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A Note on Soft Union Γ-hypernear-rings

Sabahat Ali Khan, Mohammad Yahya Abbasi and Ahmad Raza

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Abstract In this paper, we study Γ-hypernear-ring by applying the concept of soft set theory. Here, we introduce the notion of soft union(briefly, S.U.) Γ-hypernear-ring and explore some properties using Γ-hypernear-ring theoretic concepts for soft sets. Also, we define the cross product of two S.U. Γ-hypernear-rings and proved that the cross product of two S.U. Γ-hypernear-rings is also an S.U. Γ-hypernear-ring.

1 Introduction

Problems related to uncertain data, unsure article, vagueness are almost impossible to solve by classical methods. Many authors have provided different methods like Fuzzy set theory, Hesitant fuzzy set theory and Rough set theory to solve such types of problems. But in all these methods there was a difficulty to define the membership value. To handle such kind of difficulty, Molodtsov [24] introduced a new mathematical tool known as soft set theory, which is a based on the parameterization concept. After that, Maji et al. [22] defined many operations on soft sets. Soft set theory have attracted the attention of many scientists, mathematicians and researchers in a very short period of time. Also, many authors have started to work on the soft algebraic structures. i.e. Feng et al. [17, 18] on semirings and semigroups, Aktas and Cagman [4, 8] on soft groups, Ma and Zhan [21], and Zhan et al. [31] on hemirings and so on.

Dickson and Leonard [15] introduced the notion of near-ring and near-field "fields with one distributive law" (near-fields) i.e. Ideal theory plays a central role in near-rings. However in general, ideal theory in near-rings do not coincide with the usual ideals of a ring. Satyanarayana [26] introduced Γ-near-ring and obtained some of their properties. Abdullah et al. [3] introduced soft int sub Γ-near-ring and soft int Γ-ideal and gave some applications of soft int Γ-near-ring to Γ-near-ring theory.

Marty [23] gave the idea of hyperstructures as a generalization of ordinary algebraic structures. Also, he published some notes on hypergroups and used these notes in different context: algebraic functions, rational functions and non-commutative groups. A broad study of hyperstructure theory have been found in (see [5], [9, 10, 11], [19], [29], [30]). The relationship between soft set theory and hyperstructures are studied by many authors (see [1, 2], [6], [16], [25]). The concept of hypernear-ring was introduced by Dasic [12] and, then Davvaz [14] defined Γ-hypernear-rings as a generalization of the concepts of near-rings, Γ-near-rings and hypernear-rings.

Motivated by the work of Sezgin et al. [27, 28], we have introduced soft union(briefly, S.U.) Γ-hypernear-ring and S.U. Γ-hyperideal of Γ-hypernear-ring. We explore some properties using Γ-hypernear-ring theoretic concepts for soft sets and showed that how a soft set effects on a Γ-hypernear-ring by means of union and inclusion of sets. Moreover, we have defined the cross product of two S.U. Γ-hypernear-rings and proved that the cross product of two S.U. Γ-hypernear-rings(S.U. Γ-hyperideals) is also a S.U. Γ-hypernear-ring(S.U. Γ-hyperideal).

1.1 Γ-Hypernear-ring

For the definition of hypergroupoid, hypernear-ring, Γ-hypernear-ring, hyperideal, soft sets, soft subsets, we refer to [9, 13, 12, 20, 14, 7, 24]. To start with, we need the following example.

Example 1.1. [20] Let $\mathbf{N} = \{0, a, b, c\}$ with a hyperoperation $'+'$ and a binary operation defined $'\cdot'$ as follows:

+	0	a	b	c
0	{0}	{a}	{b}	{c}
a	{a}	{0, a}	{b}	{c}
b	{b}	{b}	{0, a, c}	{b, c}
c	{c}	{c}	{b,c}	{0, a, b}

·	0	a	b	c
0	0	a	b	c
a	0	a	b	c
b	0	a	b	c
c	0	a	b	c

Then $(\mathbf{N}, +, \cdot)$ is a hypernear ring.

2 Soft Union Γ-Hypernear-rings

In this section, we introduce soft union(briefly, S.U.) Γ-hypernear-ring. Then, we define S.U. Γ-hyperideal of a Γ-hypernear-ring. Also, we have investigated their related properties using soft set operations.

