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Generalized Derivations on Alternative Rings

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Abstract In this paper, we prove that if $\mathfrak{R}$ is an alternative ring having an idempotent element $e(e \neq 0, e \neq 1)$ which satisfies certain conditions and g is any multiplicative generalized derivation of $\mathfrak{R}$, then g is additive..

1 Introduction

The question of when multiplicative isomorphism is additive was first asked by Martindale III [4]. This question has become an interesting research area in associative ring theory. Later on, Daif [1], asked the same question for derivation in ring R. He defined a multiplicative derivation on R as d:R→ R such that d(ab) = d(a)b + ad(b), for all a,b ∈ R and asked when a multiplicative derivation becomes additive on R. Fortunately, in the same article, Daif gave full answer for associative ring. In [2] and [5] Ferreira et. al. has solved the similar problem for the case of alternative rings.
In [3], Hvala has given the definition of generalized derivation on ring R which is as follows: a mapping g:R→ R is said to be a generalized derivation if there exists a derivation d:R→ R such that g(xy) = g(x)y + xd(y), for all x,y ∈ R.
In the middle of the 19th century the theory of associative rings was developed. Later, algebraic structure that do not satisfy axiom of associativity were studied. Among these, there are the alternative rings which are the rings satisfying (xx)y = x(xy) and x(yy) = (xy)y. In [2] and [5] Ferreira et. al. asked the same question when a multiplicative mappings and multiplicative derivation becomes additive in an alternative rings. Ferreira et. al. discuss the answer to this question in [2] and [5]. Motivated by Daif, in this paper we discuss that when a multiplicative generalized derivation becomes additive on alternative rings.

Let $\mathfrak{R}$ be an alternative ring having a non trivial idempotent e, that is, $e^2 = e, e$ is non zero and e is not the identity element. The two sided Peirce decomposition of $\mathfrak{R}$ relative to idempotent e is of the form

$$\mathfrak{R} = \mathfrak{R}_{11} \bigoplus \mathfrak{R}_{10} \bigoplus \mathfrak{R}_{01} \bigoplus \mathfrak{R}_{00},$$

where $\mathfrak{R}_{ij}$(i,j=0,1) are the subspaces of $\mathfrak{R}$ and defined as

$$\mathfrak{R}_{ij} = \{x_{ij} \in \mathfrak{R} : ex_{ij} = ix_{ij}, x_{ij}e = jx_{ij}\}, i, j = 0, 1$$

and satisfy the following relations:
i) $\mathfrak{R}_{ij}\mathfrak{R}_{jk} \subseteq \mathfrak{R}_{ik}, \quad (i, j, k = 0, 1)$;
ii) $\mathfrak{R}_{ij}\mathfrak{R}_{ij} \subseteq \mathfrak{R}_{ji}, \quad (i, j = 0, 1)$;
iii) $\mathfrak{R}_{ij}\mathfrak{R}_{kl} = 0, \quad (j \neq k, (i, j) \neq (k, l))$.

By the definition of g, it can be easily seen that $g(0) = g(00) = g(0)0 + 0d(0) = 0$ and similarly $d(0) = 0$. Moreover, $d(e) = d(e^2) = d(e)e + ed(e)$. Now, if we express

$d(e) = a_{11} + a_{10} + a_{01} + a_{00}$, where $a_{ij} \in \mathfrak{R}_{ij}$, we have $d(e) = a_{10} + a_{01}$.
In the same way, $g(e) = g(e^2) = g(e)e + ed(e)$ and we can write $g(e) = b_{11} + b_{10} + b_{01} + b_{00}$.
Now, using the value of $g(e)$ and $d(e)$, we get

$$b_{11} + b_{10} + b_{01} + b_{00} = b_{11} + b_{01} + a_{10}.$$

From this, we can conclude that, $b_{00} = 0$ and hence $g(e) = b_{11} + a_{10} + b_{01}$.
For simplification, let f be the inner derivation of $\mathfrak{R}$ determined by the element $a_{10} - a_{01}$, that is, $f(x) = [x, a_{10} - a_{01}]$, for all $x \in \mathfrak{R}$.
Therefore

$$f(e) = [e, a_{10} - a_{01}] = a_{10} + a_{01}.$$

Now, let $F(x) = (b_{11} + b_{01})x + x(a_{10} - a_{01})$ be the generalized inner derivation determined by the two elements $b_{11} + b_{01}$ and $a_{10} - a_{01}$, so we have, $F(e) = b_{11} + b_{01} + a_{10}$.
Without any loss of generality, we can replace the derivation d by $D = d - f$ and the multiplicative generalized derivation g by the multiplicative generalized derivation $G = g - F$. This yields $D(e) = 0$ and $G(e) = 0$. By this remark, for showing that g is additive, it is sufficient to prove that G is additive.

