 - j^2}{2j} &= \frac{(2k+1)^2 - (2m+1)^2}{2(2m+1)} \\ &= \frac{(2k - 2m)(2k + 2m + 2)}{2(2m+1)} \\ &= \frac{2(k-m)(k+m+1)}{(2m+1)} \\ &= 2\omega. \text{ where } \omega = \frac{(k-m)(k+m+1)}{(2m+1)} \in \mathbb{N} \text{ for some } k \text{ and } m \end{aligned}$$

To get PPT, we remember that $\gcd(a, b) = 1$. Here b is an even integer, as number ω is a multiple of an even integer. Thus, for odd integers, a and j belongs to $\mathbb{N}_1$ satisfying the condition, $j < a(\sqrt{2} - 1)$. We get always b an even integer.This complete the proof. □

One can notice that, there are infinitely many odd integers that contains more than one OPPTs.

2.2 For $\mathbb{N}_2$ - the set of even positive integers

Let $a \in \mathbb{N}_2 - \{2\}$ be an even positive integer, then if we find $j \in \mathbb{N}_2$ such that the triplet of positive integers $\left(a, b = \frac{a^2 - j^2}{2j}, c = \frac{a^2 + j^2}{2i}\right)$, satisfy the condition, $a^2 + b^2 = c^2$. Moreover, one can notice, $c = b + j$. We call such a triple (a, b, c) is an even Pythagorean triplet, for which the smallest of all three integers is an even integer.

We can obtain the same condition as obtained in eq. (2.2), $j < a(\sqrt{2} - 1)$, for some $j \in \mathbb{N}_2$, which helps to get an even Pythagorean triplet.

The following table-1 shows, the possible multiple Pythagorean triplets for a given number $a = 12$. In which you can see the second triple, $(12, 16, 20)$ is the multiple of another triple $(4(3, 4, 5))$, which does not give a primitive triplet. In detail, we will discuss in section 4.

a	$j < a(\sqrt{2}-1)$, for some $j \in \mathbb{N}_2$	$b = \frac{a^2-j^2}{2j}$	$c = \frac{a^2+j^2}{2j} = b + j$	(a, b, c)	Remarks
12	2	35	37	$(12, 35, 37)$	PPT
	4	16	20	$(12, 16, 20)$	$4(3, 4, 5)$-Not PPT

Table-1 Multiple Pythagorean triplets

Definition 2.3. EPPT (Even Pythagorean primitive triplet): For some natural numbers a, b, and c such that a triple (a, b, c), is said to be an EPPT if, for positive even integer a, it satisfies the conditions of Definition 1.1.

The examples of EPPT are $(8, 15, 17), (12, 35, 37), \ldots\ldots$

One can notice from the above table-1, there is only one Pythagorean primitive triple in which $\gcd(a, b) = \gcd(12, 35) = 1$

Theorem 2.4. *For some choice of even integer a and $j \in \mathbb{N}_2$, satisfying, $j < a(\sqrt{2}-1)$. The integer, $b = \frac{a^2-j^2}{2j}$ is always an odd integer.*

Proof: The proof can be easily verified as presented in the above Theorem: 2.2 (To get PPT, we remember that $\gcd(a, b) = 1$). □

Now, for the class of PPT $(a, b, c), \gcd(a, b) = 1$, where $a < b$. we wish to develop PPT for some positive integer, $a, b = \frac{a^2-j^2}{2j}, c = \frac{a^2+j^2}{2j}$ with $j \in \mathbb{N}_1$ or $\mathbb{N}_2$, as the case may be varied for $a \in \mathbb{N}_1 - \{1\}$ or $\mathbb{N}_2 - \{2, 4\}$. we have come across some known facts and obvious results as follows.

(i) In the PPT, we conclude that if a is an odd or even positive integer, there exist the corresponding b is an even or odd positive integer.

(ii) The hypotenuse c is always an odd prime integer of the form, $\{4n + 5 \mid n \in \mathbb{N}\}$. Also, it is fascinating to note that exactly one of a or b is divisible by 4 .