Definition 2.1. A soft set $h_{\mathbf{N}}$ is said to be an S.U. Γ-hypernear-ring of $\mathbf{N}$ over $\mathcal{U}$ if the following conditions are satisfied:

(1) $\bigcup\limits_{\vartheta \in u\,+\,v} h_{\mathbf{N}}(\vartheta) \subseteq h_{\mathbf{N}}(u) \cup h_{\mathbf{N}}(v)$;

(2) $h_{\mathbf{N}}(-u) = h_{\mathbf{N}}(u)$;

(3) $h_{\mathbf{N}}(u\alpha v) \subseteq h_{\mathbf{N}}(v) \cup h_{\mathbf{N}}(v)$, $\forall\, u,\, v \in \mathbf{N}$ and $\alpha \in \Gamma$.

Example 2.2. Consider the Example 1.1. Let $\Gamma = \{\cdot\}$ and $\mathcal{U} = \{p, q, r, s\}$. Define a soft set $h_{\mathbf{N}} : \mathbf{N} \longrightarrow P(\mathcal{U})$ by

$$h_{\mathbf{N}}(0) = \{p, q\},\ h_{\mathbf{N}}(a) = \{p, q\},\ h_{\mathbf{N}}(b) = \{p, q, r\}$$
$$h_{\mathbf{N}}(c) = \{p, q, r\}.$$

Then we can verify that $h_{\mathbf{N}}$ is an S.U. Γ-hypernear-ring of $\mathbf{N}$ over $\mathcal{U}$.

Lemma 2.3. *Let $h_{\mathbf{N}}$ be an S.U. Γ-hypernear-ring of $\mathbf{N}$ over $\mathcal{U}$. Then $h_{\mathbf{N}}(0) \subseteq h_{\mathbf{N}}(u)$ for all $u \in \mathbf{N}$.*

Proof. Proof is straightforward. □

Theorem 2.4. *Let $h_{\mathbf{N}}$ be a soft set of $\mathbf{N}$ over $\mathcal{U}$. Then, $h_{\mathbf{N}}$ is an S.U. Γ-hypernear-ring over $\mathcal{U}$ if and only if*

(1) $\bigcup\limits_{\vartheta \in (u-v)} h_{\mathbf{N}}(\vartheta) \subseteq h_{\mathbf{N}}(u) \cup h_{\mathbf{N}}(v)$

(2) $h_{\mathbf{N}}(u\alpha v) \subseteq h_{\mathbf{N}}(u) \cup h_{\mathbf{N}}(v)$, $\forall\, u,\, v \in \mathbf{N}$ *and* $\alpha \in \Gamma$.

Proof. Let $h_{\mathbf{N}}$ be an S.U. Γ-hypernear-ring over $\mathcal{U}$. Then $h_{\mathbf{N}}(u\alpha v) \subseteq h_{\mathbf{N}}(u) \cup h_{\mathbf{N}}(v)$ and

$$\begin{aligned} \bigcup_{\vartheta \in (u-v)} h_{\mathbf{N}}(\vartheta) &\subseteq h_{\mathbf{N}}(u) \cup h_{\mathbf{N}}(-v) \\ &= h_{\mathbf{N}}(u) \cup h_{\mathbf{N}}(v) \end{aligned}$$

$\forall\, u,\, v \in \mathbf{N}$ and $\alpha \in \Gamma$. □

Theorem 2.5. *Let $h_{\mathbf{N}}$ be an S.U. Γ-hypernear-ring of $\mathbf{N}$ over $\mathcal{U}$. If $\bigcup\limits_{\vartheta \in (u+v)} h_{\mathbf{N}}(\vartheta) = h_{\mathbf{N}}(0)$ for all $u, v \in \mathbf{N}$. Then $h_{\mathbf{N}}(u) = h_{\mathbf{N}}(v)$.*

