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Self Generating of Diophantine Equation $d^2 - c^2 = b^2 - a^2$ and N-Tuples

Muneer Karama, Ayed AbdAlghany, Amer Abu-Hasheesh

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Abstract In this article, we present just two parameter formulas where the two integral parameters are also part of the solution set. We shall call them selfgenerating formulas. These formulas will then be generalized to give N-tuples when a set of (n - 2) integer is given.

1 Introduction

Diophantine equation of the forum $d^2 - c^2 = b^2 - a^2$, (same as $a^2 + c^2 = b^2 + d^2$) plays an important mathematical ideas such as the problem "Pick any point inside the square . The distances from the corners to that point have the relation $a^2 + c^2 = b^2 + d^2$ where the lines to that point are labeled a, b, c, d going clockwise" [1]. below is a figure of the square and the points.

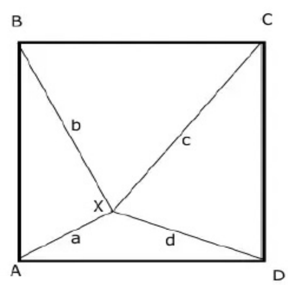

Figure 1. Geometrics representation of $a^2 + c^2 = b^2 + d^2$

Moreover, this kind of Diophantine equation related toPythagorean quadruple,Pythagorean triples, arithmetic progression among squares $b^2 - c^2 = c^2 - a^2$, and Brahmagupta Identity.

(Dickson [2], pp. 334-441) ,and (Guy [3], page 140) developed a parametric solution to $a^2 + b^2 = c^2 + d^2$ using integer parameters $(a, b, c, d) = (pr + qs, qr - ps, pr - qs, ps + qr)$, where p, q, r, s are arbitrary .Also they used Fibonacci , and Euler identities to solve this formula.

In the formulas mentioned above, either four or five variables are needed to generate four other integers (a, b, c, d). In this article, we present just two parameter formulas where the two integral parameters are also part of the solution set. We shall call them self-generating formulas. These formulas will then be generalized to give N-tuples when a set of $(n - 2)$ integer is given.

The self-generating of $d^2 - c^2 = b^2 - a^2$ formulas

We use a and b to designate the two integer parameters that will generate the solutions of $d^2 - c^2 = b^2 - a^2$. The following theorem deals with the two possible cases arising from parity conditions imposed upon a and b.

Theorem 1.1. *For positive integers a and b, where a or b or both are even (or odd), there exist c and d such that $d^2 - c^2 = b^2 - a^2$.*

Case 1: if a and b are of opposite party, then (it is useful for the reader to put in her/his mind that every odd number is a deference of two square)

$$d = \frac{b^2 - a^2 + 1}{2}, \text{ and } c = \frac{b^2 - a^2 - 1}{2}$$

Proof.

$$\begin{aligned} d^2 - c^2 =& (d-c)(d+c) \\ =& \left[\left(\frac{b^2-a^2+1}{2} - \frac{b^2-a^2-1}{2}\right)\right]\left[\left(\frac{b^2-a^2+1}{2} + \frac{b^2-a^2-1}{2}\right)\right] \\ =& \left[\frac{2}{2}\right]\left[\frac{2b^2-2a^2}{2}\right] \\ =& b^2 - a^2 \end{aligned}$$

Therefore $d^2 - c^2 = b^2 - a^2$.

Since a and b differ in parity, c and d in (1) are integers.

Corollary 1.2. *From (1), we see that* c *and* d *are consecutive integers. Therefore,* $(a, b, c, d) = 1$, *even when* $(a, b) \neq 1$

Example 1.3.

$$60^2 - 59^2 = 12^2 - 5^2$$
$$104^2 - 103^2 = 16^2 - 7^2$$
$$122^2 - 121^2 = 18^2 - 9^2$$
$$140^2 - 139^2 = 20^2 - 11^2$$

Case 2: if a and b are of same party, then

$$d = \frac{b^2-a^2}{2^{n+1}} + 2^{n-1}, \text{ and } c = \frac{b^2-a^2}{2^{n+1}} - 2^{n-1}, \text{ where } n \geq 1$$

Proof.

$$d^2 - c^2 = (d-c)(d+c) =$$
$$\left(\frac{b^2-a^2}{2^{n+1}} + 2^{n-1}\right)^2 - \left(\frac{b^2-a^2}{2^{n+1}} - 2^{n-1}\right)^2 =$$
$$\left(\frac{b^2-a^2}{2^{n+1}} + 2^{n-1} - \left(\frac{b^2-a^2}{2^{n+1}} - 2^{n-1}\right)\right)\right]\left[\left(\frac{b^2-a^2}{2^{n+1}} + 2^{n-1} + \frac{b^2-a^2}{2^{n+1}} - 2^{n-1}\right)\right]$$
$$= [2^n]\left[\frac{2b^2-2a^2}{2^{n+1}}\right]$$
$$= b^2 - a^2$$

Therefore $d^2 - c^2 = b^2 - a^2$.