(iii) It is also noted for PPT that, at least one of the positive integers a, b and c is a prime number or divisible by 5 .

Theorem 2.5. *For a triple (a, b, c), If $a \in \{\{1, 4, \}U\{2(2n-3) \mid n \in \mathbb{N}, n \geq 2\}$, never possess PPT.*

Proof: For $a = 1$ and $i = 1$, we have, $b = \frac{a^2-i^2}{2i} = \frac{1^2-1^2}{2} = 0$, which is not possible as $a < b$. And for $a = 2$, we take $i = 2$ then $b = \frac{a^2-i^2}{2i} = \frac{2^2-2^2}{4} = 0$, which is again a contradiction to the fact that, $a < b$.

If $a \in \{2(2n-3) \mid n \in \mathbb{N}, n \geq 2\}$ then a is an even positive integer. And hence, according to the fact to be PPT, we must have, $b = \frac{a^2-j^2}{2j}$ an odd positive integer along with the condition satisfying, $j < a(\sqrt{2}-1)$ such that $j \in \mathbb{N}_2$.

For $n \geq 2, n \in \mathbb{N}, b = \frac{4(2n-3)^2-4}{2(2)}$, gives $b = (2n-3)^2 - 1$, which is always an even positive integer. This violates the fact for the existence of Primitive Pythagorean triples, if for a is an even integer, then b must be an odd integer.

Therefore, even an integer is of the form $\{2(2n-3) \mid n \in \mathbb{N}, n \geq 2\}$ never possess a PPT. Thus, if the smallest integer a belongs to the set $\{1, 4, \}U\{2(2n-3) \mid n \in \mathbb{N}, n \geq 2\}$, never possess PPT. This completes the proof. □

In the shade of what we have discussed about the construction of Pythagorean triplets, we politely submit it to the reader's discretion to compare our views with those of Euclid's formula and justify mathematical generality.

In the next section, we present the position of a (shorter leg) in the different forms of a triple. Also, we find the upper limit of j, which is depending upon the odd or even value of a, which helps to find the multiple primitive Pythagorean triplets.

3 Position of a (shorter leg) in Different Forms of Triplets

Now, at this stage, it is highly necessary to explore the position of a the shorter leg to be in the different position of the three constituents of the triplet (a, b, c). Let a be an integer such that $a \in \mathbb{N} - \{1, 2, 4\}$. The following table-2 gives the three different position of a in a triple.

Case no	a	b	c
1	a	$\frac{a^2-j^2}{2j}$	$\frac{a^2+j^2}{2j}$
2	$\sqrt{j(2a+j)}$	a	$a+j$
3	$a-j$	$\sqrt{j(2a-j)}$	a

Table - 2 Different position of a in the Triplets

Case-1: For some positive integer $a, b = \frac{a^2-j^2}{2j}, c = \frac{a^2+j^2}{2j} = b + j$, with $gcd(a, b) = 1$. We take for some choice of $a \in \mathbb{N}$ such that $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2, \mathbb{N}_1 \cap \mathbb{N}_2 = \emptyset$, and $j \in \mathbb{N}_1$ or $j \in \mathbb{N}_2$. This bifurcation will distinguish to derive oddly and even PPTs, depending upon oddness or evenness of a, and j. There is also a remarkable fact, the hypotenuse remains, $c = b + j$. If for a given positive integer a, and if it does not meet in the formation of PPT then it cannot be a part of the first case of the triplet. If we observe for $a = 1$ - an odd integer and taking $j = 1$, we obtain $b = 0$, and $c = b+ j = 1$. Thus, we have a triple $(1, 0, 1)$, which cannot be a part of PPT as it violates the condition $a < b$. If we think for $a = 2$, an even integer and taking $j = 2$, we get $b = 0$, and $c = b + j = 2$, and this helps us to conclude that there cannot be a primitive Pythagorean triplet as a violation of condition, $a < b$.