Proof. Suppose that $h_{\mathbf{N}}$ is an S.U. Γ-hypernear-ring of $\mathbf{N}$ over $\mathcal{U}$ and $\bigcup_{\vartheta\in(u+v)} h_{\mathbf{N}}(\vartheta) = h_{\mathbf{N}}(0)$ for all $u, v \in \mathbf{N}$. Then, we have

$$\begin{aligned} h_{\mathbf{N}}(u) &\subseteq \bigcup_{\vartheta\in(0+u)} h_{\mathbf{N}}(\vartheta) \\ &\subseteq \bigcup_{\vartheta\in(u+v-v)} h_{\mathbf{N}}(\vartheta) \\ &= \bigcup_{\vartheta\in((u+v)-v)} h_{\mathbf{N}}(\vartheta) \\ &\subseteq \bigcup_{\vartheta\in(u+v)} h_{\mathbf{N}}(\vartheta) \cup h_{\mathbf{N}}(v) \\ &= h_{\mathbf{N}}(0) \cup h_{\mathbf{N}}(v) \\ &= h_{\mathbf{N}}(v) \end{aligned}$$

and

$$\begin{aligned} h_{\mathbf{N}}(v) &\subseteq \bigcup_{\vartheta\in(0+v)} h_{\mathbf{N}}(\vartheta) \\ &\subseteq \bigcup_{\vartheta\in((-u+u)+v)} h_{\mathbf{N}}(\vartheta) \\ &= \bigcup_{\vartheta\in(-u+(u+v)} h_{\mathbf{N}}(\vartheta) \\ &\subseteq h_{\mathbf{N}}(-u) \cup \bigcup_{\vartheta\in(u+v)} h_{\mathbf{N}}(\vartheta) \\ &= h_{\mathbf{N}}(-u) \cup h_{\mathbf{N}}(0) \\ &= h_{\mathbf{N}}(-u) \\ &= h_{\mathbf{N}}(u). \end{aligned}$$

Therefore, $h_{\mathbf{N}}(u) = h_{\mathbf{N}}(v)$. □

Corollary 2.6. *Let $h_{\mathbf{N}}$ be an S.U. Γ-hypernear-ring of $\mathbf{N}$ over $\mathcal{U}$. If $\bigcup_{\vartheta\in(u-v)} h_{\mathbf{N}}(\vartheta) = h_{\mathbf{N}}(0)$ for all $u, v \in \mathbf{N}$. Then $h_{\mathbf{N}}(u) = h_{\mathbf{N}}(v)$.*

Theorem 2.7. *Let $h_{\mathbf{N}}$ be an S.U. Γ-hypernear-ring of $\mathbf{N}$ over $\mathcal{U}$. Then for all $u \in \mathbf{N}$*

$$h_{\mathbf{N}}(u) = h_{\mathbf{N}}(0) \textit{ if and only if } \bigcup_{\vartheta\in(u+v)} h_{\mathbf{N}}(\vartheta) = \bigcup_{\vartheta\in(v+u)} h_{\mathbf{N}}(\vartheta) = h_{\mathbf{N}}(v), \forall\, v \in \mathbf{N}.$$

Proof. Assume $\bigcup_{\vartheta\in(u+v)} h_{\mathbf{N}}(\vartheta) = \bigcup_{\vartheta\in(v+u)} h_{\mathbf{N}}(\vartheta) = h_{\mathbf{N}}(v)\ \forall\, v \in \mathbf{N}$. By putting $v = 0$, we have $\bigcup_{\vartheta\in(u+0)} h_{\mathbf{N}}(\vartheta) = h_{\mathbf{N}}(0)$. It implies $h_{\mathbf{N}}(u) = h_{\mathbf{N}}(0)$.

Conversely, suppose that $h_{\mathbf{N}}(u) = h_{\mathbf{N}}(0)$. By Lemma 2.3, we have $h_{\mathbf{N}}(0) \subseteq h_{\mathbf{N}}(u) \subseteq h_{\mathbf{N}}(v)$ $\forall\, v \in \mathbf{N}$. Thus, we have

$$\begin{aligned} \bigcup_{\vartheta\in(u\,+\,v)} h_{\mathbf{N}}(\vartheta) &\subseteq h_{\mathbf{N}}(u) \cup h_{\mathbf{N}}(v) \\ &\subseteq h_{\mathbf{N}}(v)\ \forall\, v \in \mathbf{N}. \end{aligned} \tag{2.1}$$