2 ADDITIVITY OF MULTIPLICATIVE GENERALIZED DERIVATION

In this note, to prove that multiplicative generalized derivation G:$\mathfrak{R} \to \mathfrak{R}$ is additive, we assume that the following conditions hold:
(C1) $x(\mathfrak{R}e) = 0$ implies $x = 0$;
(C2) $(e\mathfrak{R})x = 0$ implies $x = 0$;
(C3) $(xe)((1-e)\mathfrak{R}e) = 0$ implies $xe = 0$.

To prove our main theorem, we need the following series of lemmas:

Lemma 2.1. *([5], Lemma 4.1)* $D(\mathfrak{R}_{ij}) \subseteq \mathfrak{R}_{ij}, i, j = 0, 1.$

Lemma 2.2. *Let* $x, y \in \mathfrak{R}$ *such that* $xy = 0$. *Then*

$$(G(x+z))y = (G(x) + G(z))y, \text{ for all } z \in \mathfrak{R}.$$

Proof. As $xy = 0$, we have $0 = G(xy) = G(x)y + xD(y)$, so $G(x)y = -xD(y)$
Moreover

$$\begin{aligned}(G(x+z))y &= G((x+z)y) - (x+z)D(y)\\ &= G(zy) - xD(y) - zD(y)\\ &= G(z)y + zD(y) + G(x)y - zD(y)\end{aligned}$$

Hence, $(G(x+z))y = (G(x) + G(z))y$. □

Lemma 2.3. $G(\mathfrak{R}_{1n}) \subseteq \mathfrak{R}_{1n}, n = 0, 1; G(\mathfrak{R}_{01}) \subseteq \mathfrak{R}_{11} + \mathfrak{R}_{01}, G(\mathfrak{R}_{11} + \mathfrak{R}_{01}) \subseteq \mathfrak{R}_{11} + \mathfrak{R}_{01}$ *and* $G(\mathfrak{R}_{10}) \subseteq \mathfrak{R}_{10} + \mathfrak{R}_{00}$. *Moreover,* G *is additive on* $\mathfrak{R}_{1n}$ *and* $G(x_{11} + x_{10}) = G(x_{11}) + G(x_{10})$, *for every* $x_{11} \in \mathfrak{R}_{11}$ *and* $x_{10} \in \mathfrak{R}_{10}$.

Proof. Since, $G(xy) = G(x)y + xD(y)$, for every $x, y \in \mathfrak{R}$ and it follows that for any $x_{1n} \in \mathfrak{R}_{1n}, n = 0, 1$, we have $G(x_{1n}) = G(ex_{1n}) = G(e)x_{1n} + eD(x_{1n}) = D(x_{1n})$, as $G(e) = 0$ and $D(R_{1n}) \subseteq R_{1n}$. So we have that $G \mid_{\mathfrak{R}_{1n}} = D \mid_{\mathfrak{R}_{1n}}, n = 0, 1$ and hence G is additive on $\mathfrak{R}_{1n}, n = 0, 1$, since D is.

Moreover, with the same argument, using Lemma 2.1, we have if $x_{11} \in \mathfrak{R}_{11}$ and $x_{10} \in \mathfrak{R}_{10}$, then

$$\begin{aligned} G(x_{11}+x_{10}) &= G(e(x_{11}+x_{10})) \\ &= G(e)(x_{11}+x_{10}) + eD(x_{11}+x_{10}) \\ &= e(D(x_{11})+D(x_{10})) \\ &= D(x_{11})+D(x_{10}) \\ &= G(x_{11})+G(x_{10}) \end{aligned}$$

Now, let x_{01} be any element in $\mathfrak{R}_{01}$, we have $G(x_{01}) = G(x_{01}e) = G(x_{01})e + x_{01}D(e)$. As $D(e) = 0$, we get, $G(x_{01}) = G(x_{01})e$. Now, if $G(x_{01}) = y_{11} + y_{10} + y_{01} + y_{00}$, we have $G(x_{01}) = y_{11} + y_{01} \in \mathfrak{R}_{11} + \mathfrak{R}_{01}$. Hence, $G(\mathfrak{R}_{01}) \subseteq \mathfrak{R}_{11} + \mathfrak{R}_{01}$.
Again, since $y_{11} \in \mathfrak{R}_{11}$ and $y_{01} \in \mathfrak{R}_{01}$, then

$$\begin{aligned} G(y_{11}+y_{01}) &= G((y_{11}+y_{01})e) \\ &= G(y_{11}+y_{01})e + (y_{11}+y_{01})D(e) \\ &= (G(y_{11}+y_{01}))e \in \mathfrak{R}_{11}+\mathfrak{R}_{01} \end{aligned}$$

