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ON THE EXPONENTIAL DIOPHANTINE EQUATION $p^x - 2^y = z^2$ WITH $p = k^2 + 4$ A PRIME NUMBER

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Abstract We are interested in finding non-trivial non-negative integer solutions for the Diophantine equation $p^x - 2^y = z^2$, where $p = k^2 + 4$ is a prime number and $k \geq 1$. We show that all the non-trivial non-negative integer solutions of this equation are given by $(1, 2, k)$ if $p \geq 13$ and $(1, 0, 2), (1, 2, 1), (3, 2, 11)$ if $p = 5$. The proofs are based on the use of the Catalan-Mihăilescu Theorem (old Catalan conjecture) and the properties of the modular arithmetic. In addition, we prove that equations of type $p^x - 2^y = w^{2^u}$ with $u \geq 2$ do not have positive integer solutions if $p \geq 13$. Moreover we find a single positive solution when $p = 5$, the solution $(1, 2, 1)$.

1 Introduction

Diophantine equations of the form $a^x + b^y = c^z$ have been studied by numerous mathematicians for many decades and by a variety of methods. One of the first references to these equations was given by Fermat-Euler (see [3]), showing that the equation $a^2 + 2 = c^3$ has only a positive integer solution $(a, c) = (5, 3)$. Scott [14] proved that if $a > 1$ and $b > 1$ satisfy $\gcd(a, b) = 1$ and c is prime, then the equation $a^x + b^y = c^z$ has at most two solutions in positive integers (x, y, z), when $c \neq 2$, and at most one solution (x, y, z) when $c = 2$, except for two cases (taking $a < b$): $(a, b, c) = (3, 5, 2)$, which has exactly three solutions $(x, y, z) = (1, 1, 3), (3, 1, 5), (1, 3, 7)$ and $(a, b, c) = (3, 13, 2)$, which has exactly two solutions $(x, y, z) = (1, 1, 4), (5, 1, 8)$ ([4], Section D9, p. 87). In 2007, Acu [1] solved the equation for $a = 2$, $b = 5$ and $z = 2$. The non-negative integer solutions to the equation are $(x, y, c) \in \{(3, 0, 3), (2, 1, 3)\}$. In 2017, Asthana and Singh [2] determined the solution set of the Diophantine equation $3^x + 13^y = z^2$, given by $\{(1, 0, 2); (1, 1, 4); (3, 2, 14); (5, 1, 16)\}$. Rabago [12] and Sroysang [15] studied the equations $2^x + 17^y = z^2$ and $2^x + 19^y = z^2$, respectively. Then A. Suvarnamani, A. Singta and S. Chotchaisthit ([16]) found solutions of two diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$. In 2019, the equation $2^x - 3^y = z^2$ was studied at [18]. In this paper, we use elementary methods, Catalan-Mihăilescu Theorem (Theorem 2.1) and Theorem 3.6 of Keskin and Duman [6] to study the exponential Diophantine equations of the form $p^x - 2^y = z^2$, where $p = k^2 + 4$ are prime numbers, $(x, y, z) \in \mathbb{N}^3$ e $k \in \mathbb{N}^*$.

2 Notation and Preliminary Results

Denote by $\mathbb{Z}$ be the set of *integer number* and let $\mathbb{N}$ be the set of all positive integers together with the number 0, that is, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, such a set will be called the set of *natural numbers*. Define $\mathbb{N}^* = \mathbb{N} \backslash \{0\}$ and $\mathbb{N}^q = \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$ as the *Cartesian product* of q copies of $\mathbb{N}$. We will use the $\equiv$ symbol for congruence module m and $a \equiv b \pmod{m}$ means that a is congruent to b module m, we will also use the notation $a \mid b$ to say that a divides b and $a^l \mid\mid b$ to say that a^l is the greatest power of a that divides b. For the case where a does not divide b we will use the notation $a \nmid b$. The set of all non-negative integer solutions to the equation $p^x - 2^y = z^2$ will be said simply the *solution set of the equation*, i.e, the set $\{(x, y, z) \in \mathbb{N}^3 \mid p^x - 2^y = z^2\}$.