Now

$$\begin{aligned} h_{\mathbf{N}}(v) &\subseteq \bigcup_{\vartheta\in(0\,+\,v)} h_{\mathbf{N}}(\vartheta) \\ &\subseteq \bigcup_{\vartheta\in(-u+u)\,+\,v} h_{\mathbf{N}}(\vartheta) \\ &= \bigcup_{\vartheta\in-u\,+\,(u\,+\,v)} h_{\mathbf{N}}(\vartheta) \\ &\subseteq h_{\mathbf{N}}(-u) \cup \bigcup_{\vartheta\in u\,+\,v} h_{\mathbf{N}}(\vartheta) \\ &\subseteq h_{\mathbf{N}}(u) \cup \bigcup_{\vartheta\in u\,+\,v} h_{\mathbf{N}}(\vartheta). \end{aligned} \tag{2.2}$$

As $h_{\mathbf{N}}(u) \subseteq h_{\mathbf{N}}(v)\ \forall\ v \in \mathbf{N}$. It implies that $h_{\mathbf{N}}(u) \subseteq h_{\mathbf{N}}(\vartheta)\ \forall\ \vartheta \in u\ +\ v$. Therefore, $h_{\mathbf{N}}(u) \subseteq \bigcup_{\vartheta\in u\,+\,v} h_{\mathbf{N}}(\vartheta)$. Thus, from (2.2)

$$h_{\mathbf{N}}(v) \subseteq \bigcup_{\vartheta\in u\,+\,v} h_{\mathbf{N}}(\vartheta). \tag{2.3}$$

Now from (2.1) and (2.3), we have

$$h_{\mathbf{N}}(v) = \bigcup_{\vartheta\in u\,+\,v} h_{\mathbf{N}}(\vartheta). \tag{2.4}$$

Also, we have

$$\begin{aligned} \bigcup_{\vartheta\in(v\,+\,u)} h_{\mathbf{N}}(\vartheta) &\subseteq \bigcup_{\vartheta\in(v\,+\,u)\,+\,0} h_{\mathbf{N}}(\vartheta) \\ &\subseteq \bigcup_{\vartheta\in(v\,+\,u)\,+\,(v-v)} h_{\mathbf{N}}(\vartheta) \\ &= \bigcup_{\vartheta\in(v\,+\,(u\,+\,v))\,-\,v} h_{\mathbf{N}}(\vartheta) \\ &\subseteq h_{\mathbf{N}}(v) \cup \bigcup_{\vartheta\in u\,+\,v} h_{\mathbf{N}}(\vartheta) \cup h_{\mathbf{N}}(v) \\ &= h_{\mathbf{N}}(v) \cup \bigcup_{\vartheta\in u\,+\,v} h_{\mathbf{N}}(\vartheta) \\ &= h_{\mathbf{N}}(v). \end{aligned}$$

and

$$\begin{aligned} h_{\mathbf{N}}(v) &\subseteq \bigcup_{\vartheta\in v\,+\,0} h_{\mathbf{N}}(\vartheta) \\ &= \bigcup_{\vartheta\in v\,+\,(u\,-\,u)} h_{\mathbf{N}}(\vartheta) \\ &= \bigcup_{\vartheta\in(v\,+\,u)\,-\,u} h_{\mathbf{N}}(\vartheta) \\ &\subseteq \bigcup_{\vartheta\in v\,+\,u} h_{\mathbf{N}}(\vartheta) \cup h_{\mathbf{N}}(u) \\ &= \bigcup_{\vartheta\in v\,+\,u} h_{\mathbf{N}}(\vartheta). \end{aligned}$$

Therefore,

$$h_{\mathbf{N}}(v) = \bigcup_{\vartheta\in v\,+\,u} h_{\mathbf{N}}(\vartheta). \tag{2.5}$$

From (2.4) and (2.5), we have $\bigcup_{\vartheta\in u\,+\,v} h_{\mathbf{N}}(\vartheta) = \bigcup_{\vartheta\in v\,+\,u} h_{\mathbf{N}}(\vartheta) = h_{\mathbf{N}}(v)\ \forall\ v \in \mathbf{N}$. □

Definition 2.8. Let $h_{\mathbf{N}}$ be an S.U. Γ-hypernear-ring of $\mathbf{N}$ over $\mathcal{U}$. Then $h_{\mathbf{N}}$ is called an S.U. Γ-hyperideal of $\mathbf{N}$ over $\mathcal{U}$ if it satisfies the following conditions:

(1) $\bigcup_{\vartheta \in u + v - u} h_{\mathbf{N}}(\vartheta) \subseteq h_{\mathbf{N}}(v)$,

(2) $h_{\mathbf{N}}(u\gamma v) \subseteq h_{\mathbf{N}}(u)$, and

(3) $\bigcup_{\vartheta \in (u\alpha(v + w) - u\beta v)} h_{\mathbf{N}}(\vartheta) \subseteq h_{\mathbf{N}}(w), \forall\, u, v, w \in \mathbf{N}$ and $\alpha, \beta, \gamma \in \Gamma$.