Since a and b same in parity, c and d in (2) are integers.

Corollary 1.4. *If* $b - a \equiv 0 (\mathrm{mod} 4)$, $\frac{b^2-a^2}{2^{n+1}}$ *is even or odd, c and* d *are consecutive odd integers always, so* $(a, b, c, d) = 1$.

Example 1.5.

$$11^2 - 9^2 = 7^2 - 3^2$$
$$13^2 - 11^2 = 8^2 - 4^2$$
$$15^2 - 13^2 = 9^2 - 5^2$$
$$17^2 - 15^2 = 10^2 - 6^2$$
$$62^2 - 46^2 = 48^2 - 24^2$$

2 Self generating n-tuples

Definition of self generating n-tuples in this paper means, we can use a list of n in case 2 to solve $d^2 - c^2 = b^2 - a^2$ for example if $n = 1$, then we have ; $d = \frac{b^2-a^2}{2^2} + 2^0$, and $c = \frac{b^2-a^2}{2^2} - 2^0$, and so one, which yelled to new chain of squares when n $= 2, 3, ..$, i.e. the ideas and methods of proof for the self-generating $d^2 - c^2 = b^2 - a^2$ can be generalized to the N-tuple case. We need to find formulas for generating integer N-tuples $(a_1, a_2, \ldots, a_n)$ when given a set of integer values for the $(n-2)$ members of the "parameter set" $S = (a_1, a_2, \ldots, a_{n-2})$.

Comparable to the parity conditions imposed on the self-generating $d^2-c^2 = b^2-a^2$ formulas, we introduce the variable T. Theorem 2: let $S = (a_1, a_2, \ldots, a_{n-2})$, where a_1 is an integer , let $T =$ number of odd (or even) numbers in S. If $T \not\equiv 2(\text{mod}4)$, then there exist integers a_{n-1}, and a_n such that

$$a_n^2 - a_{n-1}^2 = a_{n-2}^2 - a_{n-3}^2 = \ldots = a_2^2 - a_1^2$$

Proof. Let $T \equiv 0(\text{mod}4)$, then, setting

$$a_n = \frac{a_{n-2}^2 - a_{n-3}^2 + 2^{n-1} = \cdots = a_2^2 - a_1^2 + 2^{n-1}}{2^{n+1}}$$

And

$$a_{n-1} = \frac{a_{n-2}^2 - a_{n-3}^2 - 2^{n-1} = \cdots = a_2^2 - a_1^2 - 2^{n-1}}{2^{n+1}}$$

We have $a_n^2 - a_{n-1}^2 = (a_n - a_{n-1})(a_n + a_{n-1}) = \left[2^{n-1}\right]\left[\frac{2\left(a_{n-2}^2 - a_{n-3}^2 = \cdots = a_2^2 - a_1^2\right)}{2^{n+1}}\right] = a_n^2 - a_{n-1}^2 = a_{n-2}^2 - a_{n-3}^2 = \cdots = a_2^2 - a_1^2$.

Example 2.1.

$$433^2 - 431^2 = 218^2 - 214^2 = 112^2 - 104^2 = 62^2 - 46^2 = 48^2 - 24^2$$
$$= 43^2 - 11^2$$

References

[1] Dickson, L. E. "Pell Equation; $ax^2 + bx + c$ Made a Square" and "Further Single Equations of the Second Degree." Chs. 12-13 in History of the Theory of Numbers, Vol. 2: Diophantine Analysis. New York: Dover, pp. 334-441, 2005.

[2] Guy, R. K. Unsolved Problems in Number Theory, 2nd ed. New York: Springer-Verlag, 1994.

[3] https://math.stackexchange.com/questions/153603/diophantine-equation-a2b2c2d2?noredirecthttps://math.stackexchange.c equation-a2b2c2d2?noredirect $= 1\&lq = 1$.

Author information

Muneer Karama, Ayed AbdAlghany, Amer Abu-Hasheesh, Department of Applied Mathematics and Physics, Palestine Polytechnic University, Hebron, Palestine, Palestine.
E-mail: muneerk@ppu.edu, Hasheesh_a@ppu.edu, ayed42@ppu.edu

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