The following theorem was proved by Mihăilescu in [9] and is written in the form of his famous conjecture.

Theorem 2.1. *(Catalan-Mihăilescu Theorem) The only solution* $(a,b,x,y) \in \mathbb{N}^4$ *for the Diophantine equation* $a^x - b^y = 1$*, where* $a,b,x,y > 1$*, is* $(3,2,2,3)$.

Remark 2.2. The equation $a^x - z^2 = 1$ has no positive integer solutions if $a,x,z > 1$.

We need the important result obtained by Sury [17]. This result was obtained first by Nagell [10], but the proof is not elementary, while the Sury's proof is elementary.

Theorem 2.3. *The Diophantine equation* $z^2 + 2 = y^x, x > 1$ *has only the solutions* $(z,y,x) = (\pm 5, 3, 3)$.

The following lemma will be used in the proof of the main theorems.

Lemma 2.4. *There is no* $w \in \mathbb{Z}$ *such that* $w^2 \equiv 3 \pmod 4$.

Proof. It is sufficient to verify for residual classes in $\mathbb{Z}/4\mathbb{Z}$ (the ring of integers modulo 4), i.e., just check for $w = 0,1,2,3$. As $1^2 = 1, 2^2 = 4, 3^2 = 9$ and $0^2 = 0$, it follows that $w^2 \equiv 1$ or $0 \pmod 4$. □

Let $k,s > 0$ natural numbers. The generalized Fibonacci and Lucas sequences $(U_n(k,s))$ and $(V_n(k,s))$ are defined as follows.

$$U_0(k,s) = 0, U_1(k,s) = 1, U_{n+1}(k,s) = kU_n(k,s) + sU_{n-1}(k,s) \text{ if } n \geq 1 \text{ and}$$

$$V_0(k,s) = 2, V_1(k,s) = k, V_{n+1}(k,s) = kV_n(k,s) + sV_{n-1}(k,s) \text{ if } n \geq 1.$$

For $k = s = 1$, the sequences $(U_n(1,1))$ and $(V_n(1,1))$ are called Fibonacci and Lucas sequences. For more information about generalized Fibonacci and Lucas sequences, one can consult [5, 8, 13].

Consider the following equation about positive integers

$$z^2 - dy^2 = -4, \text{ where } d = k^2 + 4 \text{ it is not a perfect square.} \tag{2.1}$$

Theorem 2.5 (Theorem 3.6,[6]). *Let* k, d *be positive integers such that* $k > 1$ *and* $d = k^2 + 4$*. Then all positive integer solutions of the equation* (2.1) *are given by* $(x,y) = (V_{2n-1}(k,1), U_{2n-1}(k,1))$*, with* $n \geq 1$.

Definition 2.6. Let n be a positive integer. The p-adic order of n, denoted by $v_p(n)$, is defined to be the natural l such that $p^l \,||\, n$, i. e., $n = p^l m$, with $\gcd(p,m) = 1$.

Theorem 2.7 (Theorem 1.3, [7]). *Let* $k \geq 1$ *be a integer number and let* $p \neq 2$ *be a prime number such that* $p|(k^2+4)$*. Then* $v_p(U_n(k,1)) = v_p(n)$.

Now we are ready to prove the following proposition.

Proposition 2.8. *Let* $p = k^2 + 4$ *be a prime number for some integer* $k \geq 3$*. The Diophantine equation*

$$z^2 - p(p^l)^2 = z^2 - p^{2l+1} = -4, \; l \in \mathbb{N}^*, \tag{2.2}$$

has no positive integer solutions (z,l).