If $h_{\mathbf{N}}$ is an S.U. Γ-hypernear-ring of $\mathbf{N}$ over $\mathcal{U}$ such that $h_{\mathbf{N}}$ satisfied the condition (1) and (2), then $h_{\mathbf{N}}$ is called an S.U. right hyperideal of $\mathbf{N}$ over $\mathcal{U}$ and if $h_{\mathbf{N}}$ satisfied the condition (1) and (3), then $h_{\mathbf{N}}$ is called an S.U. left hyperideal of $\mathbf{N}$ over $\mathcal{U}$.

Theorem 2.9. *If $h_{\mathbf{N}}$, $k_{\mathbf{N}}$ are two S.U. Γ-hypernear-rings over $\mathcal{U}$. Then $h_{\mathbf{N}} \tilde{\bigcup} k_{\mathbf{N}}$ is an S.U. Γ-hypernear-ring over $\mathcal{U}$.*

Proof. Let $x, y \in \mathbf{N}$ and $\alpha \in \Gamma$. Then,

$$\begin{aligned}
\bigcup_{\vartheta \in u-v} (h_{\mathbf{N}} \tilde{\bigcup} k_{\mathbf{N}})(\vartheta) &= \bigcup_{\vartheta \in u-v} \big[h_{\mathbf{N}}(\vartheta) \cup k_{\mathbf{N}}(\vartheta)\big] \\
&\subseteq \bigcup_{\vartheta \in u-v} h_{\mathbf{N}}(\vartheta) \cup \bigcup_{\vartheta \in u-v} k_{\mathbf{N}}(\vartheta) \\
&\subseteq \big[h_{\mathbf{N}}(u) \cup h_{\mathbf{N}}(v)\big] \cup \big[k_{\mathbf{N}}(u) \cup k_{\mathbf{N}}(v)\big] \\
&= \big[h_{\mathbf{N}}(u) \cup k_{\mathbf{N}}(u)\big] \cup \big[h_{\mathbf{N}}(v) \cup k_{\mathbf{N}}(v)\big] \\
&= \big[(h_{\mathbf{N}} \tilde{\bigcup} k_{\mathbf{N}})(u)\big] \cup \big[(h_{\mathbf{N}} \tilde{\bigcup} k_{\mathbf{N}})(v)\big]
\end{aligned}$$

and

$$\begin{aligned}
(h_{\mathbf{N}} \tilde{\bigcup} k_{\mathbf{N}})(u\alpha v) &= h_{\mathbf{N}}(u\alpha v) \cup k_{\mathbf{N}}(u\alpha v) \\
&\subseteq \big[h_{\mathbf{N}}(u) \cup h_{\mathbf{N}}(v)\big] \cup \big[k_{\mathbf{N}}(u) \cup k_{\mathbf{N}}(v)\big] \\
&= \big[h_{\mathbf{N}}(u) \cup k_{\mathbf{N}}(u)\big] \cup \big[h_{\mathbf{N}}(u) \cup k_{\mathbf{N}}(v)\big] \\
&= \big[(h_{\mathbf{N}} \tilde{\bigcup} k_{\mathbf{N}})(u)\big] \cup \big[(h_{\mathbf{N}} \tilde{\bigcup} k_{\mathbf{N}})(v)\big].
\end{aligned}$$

Therefore, $h_{\mathbf{N}} \tilde{\bigcup} k_{\mathbf{N}}$ is an S.U. Γ-hypernear-ring over $\mathcal{U}$. □