So, we get $G(\mathfrak{R}_{11} + \mathfrak{R}_{01}) \subseteq \mathfrak{R}_{11} + \mathfrak{R}_{01}$.
Finally, let $x_{00} \in \mathfrak{R}_{00}$, we write, $G(x_{00}) = z_{11} + z_{10} + z_{01} + z_{00}$, then

$$0 = G(x_{00}e) = (G(x_{00}))e + x_{00}D(e) = (G(x_{00}))e = z_{11} + z_{01}.$$

So, $G(x_{00}) = z_{10} + z_{00} \in \mathfrak{R}_{10} + \mathfrak{R}_{00}$, hence $G(\mathfrak{R}_{00}) \subseteq \mathfrak{R}_{10} + \mathfrak{R}_{00}$. □

Lemma 2.4. *For any $x_{11} \in \mathfrak{R}_{11}$ and $x_{01} \in \mathfrak{R}_{01}$, we have*

$$G(x_{11}+x_{01}) = G(x_{11}) + G(x_{01}).$$

Proof. Let us consider an arbitrary element $t_{01} \in \mathfrak{R}_{01}$. As $x_{11}t_{01} = 0$, using Lemma 2.2, we have

$$\{G(x_{11}+x_{01})\}t_{01} = \{G(x_{11}) + G(x_{01})\}t_{01}$$

which implies

$$\{G(x_{11}+x_{01}) - G(x_{11}) - G(x_{01})\}t_{01} = 0$$

As t_{01} is arbitrary, we can write

$$\{G(x_{11}+x_{01}) - G(x_{11}) - G(x_{01})\}\mathfrak{R}_{01} = 0$$

Now, using condition (C3), we have

$$G(x_{11}+x_{01}) = G(x_{11}) + G(x_{01}).$$

□

Lemma 2.5. *G is additive on $\mathfrak{R}_{01}$.*

Proof. Let $x_{01}, y_{01} \in \mathfrak{R}_{01}$. For an element $t_{01} \in \mathfrak{R}_{01}$, we observe that

$$\begin{aligned} \{G(x_{01}+y_{01})\}t_{01} &= G((x_{01}+y_{01})t_{01}) - (x_{01}+y_{01})D(t_{01}) \\ &= G(x_{01}t_{01}+y_{01}t_{01}) - (x_{01}+y_{01})D(t_{01}) \\ &= G(x_{01}t_{01}) + G(y_{01}t_{01}) - (x_{01}+y_{01})D(t_{01}) \\ &= G(x_{01})t_{01} + x_{01}D(t_{01}) + G(y_{01})t_{01} \\ &\quad + y_{01}D(t_{01}) - x_{01}D(t_{01}) - y_{01}D(t_{01}) \\ &= \{G(x_{01})+G(y_{01})\}t_{01} \end{aligned}$$

This implies that

$$\{G(x_{01}+y_{01})-G(x_{01})-G(y_{01})\}t_{01}=0$$

Since t_{01} is arbitrary, using Lemma 2.3 and condition (C3), we have

$$G(x_{01}+y_{01})=G(x_{01})+G(y_{01}).$$

□

Lemma 2.6. *G is additive on $\mathfrak{R}_{11}+\mathfrak{R}_{01}=\mathfrak{R}e$.*

Proof. Let us consider two elements $x_{11}+x_{01}, y_{11}+y_{01}\in\mathfrak{R}_{11}+\mathfrak{R}_{01}$ for all $x_{11}, y_{11}\in\mathfrak{R}_{11}$ and $x_{01}, y_{01}\in\mathfrak{R}_{01}$, by Lemmas 2.3, 2.4 and 2.5, we have

$$\begin{aligned} G((x_{11}+x_{01})+(y_{11}+y_{01})) &= G((x_{11}+y_{11})+(x_{01}+y_{01}))\\ &= G(x_{11}+y_{11})+G(x_{01}+y_{01})\\ &= G(x_{11})+G(y_{11})+G(x_{01})+G(y_{01})\\ &= G(x_{11}+x_{01})+G(y_{11}+y_{01}) \end{aligned}$$