Proof. First, we consider the Diophantine equation $z^2 - py^2 = -4$. According to Theorem 2.5, this equation has solution $(z,y) = (V_{2n-1}(k,1), U_{2n-1}(k,1))$ for each $n \in \mathbb{N}^*$. A solution of $z^2 - py^2 = -4$ provide a solution for (2.2) if $U_{2n-1}(k,1) = p^l$. Now, we have $v_p(U_{2n-1}(k,1)) = v_p(2n-1) = l$, that is, $2n - 1 = mp^l$, with $\gcd(m,p) = 1$, by Theorem 2.7.

Claim: $U_n(k,1) > n$, for $n \geq 2$. We proof this claim by induction on n. Clearly the claim is true for $n = 2$ since $U_2(k,1) = k > n = 2$. Now, $U_{n+1}(k,1) = kU_n(k,1) + U_{n-1}(k,1) > kn + 1 \geq n + 1$.

If $2n - 1 = mp^l$, then by claim $U_{mp^l}(k,1) > mp^l$, that is, $U_{2n-1}(k,1) \neq p^l$ for any l. Therefore, the equation (2.2) has no positive integer solutions. □

Remark 2.9. We note that $k^2 + 4$ is not a prime number if $k \geq 3$ is a perfect square. To see this, put $k = w^2$ and so

$$k^2 + 4 = w^4 + 4 = (w^2+2)^2 - 4w^2 = \left(w^2 + 2 - 2w\right)\left(w^2 + 2 + 2w\right).$$

It is clear that $w^2 + 2 \pm 2w > 1$, because $w \geq 2$.

3 Main Theorems

Theorem 3.1. *If $p = k^2 + 4$ is a prime number for some $k \geq 3$ then the solution set of the Diophantine equation*

$$p^x - 2^y = z^2, (x, y, z) \in \mathbb{N}^3, \tag{3.1}$$

is given by $\{(0, 0, 0), (1, 2, k)\}$.

Proof. The proof we will be done by testing several cases. We will assume that x, y, z are natural numbers such that they satisfy the equation $p^x - 2^y = z^2$.

Case 1. ($x = 0$). Consider two subcases $y = 0$ and $y \geq 1$.

Case i. If $y = 0$, then $z^2 = 0$ and so $(0, 0, 0)$ is a solution of the equation.

Case ii. If $y \geq 1$, then $z^2 = 1 - 2^y < -1$ which an absurd.

Case 2. ($x = 1$). In this case we have $p - 2^y = z^2$ and as $z \geq 0$ we must have $p \geq 2^y$ which implies that $y \in \{0, 1, \dots, \lfloor log_2 p \rfloor\}$ where $\lfloor u \rfloor$ means the largest integer less than or equal to u. Let us divide this case into the subcases $y = 0, y = 1, y = 2$ and $y \geq 3$.

Case i. If $y = 0$, then $p - 1 = z^2 = k^2 + 3$, so we have $(z - k) \cdot (z + k) = 3$, whence we conclude that $k = 1$, which is a contradiction with $k \geq 3$.

Case ii. If $y = 1$, then $p - 2 = z^2 = k^2 + 2$, so we have $(z - k) \cdot (z + k) = 2$, whence we conclude that $k = \frac{3}{2}$, which is a contradiction with $k \geq 3$.

Case iii. If $y = 2$, then $p - 2 = z^2 = k^2$, so $z = k$ and a solution to the equation is equal to $(1, 2, k)$.

Case iv. If $y \geq 3$, we will rewrite the equation $p - 2^y = z^2 = k^2 + 4 - 2^y$ as $(z - k) \cdot (z + k) = 4 \cdot (1 - 2^{y-2})$. According to the hypotheses z and k are both odd numbers, i. e., there are non-zero integers a and b such that $z - k = 2a$ and $z + k = 2b$. It follows that $ab + 2^{y-2} = 1$. In the previous conditions $k = b - a$, and hence either a or b is even, because k is odd. Then we have a contradiction because the left side of equation $ab + 2^{y-2} = 1$ is even while the right side is odd.

Therefore, $(1, 2, k)$ is the only solution to the equation of the statement in this case.