Theorem 2.10. *If $h_{\mathbf{N}}$ and $k_{\mathbf{N}}$ are S.U. Γ-hyperideals of $\mathbf{N}$ over $\mathcal{U}$. Then $h_{\mathbf{N}} \tilde{\bigcup} k_{\mathbf{N}}$ is an S.U. Γ-hyperideal of $\mathbf{N}$ over $\mathcal{U}$.*

Proof. Proof is straightforward. □

Theorem 2.11. *If $h_{\mathbf{N}}$ is an S.U. Γ-hyperideal of $\mathbf{N}$ over $\mathcal{U}$, then $\mathbf{N}_h = \{u \in \mathbf{N} : h_{\mathbf{N}}(u) = h_{\mathbf{N}}(0)\}$ is a Γ-hyperideal of $\mathbf{N}$.*

Proof. $\mathbf{N}_h$ is non-empty, since $0 \in \mathbf{N}_h$. Now, we claim that $\mathbf{N}_h$ is a Γ-hyperideal of $\mathbf{N}$. To prove our claim, we have to show that

1. $\mathbf{N}_h$ is a sub-hypergroup of $\mathbf{N}$,
2. $n + u - n \subseteq \mathbf{N}_h$,
3. $u\alpha n \in \mathbf{N}_h$ and
4. $n\alpha(s + u) - n\beta s \subseteq \mathbf{N}_h$.

Let $u, v \in \mathbf{N}_h$, then $h_{\mathbf{N}}(u) = h_{\mathbf{N}}(v) = h_{\mathbf{N}}(0)$. By Lemma 2.3, $h_{\mathbf{N}}(0) \subseteq \bigcup_{\vartheta \in u-v} h_{\mathbf{N}}(\vartheta)$, $h_{\mathbf{N}}(0) \subseteq \bigcup_{\vartheta \in n + u - n} h_{\mathbf{N}}(\vartheta)$, $h_{\mathbf{N}}(0) \subseteq h_{\mathbf{N}}(u\alpha n)$ and $h_{\mathbf{N}}(0) \subseteq \bigcup_{\vartheta \in (n\alpha(s + u) - n\beta s)} h_{\mathbf{N}}(\vartheta)$ for all $u, v \in \mathbf{N}_h$, $n, s \in \mathbf{N}$ and $\alpha, \beta \in \Gamma$. As $h_{\mathbf{N}}$ is an S.U. Γ-hyperideal of $\mathbf{N}$ over $\mathcal{U}$, therefore for all $u, v \in \mathbf{N}_h$ and $n, s \in \mathbf{N}$, we have
(1). $\bigcup_{\vartheta \in u-v} h_{\mathbf{N}}(\vartheta) \subseteq h_{\mathbf{N}}(u) \cap h_{\mathbf{N}}(v) = h_{\mathbf{N}}(0)$, (2). $\bigcup_{\vartheta \in n + u - n} h_{\mathbf{N}}(\vartheta) \subseteq h_{\mathbf{N}}(u) = h_{\mathbf{N}}(0)$,
(3). $h_{\mathbf{N}}(u\alpha n) \subseteq h_{\mathbf{N}}(n) = h_{\mathbf{N}}(0)$ and (4). $\bigcup_{\vartheta \in (n\alpha(s + u) - n\beta s)} h_{\mathbf{N}}(\vartheta) \subseteq h_{\mathbf{N}}(u) = h_{\mathbf{N}}(0)$.

Therefore,

1. $\bigcup_{\vartheta \in u-v} h_{\mathbf{N}}(\vartheta) = h_{\mathbf{N}}(0)$,

2. $\bigcup_{\vartheta\in n\,+\,u\,-\,n} h_{\mathbf{N}}(\vartheta) = h_{\mathbf{N}}(0)$,
3. $h_{\mathbf{N}}(u\alpha n) = h_{\mathbf{N}}(0)$ and
4. $\bigcup_{\vartheta\in(n\alpha(s\,+\,u)\,-\,n\beta s)} h_{\mathbf{N}}(\vartheta) = h_{\mathbf{N}}(0)$.

Hence, $\mathbf{N}_h$ is an Γ-hyperideal of $\mathbf{N}$. □

Definition 2.12. Let $h_{\mathbf{N}}$ be a soft set of $\mathbf{N}$ over $\mathcal{U}$. Then the set $U(h_{\mathbf{N}}, \delta) = \{u \in \mathbf{N} : h_{\mathbf{N}}(u) \subseteq \delta\}$, where $\delta \subseteq \mathcal{U}$, is called lower δ-inclusion of $h_{\mathbf{N}}$.