Case-2: We discuss the case when a takes a middle position in a triple. It means that the value of a as the greater one of the two legs, in such a way the triplets will take up the form as follows,

$$\left(\sqrt{j(2a+j)}, a, a+j\right); \sqrt{j(2a+j)} < a < a+j. \tag{3.1}$$

It is Importance to obtain whether, for a given integer value of a triple, the smaller leg must have existed. The fair point to be noted, if a is an even integer, then the value of j must be an odd positive integer or if a is an odd integer, the value of j must be an even positive integer. And, the hypotenuse, $c = a + j$ remains the same as we have discussed in the above case-1.

Let us consider two examples to understand the same. Letting the middle one position $a = 4$, and $j = 1$ then according to the case-2, the first place is $\sqrt{j(2a+j)} = \sqrt{1(2(4)+1)} = 3$, and the hypotenuse, $c = a + j = 5$. Therefore, for a given value $a = 4$ of a leg of right triangles, there exist a smaller leg with its length 3 and the largest one hypotenuse is $a + j = 5$. Thus, we have the triple $(3, 4, 5)$ is a primitive Pythagorean triplet.

Now, for $a = 5$, according to this positive value of a there does not exist the integer value of j, for which the first leg, $\sqrt{j(2a+j)}$ an integer smaller than $a = 5$. In this case, where the selected integer is an occupant of the second position, we have the reversal in the pattern and consider the value of j quite opposite to the nature of a in respect to oddness or evenness.

To conclude, we can find the upper limit of j, depending upon the odd or even value of a. Therefore, we have,

$$\sqrt{j(2a+j)} < a$$
$$\Rightarrow j(2a+j) < a^2$$
$$\Rightarrow 0 < a^2 - 2ja - j^2$$
$$\Rightarrow a^2 - 2ja + j^2 > 2j^2$$
$$\Rightarrow (a-j)^2 > 2j^2.$$

From this inequality, the only possible concerned inequality is,

$$a - j > (\sqrt{2})j \Leftrightarrow a > (\sqrt{2}+1)j \Leftrightarrow j < a(\sqrt{2}-1) \tag{3.2}$$

As $\sqrt{2}-1 < 1$, it convey that $j < a$ for which $j \in \mathbb{N}$. It can be approximated by assuming $j < \frac{a}{2}$.

Case-3: We consider the case when the given integral value of a deal as third component (hypotenuse) of triple or hypotenuse of a right triangle in such a way, the triple takes up the form as

$$\left(a-j, \sqrt{j(2a-j)}, a\right); a-j < \sqrt{j(2a-j)} < a \tag{3.3}$$

It is a noteworthy point that, if a is an odd positive integer then j is necessarily an even positive integer, and such a value of a and j produces an odd Pythagorean Primitive triplet. In case, if a is an even positive integer then the given triple cannot be a PPT. For an even value of a we have to take an even integer value of j. In this case, also we can obtain the condition on j, depending upon the given value of a.

As $a - j \in \mathbb{N}$, it follows that $a > j$. Also, the inequality,

$$a - j < \sqrt{j(2a-j)}, \text{ and } \sqrt{j(2a-j)} < a.$$
$$\therefore (a-j)^2 < j(2a-j) \Rightarrow (a-2j)^2 < 2j^2.$$

The possible inequality holds as:

$$\Rightarrow a - 2j < \sqrt{2}j. \tag{3.4}$$

$$\therefore a < \sqrt{2}(\sqrt{2}+1)j. \tag{3.5}$$

$$\Leftrightarrow j > \frac{a}{2}(\sqrt{2}-1)(\sqrt{2}) \Leftrightarrow j > \left[\frac{a}{2}(2-\sqrt{2})\right]. \tag{3.6}$$

We can approximate as $j > 0.3a$, by assuming $\sqrt{2} \cong 1.4$

Remark 3.1. For an odd PPT, if a is an odd positive integer then corresponding some choice of $j = (2n-1)^2 = 1, 9, 25, 49, 81, \ldots < \frac{a}{2}$ and for even PPT, it a is an even integer then corresponding to the choice of $i = 2n^2$, where $n = 1, 2, 3, 4 \ldots .. < \frac{a}{2}$.

In the next section, the investigation of multiple primitive Pythagorean triples and their graphical representation have been explored.