Case 3. ($x > 1$, $y = 0$). In this case the equation (3.1) is reduced to $p^x - z^2 = 1$. According to Remark 2.2, the last equation has no positive solutions if $z > 1$. If $z = 0$, we have $p^x = 1$ which implies that $x = 0$, a contradiction since $x > 1$. If $z = 1$, we have $p^x = 2$, a contradiction because $p \geq 13$. Therefore, there is no solution of (3.1) in this case.

Case 4. ($x > 1$ and $y = 1$). In this case the equation (3.1) is reduced to $p^x = z^2 + 2$. According to Theorem 2.3, there is no positive integer solutions if $x > 1$.

Case 5. ($x > 1$ even and $y > 1$). We will show that the equation (3.1) has no positive integer solutions in this case also. We observe that there is $t \in \mathbb{N}^*$ such that

$$p^{2t} - z^2 = 2^y = (p^t - z) \cdot (p^t + z) = 2^y.$$

Since $p^t - z < p^t + z$ are both even numbers, there exists $\alpha \in \mathbb{N}$ such that $p^t - z = 2^\alpha$ and $p^t + z = 2^{y-\alpha}$ from which one obtain

$$2.p^t = 2^\alpha + 2^{y-\alpha} = 2^\alpha(1 + 2^{y-2\alpha}). \tag{3.2}$$

Let us divide this case into the subcases $\alpha = 0, \alpha = 1$ and $\alpha \geq 2$.

If $\alpha = 0$ we have $2.p^t = 1 + 2^y$ which is an absurd for $y > 1$. If $\alpha \geq 2$ we have $p^t = 2^{\alpha-1}(1 + 2^{y-2\alpha})$ which is also an absurd because the left side is an odd number and the right one is even. If $\alpha = 1$ the equality (3.2) is reduced to $p^t - 2^{y-2} = 1$. Now one apply Theorem 2.1 to conclude that $t = 1$ or $y \in \{2, 3\}$. We divide the subcase $\alpha = 1$ into the two subcases.

Case i. If $t = 1$, then $2^{y-2} = p - 1 = k^2 + 3$. If $y = 2, 3$ or 4 we have $k^2 = -2, k^2 = -1$ or $k^2 = 1$, which is a contradiction with $k \geq 3$. Now, we suppose $y \geq 5$ we have $2^{y-2} \equiv 0 \pmod 4$ and therefore $k^2 \equiv 1 \pmod 4$. Thus $k = 4w + 1$ or $k = 4w + 3$ for some $w \in \mathbb{N}$. If $k = 4w + 1$, then $2^{y-2} = k^2 + 3$ reduces to $2^{y-4} = 4w^2 + 2w + 1$, an absurd. If $k = 4w + 3$, then $2^{y-2} = k^2 + 3$ reduces to $2^{y-4} = 4w^2 + 6w + 3$, an absurd.

Case ii. If $y \in \{2, 3\}$, then $p^t \in \{2, 3\}$, which is a contradiction because $p^t \geq 13$.

Case 6. ($x > 1$ odd and $y > 1$). We will also show that the equation (3.1) has no positive integer solutions in this case. There is $s \in \mathbb{N}^*$ such that the equation (3.1) is reduced to $p^{2s+1} - 2^y = z^2$. Since $p = k^2 + 4$ is an odd prime we have $p \equiv 3 \pmod 4$ or $p \equiv 1 \pmod 4$. If $p \equiv 3 \pmod 4$, then $k^2 \equiv 3 \pmod 4$, a contradiction by Lemma 2.4. Therefore $p \equiv 1 \pmod 4$. We will consider three sub-cases from now on: $y \geq 4$ even, $y \geq 3$ odd and $y = 2$.

Case i. First consider $y \geq 4$ even, i.e., $y = 2u$ where $u \geq 2$. We have the equation $p^{2s+1} - 2^{2u} = z^2$. Of course, z must be odd, so $z^2 \equiv 1 \pmod 4$ which implies that $z \equiv 1 \pmod 4$ or $z \equiv 3 \pmod 4$. Note that $p^{2s+1} - 2^{2u} = p.p^{2s} - 2^{2u} = (p - 4).p^{2s} + 4.p^{2s} - 4^u = z^2$ and as $p = k^2 + 4$ we get the equation $(k.p^s - z) \cdot (k.p^s + z) = 4(4^{u-1} - p^{2s})$.