Theorem 2.13. *Let* $\mathbf{N}$ *be* Γ*-hypernear-ring and* $h_{\mathbf{N}}$ *a soft set of* $\mathbf{N}$ *over* $\mathcal{U}$*, and* δ *be a subset of* $\mathcal{U}$ *such that* $\emptyset \subseteq \delta \subseteq h_{\mathbf{N}}(0)$*. If* $h_{\mathbf{N}}$ *is an S.U.* Γ*-hyperideal of* $\mathbf{N}$ *over* $\mathcal{U}$*, then* $U(h_{\mathbf{N}}, \delta)$ *is a* Γ*-hyperideal of* $\mathbf{N}$.

Proof. As $h_{\mathbf{N}}(0) \subseteq \delta$, then $0 \in U(h_{\mathbf{N}}, \delta)$ and $\emptyset \neq U(h_{\mathbf{N}}, \delta) \subseteq \mathbf{N}$. If $u, v \in U(h_{\mathbf{N}}, \delta)$, then $h_{\mathbf{N}}(u) \subseteq \delta$ and $h_{\mathbf{N}}(v) \subseteq \delta$. We have to prove that (1) $u - v \subseteq U(h_{\mathbf{N}}, \delta)$, (2) $n + u - n \subseteq U(h_{\mathbf{N}}, \delta)$, (3) $u\alpha n \in U(h_{\mathbf{N}}, \delta)$ and (4) $n\alpha(s + u) - n\beta s \subseteq U(h_{\mathbf{N}}, \delta)$ for all $u, v \in U(h_{\mathbf{N}}, \delta)$, $n, s \in N$ and $\alpha, \beta \in \Gamma$. Now, $h_{\mathbf{N}}$ is an S.U. Γ-hyperideal of $\mathbf{N}$ over $\mathcal{U}$, so (1) $\bigcup_{\vartheta\in u-v} h_{\mathbf{N}}(\vartheta) \subseteq h_{\mathbf{N}}(u) \cup h_{\mathbf{N}}(v) \subseteq \delta \cup \delta$ implies $u - v \subseteq U(h_{\mathbf{N}}$, (2) $\bigcup_{\vartheta\in n\,+\,u\,-\,n} h_{\mathbf{N}}(\vartheta) \subseteq h_{\mathbf{N}}(u) \subseteq \delta$ implies $n + u - n \subseteq U(h_{\mathbf{N}}, \delta)$, (3) $h_{\mathbf{N}}(u\alpha n) \subseteq h_{\mathbf{N}}(n) \subseteq \delta$ implies $u\alpha n \in U(h_{\mathbf{N}}, \delta)$ and (4) $\bigcup_{\vartheta\in(n\alpha(s\,+\,u)\,-\,n\beta s)} h_{\mathbf{N}}(\vartheta) \subseteq h_{\mathbf{N}}(u) \subseteq \delta$ implies $n\alpha(s + u) - n\beta s \subseteq U(h_{\mathbf{N}}, \delta)$ for all $u, v \in U(h_{\mathbf{N}}, \delta)$, $n, s \in N$ and $\alpha, \beta \in \Gamma$. Hence, $U(h_{\mathbf{N}}, \delta)$ is a Γ-hyperideal of $\mathbf{N}$. □

Theorem 2.14. *Let* $(\mathbf{M}, +_1, \Gamma_1)$ *and* $(\mathbf{N}, +_2, \Gamma_2)$ *be two* Γ*-hypernear-rings. If the hyperoperation* $\bigoplus$ *and operation* $\bigodot$ *on* $\mathbf{M} \times \mathbf{N}$ *are defined as*
(1) $(u_1, v_1) \bigoplus (u_2, v_2) = \{(u, v) : u \in u_1 +_1 u_2, v \in v_1 +_2 v_2\}$, $\forall (u_1, v_1)$ *and* $(u_2, v_2) \in \mathbf{M} \times \mathbf{N}$
(2) $(u_1, v_1) \bigodot (u_2, v_2) = (u_1\alpha_1 u_2, v_1\alpha_2 v_2)$. $\forall \alpha_1 \in \Gamma_1$ *and* $\alpha_2 \in \Gamma_2$.
Then their product $\mathbf{M} \times \mathbf{N}$ *is a* Γ*-hypernear-ring.*