4 MPPT (Multiple Primitive Pythagorean Triplets) and Its Statistics.

In this section, we investigate MPPT and their graphical representation. Many positive integers possess more than one PPT. We have bifurcated them in two ways.

4.1 For a triple (a, b, c). If a is of the form $4(2n+3)$, where $n \in \mathbb{N}$, the triple (a, b, c) possess at least one primitive triplet.

Let $a = 4(2n+3)$ be an even positive integer of the infinite sequence $\{20, 28, 36, \ldots\ldots\}$. We have $a^2 = 16(2n+3)^2$, and for $b = \frac{a^2-j^2}{2j}$; as discussed in Theorem 2.1, we must have a choice of $j = 2n^2 = 2, 8, 18, \ldots < \frac{a}{2}$, where $n = 1, 2, 3, 4 \ldots$ such that $j \in \mathbb{N}_2$ and hence, $b \in \mathbb{N}_1$. As a consequence, the hypotenuse, $c = b + j = \frac{a^2+j^2}{2j}$.

As a^2 is an odd multiple of 16 , and for $b \in \mathbb{N}_1$, the value of $j = 2n^2 = 2, 8, \ldots \in \mathbb{N}_2$ will fit better. On $j = 2$ and $j = 8$, we will have two PPTs. The following table-3 gives how an even multiple primitive triples can be formed.

a	$j = 2n^2$ $= 2, 8, 18, \ldots \in \mathbb{N}_2 < \frac{a}{2}$	$b = \frac{a^2-j^2}{2j}$	$c = \frac{a^2+j^2}{2j} = b + j$	(a, b, c)	Remarks
20	2	99	101	$(20, 99, 101)$	2-PPT
	8	21	29	$(20, 21, 29)$	
60	2	899	901	$(60, 899, 901)$	3-PPT
	8	221	229	$(60, 221, 229)$	
	18	91	109	$(60, 91, 109)$	

Table - 3 Even multiple PPTs

One can notice from the table-3, that for the first leg, $a = 20$, we have two different Pythagorean primitive triplets, $(20, 99, 101)$ and $(20, 21, 29)$. On $a = 60$, we have three different Pythagorean primitive triplets, $(60, 899, 901)$, $(60, 221, 229)$ and $(60, 91, 109)$, which satisfying, $a^2 + b^2 = c^2$. Now, we consider some even numbers, which possess the first leg of the triple of the form $a = 4(2n + 3), n \in N$, have multiple numbers of triplets corresponding to some value of $j = 2n^2 = 2, 8, 18, \ldots \in \mathbb{N}_2 < \frac{a}{2}$, in the following table- 4.

Numbers of form $a = 4(2n + 3)$	Numbers of Triplets	Value of $j \in \mathbb{N}_2$
$20, 28, 36, 44, 52, 68, 76,$ $92, 100, 108, 116, 124 \ldots\ldots$	2	2,8
$60, 84, 132, 140, 156 \ldots..$	3	$(2, 8, 18), (2, 8, 50) \ldots$
$204, 228, 276, \ldots\ldots.$	4	$(2, 8, 18, 72), (2, 8, 50, 200) \ldots.$
$780, 924, \ldots\ldots$	6	$(2, 8, 18, 72, 98, 242), \ldots$

Table -4 Number of PTs corresponding to even number of form $4(2n + 3)$

Now, we see the graphical portrait of the above table-4, which has been tabulated on a wider range. We graph this data in table- 4 to give a broader view. The following observations of the table-5 are associated with the graphical representation.

We consider integers up to 1012 and notice 125 positive integers are of from $4(2n + 3)$. We tabulated several triplets corresponding to integer in table-5 as follows, which can be observed from table-4.

No. of triplets (Based on j)	2	3	4	6	7#
No of Integers	62	32	27	3	1

Table -5 -Number of triples, the first leg of the form, $4(2n + 3)$ *up to 1012*

At this point, to shed the light on our efforts in the same direction, we see an important observation from table-5 and figure-1 as follows:

(i) Extensive study to the movement of the pattern up to 5000 positive integers has noted that 'there is no such integer, which Possesses 5 PPTs.'