Hence G is additive on $\mathfrak{R}e$. □

Lemma 2.7. *G is additive on $\mathfrak{R}e+\mathfrak{R}_{10}=\mathfrak{R}_{11}+\mathfrak{R}_{01}+\mathfrak{R}_{10}$.*

Proof. Let $x_{ij}\in\mathfrak{R}_{ij}$. For an element $t_{11}\in\mathfrak{R}_{11}$, as $x_{10}t_{11}=0$, using Lemmas 2.2 and 2.4, we find that

$$\begin{aligned} G(x_{11}+x_{01}+x_{10})t_{11} &= G(x_{10}+(x_{11}+x_{01}))t_{11}\\ &= G(x_{10})t_{11}+(G(x_{11}+x_{01}))t_{11}\\ &= G(x_{10})t_{11}+G(x_{11})t_{11}+G(x_{01})t_{11} \end{aligned}$$

which implies that

$$\{G(x_{11}+x_{01}+x_{10})-G(x_{11})-G(x_{01})-G(x_{10})\}t_{11}=0 \tag{2.1}$$

Again, consider an element $t_{01}\in\mathfrak{R}_{01}$. As $x_{11}t_{01}=0$, using Lemmas 2.2 and 2.3, we have

$$\begin{aligned} G(x_{11}+x_{01}+x_{10})t_{01} &= G(x_{11}+(x_{10}+x_{01}))t_{01}\\ &= G(x_{11})t_{01}+G(x_{10}+x_{01})t_{01}\\ &= G(x_{11})t_{01}+G((x_{10}+x_{01})t_{01})-(x_{10}+x_{01})D(t_{01})\\ &= G(x_{11})t_{01}+G(x_{10}t_{01}+x_{01}t_{01})-(x_{10}+x_{01})D(t_{01})\\ &= G(x_{11})t_{01}+G(x_{10}t_{01})+G(x_{01}t_{01})-(x_{10}+x_{01})D(t_{01})\\ &= G(x_{11})t_{01}+G(x_{10})t_{01}+x_{10}D(t_{01})+G(x_{01})t_{01}\\ &\quad +x_{01}D(t_{01})-(x_{10}+x_{01})D(t_{01})\\ &= G(x_{11})t_{01}+G(x_{10})t_{01}+G(x_{01})t_{01}. \end{aligned}$$

$$\{G(x_{11}+x_{01}+x_{10})-G(x_{11})-G(x_{01})-G(x_{10})\}t_{01}=0 \tag{2.2}$$

Now, combining equation (2.1) and (2.2), we get

$$\{G(x_{11}+x_{01}+x_{10})-G(x_{11})-G(x_{01})-G(x_{10})\}(t_{11}+t_{01})=0$$

As t_{11} and t_{01} are arbitrary, we have

$$\{G(x_{11}+x_{01}+x_{10})-G(x_{11})-G(x_{01})-G(x_{10})\}\mathfrak{R}e=0$$

Now, using condition (C1), we have

$$G(x_{11} + x_{01} + x_{10}) = G(x_{11}) + G(x_{01}) + G(x_{10}).$$

□

Now, we are in the position to state and prove our main theorem.

Theorem 2.8. *Let $\mathfrak{R}$ be an alternative ring having a non trivial idempotent e which satisfies the conditions (C1), (C2) and (C3). If g is any multiplicative generalized derivation on $\mathfrak{R}$, then g is additive.*

Proof. Let x and y be any two elements of $\mathfrak{R}$. Consider $G(x) + G(y)$. Take an element t in $\mathfrak{R}e = \mathfrak{R}_{11} + \mathfrak{R}_{01}$. Thus, xt and yt are the elements of $\mathfrak{R}e + \mathfrak{R}_{10}$.
Using lemma (2.7), we can obtain

$$\begin{aligned}
(G(x) + G(y))t &= G(x)t + G(y)t \\
&= G(xt) - xD(t) + G(yt) - yD(t) \\
&= G(xt) + G(yt) - (x + y)D(t) \\
&= G(xt + yt) - (x + y)D(t) \\
&= G((x + y)t) - (x + y)D(t) \\
&= G(x + y)t + (x + y)D(t) - (x + y)D(t) \\
&= G(x + y)t
\end{aligned}$$

Hence

$$\{G(x) + G(y) - G(x + y)\}t = 0$$

Now, since t is arbitrary, using condition (C1), we have

$$G(x + y) = G(x) + G(y)$$

Hence, G is additive on $\mathfrak{R}$. □

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