- If $z \equiv 1 \pmod 4$, then $(k.p^s - z) \equiv 0$ or $2 \pmod 4$ and $(k.p^s + z) \equiv 2$ or $0 \pmod 4$, because $p \equiv 1 \pmod 4$ implies $k \equiv 1$ or $3 \pmod 4$. So there are non-zero integers a and b such that $(k.p^s - z) = 4a$ or $2a$ where $\gcd(a, 2) = 1$ and $(k.p^s + z) = 2b$ or $4b$ where $\gcd(b, 2) = 1$. In any case, we have $8|(k.p^s - z) \cdot (k.p^s + z)$, but $8 \nmid 4(4^{u-1} - p^{2s})$, because p is odd, an absurd.

- Let $z \equiv 3 \pmod 4$. Analogously to the case $z \equiv 1 \pmod 4$ we have $(k.p^s - z) \equiv 2$ or $0 \pmod 4$ and $(k.p^s + z) \equiv 0$ or $2 \pmod 4$. The result follows as in the previous case.

Case ii. For $y \geq 3$ odd. There is $w \in \mathbb{N}^*$ such that the equation (3.1) is reduced to $p^{2s+1} - 2^{2w+1} = z^2$. Clearly z^2 is not even because the left side of the expression is always odd. So again $z \equiv 1 \pmod 4$ or $z \equiv 3 \pmod 4$. By manipulating as in the previous case, we obtain $(k.p^s - z) \cdot (k.p^s + z) = 4(2 \cdot 4^{w-1} - p^{2s})$. We have a contradiction as in the previous case.

Case iii. Let $y = 2$. If $z = 0$, we have $p^x = 4$ which is impossible since $p \geq 13$ and $x > 1$. If $z = 1$, we have $p^x = 5$, which is a contradiction for the same reason. From now on, let $z > 1$. In this case, the integers $x, z > 1$ must satisfy $z^2 - p^x = -4$. By Proposition 2.8, this equation has no positive integer solutions.

□

Corollary 3.2. *Let $p = k^2 + 4$ be a prime number for some integer $k \geq 3$. Then $(0, 0, 0)$ is the unique non-negative integer solution of the Diophantine equation*

$$p^x - 2^y = w^{2^u}, (x, y, w) \in \mathbb{N}^3 \text{ and } u \geq 2. \tag{3.3}$$

Proof. We write $p^x - 2^y = (w^{2^{u-1}})^2, u \geq 2$. According to Theorem 3.1 we have $w^{2^{u-1}} = 0$ or $w^{2^{u-1}} = k$. In the first case, $w = 0$, thus finding the trivial solution for equation (3.3). In the second case, it follows from Remark 2.9 that k is not a perfect square. Therefore we can choose a prime factor q of k such that q^2 does not divide k. Now, the polynomial $f(x) = x^{2^{u-1}} - k$ is irreducible in $\mathbb{Z}[x]$ (polynomial ring with coefficients in $\mathbb{Z}$) by Eisenstein criterion for prime q. So $k^{\frac{1}{2^{u-1}}} \xi^j \notin \mathbb{Z}$, for all $j \in \{0, 1, \ldots, 2^{u-1} - 1\}$, where ξ is a primitive 2^{u-1}-th root of unity. Therefore, $p^x - 2^y = w^{2^u}$ has no non-trivial solution. □

Remark 3.3. Another proof for the previous corollary is by induction on u.

As an example if $p = 29$ we have $k = 5$ and $(0,0,0)$ is the unique non-negative integer solution of $29^x - 2^y = w^4$. Another example is the following: if $p = 13$, we have $k = 3$ and the only non-negative integer solution of $13^x - 2^y = w^8$ is $(0,0,0)$.