(ii) # The positive integer 660 has 7 PPTs. Also, we note that for the initial ten integers multiple of 660 shows diverse nature in creating triplets.

(iii) The positive numbers 2640 [=660×4] and 3960 [=660×6] have 6 PPTs, And the number 4620 [=660×7] has 13 PPTs and others have 7 primitive triplets.

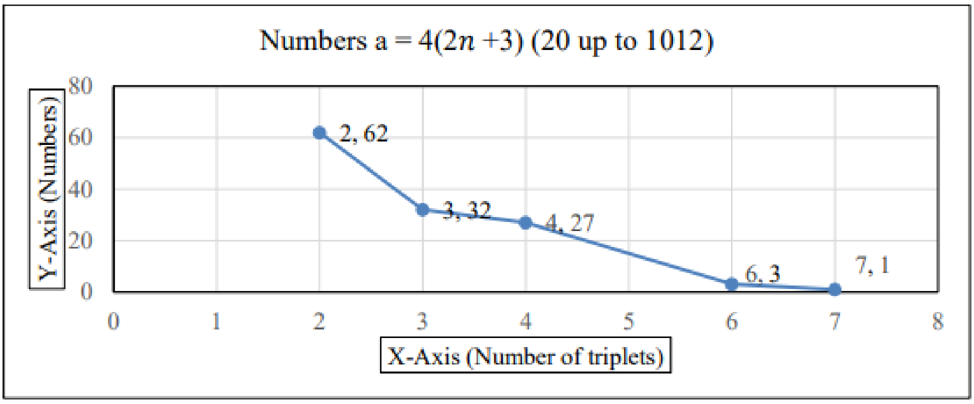

Figure 1. *Graphical Portrait of even no.* $4(2n+3)$ *of PPTs*

4.2 For a triple (a, b, c), If $a \in \{3(2n+9) \mid n \in \mathbb{N} \cup \{0\}\} = \{27, 33, 39, 45, \ldots\}$, have one or more than one PPTs.

Let $a = 3(2n+9)$ be an odd integer, and taking $j = (2n-1)^2 = \{1, 9, ..\} \in \mathbb{N}_1$. As we have discussed in theorem 2.2, we must have $b = \frac{a^2-j^2}{2j}$ an even positive integer. As a consequence, the hypotenuse, $c = b + j = \frac{a^2+j^2}{2j}$. The following table-6 gives how the odd multiple primitive triples can be formed.

a	$j = (2n-1)^2$ $= 1, 9, 25, \ldots$ $\in \mathbb{N}_1 < \frac{a}{2}$	$b = \frac{a^2-j^2}{2j}$	$c = \frac{a^2+j^2}{2j} = b + j$	(a, b, c)	Remarks
33	1	544	545	$(33, 544, 545)$	2-PPTs
	9	56	65	$(33, 56, 65)$	
105	1	5512	5513	$(105, 5512, 5513)$	3-PPTs
	9	608	617	$(105, 608, 617)$	
	25	208	233	$(105, 208, 233)$	

Table -6 Odd multiple PPT

Thus, for the first leg of the triple $a = 33$, we have two primitive triples $(33, 544, 545)$ and $(33, 56, 65)$. On for $a = 105$, we have three primitive triples $(105, 5512, 5513), (105, 608, 617)$ and $(105, 208, 233)$, which follows, $a^2 + b^2 = c^2$ As a special case, when $n = 3p$, along with $p \in \{\{1, 2, 3, 4, 5, 7\} U \{8, 12, 18, 24, 27\} U \{33, 36, 39, 42 \ldots\}\}$ then $a = 3(2n+9)$ possess more than one PPTs. We tabulate the first leg of the triple with Positive integers are of the form, $a = 3(2n+9)$ in table-7, have a possible number of triplets corresponding to the value of some positive integers $j = (2n-1)^2 = 1, 9, 25, 49, \ldots \in \mathbb{N}_1 < \frac{a}{2}$