Theorem 3.4. *The set* $\{(0,0,0); (1,0,2); (1,2,1); (3,2,11)\}$ *is the solution set of the Diophantine equation*

$$5^x - 2^y = z^2, (x,y,z) \in \mathbb{N}^3. \tag{3.4}$$

Proof. Let $(x,y,z) \in \mathbb{N}^3$ be a solution of the equation (3.4). Similarly to the Case 1 of Theorem 3.1 one obtain that $(0,0,0)$ is the unique solution of (3.4) in the case $x = 0$. Similar to Case 2 of Theorem 3.1 we have $y = 0, y = 1$ or $y = 2$, then we get $z^2 = 4, z^2 = 3$ or $z^2 = 1$ and we find the solutions $(1,0,2)$ and $(1,2,1)$. Cases 3, 4 and 5 are similar to Theorem 3.1 and we did not find non-negative integer solutions in these cases. It remains for us to analyze the analogue of Case 6, for that we will use the same notations of what was done before.

Case 6 ($x > 1$ odd and $y > 1$). Let us divide this case into the subcases $y \geq 4$ even, $y \geq 3$ odd and $y = 2$.
First consider $y \geq 4$ even and take $y = 2u$ and $x = 2s + 1$ where $u \geq 2, s \geq 1$. In this case, equation (3.4) is reduced to the following $5^{2s+1} - 2^{2u} = z^2$. So, $z^2 \equiv 1 \pmod 4$ which implies that $z \equiv 1 \pmod 4$ or $z \equiv 3 \pmod 4$. After manipulation, the previous equation can be rewritten as

$$(5^s - z) \cdot (5^s + z) = 4(4^{u-1} - 5^{2s}).$$

If $z \equiv 1 \pmod 4$, then $(5^s - z) \equiv 0 \pmod 4$ and $(5^s + z) \equiv 2 \pmod 4$. Thus $(5^s - z) = 4a$ and $5^s + z = 2b$ for some nonzero integers a and b such that b is an odd number. Therefore we have $8|(5^{2s} - z^2)$, but $8 \nmid [4(4^{u-1} - 5^{2s})]$, a contradiction.

If $z \equiv 3 \pmod 4$, then $(5^s - z) \equiv 2 \pmod 4$ and $(5^s + z) \equiv 0 \pmod 4$ and the result is similar to the case $z \equiv 1 \pmod 4$. .

Now consider the second subcase, that is, for $y \geq 3$ odd. Take $y = 2w + 1$ and $x = 2s + 1$ where $w, s \geq 1$. We have $5^{2s+1} - 2^{2w+1} = z^2$ and clearly z^2 is not even because the left side of the expression is always odd. So again $z \equiv 1 \pmod 4$ or $z \equiv 3 \pmod 4$. Analogous to the previous case,

$$(5^s - z) \cdot (5^s + z) = 4(2 \cdot 4^{w-1} - 5^{2s}).$$

As in the previous subcase, we conclude that $8|(5^{2s} - z^2)$, but $8 \nmid [4(2 \cdot 4^{w-1} - 5^{2s})]$, a contradiction.

In the last sub-case, consider $y = 2$. If $z = 0$, we have $5^x = 4$ which is impossible on the integers. If $z = 1$, we have $x = 1$, an absurd. From now on, let $z > 1$. In this case, the integers $x, z > 1$ must satisfy $5^x - 4 = z^2$. In [11], Nagell showed that the only positive integer solution is $(3,2,11)$. So in Case 6 we find only solution $(3,2,11)$. □

Corollary 3.5. *The solution set of the Diophantine equation* $5^x - 2^y = w^{2^u}$, *with* $u \geq 2$ *and* $(x,y,w) \in \mathbb{N}^3$, *is given by* $\{(0,0,0); (1,2,1)\}$.

Proof. The proof is similar to Corollary 3.2, noting that in the first step $w^{2^{u-1}} \in \{0,2,1,11\}$. □

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