Positive integers of the form: $a = 3(2n+9)$	No. of Primitive Triplets	Some values of $j \in \mathbb{N}_1$
$27, 45, 63, 81.99, 117, 153, 171,$ $243, 351, 459, 513, 621, 675,$ $729, 783, 837, 891, 945, 999, \ldots$	1	1
$33, 39, 51, 57, \ldots, 135, \ldots .567, \ldots$	2	$(1, 9), (1, 25), (1, 49), \ldots$
$105, 165, 195, 231, \ldots \ldots .$	3	$(1, 9, 25), (1, 9, 49), \ldots$
$315, 429, \ldots \ldots$	4	$(1, 25, 49, 81),$ $(1, 9, 121, 169), \ldots$

Table - 7 Number of PTs corresponding to an odd number of form $3(2n+9)$

We have through the integers 27 up to 885,143 numbers from the sequence $3(2n + 9)$. Based upon, the tabulated number of triples corresponding to integer in table-8 as follows.

No. of triplets (Based on j)	1	2	3	4
No of Integers	16	88	25	14

Table - 8 - Number of triples, the first leg of the form, 3(2n+9) up to 885 .

In the view of the above records of table-8, we have drawn the graphical representation (figure-2) as follows, which will help to investigations some open questions.

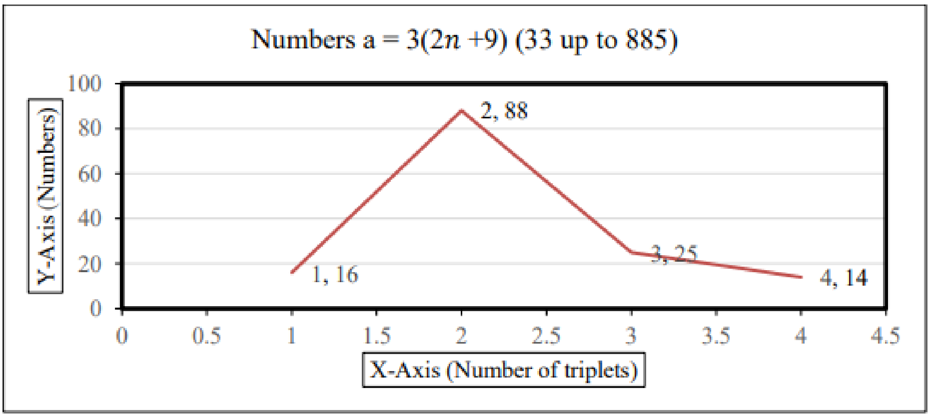

Figure 2. *Graphical Portrait of odd no.* $3(2n + 9)$ *of PPTs*

5 Conclusion

An important frame of works has been produced, over the centuries, around the properties of Pythagorean triples. In more recent times, a renewed interest withinside the generation of Pythagorean triples has been inspired through their numerous real-lifestyles applications, including cryptography and data security. The set of triples that may be generated by the use of the classical Euclid's system does not provide output about strict MPPTs (multiple primitive Pythagorean triples). While numerous methods have been explored to discover alternatives to characterise and generate primitive Pythagorean triples, the majority of these focuses on this article is Multiple primitive triples only. This paper has provided a new approach for producing all Pythagorean triples, both primitive and non-primitives. A new parameterisation of Pythagorean triples has been introduced in the general form of odd or even class of MPPTs. We have given a distinct type of triples having shorter leg 'a'', and have been detected that some positive integers have distinctive triples, whereas some have over one triplet. A variety of them have thirteen primitive triplets and some integers never possess any PPTs noted. Moreover, we have obtained some special attention, which can be considered as open questions and for future works as follows:

(i) For any $n \in N$, are there any positive integers of $4(2n + 3)$ and $3(2n + 9)$ possess 5 PPTs?

(ii) For any $n \in N$, are there any positive integers of from $4(2n + 3)$ possess only one PPT?

(iii) Does the number of positive integers of triples keep on increasing as the number of form $4(2n + 3)$ and $3(2n + 9)$, for some $n \in N$, increase?

(iv) What is the general pattern of numbers of the form of $3(2n + 9)$, that consists of exactly one PPT?

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