Proof. Proof is straightforward. □

Definition 2.15. Let $\mathbf{N}$, $\mathbf{M}$ be two Γ-hypernear-ring and $g_{\mathbf{N}}$ an S.U. Γ-hypernear-ring of $\mathbf{N}$ over $\mathcal{U}$, $k_{\mathbf{M}}$ an S.U. Γ-hypernear-rings of $\mathbf{M}$ over $\mathcal{U}$. Then the cross product of $g_{\mathbf{N}}$ and $k_{\mathbf{M}}$ is defined as $h_{\mathbf{N}\times\mathbf{M}} = g_{\mathbf{N}} \times k_{\mathbf{M}}$, where $h_{\mathbf{N}\times\mathbf{M}}(u, v) = g_{\mathbf{N}}(u) \times k_{\mathbf{M}}(v)$ for all $(u, v) \in \mathbf{N} \times \mathbf{M}$.

Theorem 2.16. *If* $g_{\mathbf{N}}$ *is an S.U.* Γ*-hypernear-ring of* $\mathbf{N}$ *over* $\mathcal{U}$ *and* $k_{\mathbf{M}}$ *is an S.U.* Γ*-hypernear-ring of* $\mathbf{M}$ *over* $\mathcal{U}$*. Then the cross product* $h_{\mathbf{N}\times\mathbf{M}}$ *is an S.U.* Γ*-hypernear-ring of* $\mathbf{N} \times \mathbf{M}$ *over* $\mathcal{U} \times \mathcal{U}$.

Proof. Let $(u_1, v_1), (u_2, v_2) \in \mathbf{N} \times \mathbf{M}$ and $\alpha_1 \in \Gamma_1$, $\alpha_2 \in \Gamma_2$. Then

$$\begin{aligned}
\bigcup_{(\vartheta_1,\vartheta_2)\in(u_1,v_1)\,\ominus\,(u_2,v_2)} h_{\mathbf{N}\times\mathbf{M}}(\vartheta_1, \vartheta_2) &= \bigcup_{(\vartheta_1,\vartheta_2)\in(u_1\,-_1\,u_2)\,\times\,(v_1\,-_2\,v_2)} h_{\mathbf{N}\times\mathbf{M}}(\vartheta_1, \vartheta_2) \\
&= \bigcup_{\vartheta_1\in(u_1\,-_1\,u_2),\vartheta_2\in(v_1\,-_2\,v_2)} g_{\mathbf{N}}(\vartheta_1) \times k_{\mathbf{M}}(\vartheta_2) \\
&= \bigcup_{\vartheta_1\in(u_1\,-_1\,u_2)} g_{\mathbf{N}}(\vartheta_1) \times \bigcup_{\vartheta_2\in(v_1\,-_2\,v_2)} k_{\mathbf{M}}(\vartheta_2) \\
&\subseteq \big[g_{\mathbf{N}}(u_1) \cup g_{\mathbf{N}}(u_2)\big] \times \big[k_{\mathbf{M}}(v_1) \cup k_{\mathbf{M}}(v_2)\big] \\
&= \big[g_{\mathbf{N}}(u_1) \times k_{\mathbf{M}}(v_1)\big] \cup \big[g_{\mathbf{N}}(u_2) \times k_{\mathbf{M}}(v_2)\big] \\
&= h_{\mathbf{N}\times\mathbf{M}}(u_1, v_1) \cup h_{\mathbf{N}\times\mathbf{M}}(u_2, v_2).
\end{aligned}$$

and

$$\begin{aligned}
h_{\mathbf{N}\times\mathbf{M}}((u_1, v_1) \bigodot (u_2, v_2)) &= h_{\mathbf{N}\times\mathbf{M}}(u_1\alpha_1 u_2, v_1\alpha_2 v_2) \\
&= g_{\mathbf{N}}(u_1\alpha_1 u_2) \times k_{\mathbf{M}}(v_1\alpha_2 v_2) \\
&\subseteq \big[g_{\mathbf{N}}(u_1) \cup g_{\mathbf{N}}(u_2)\big] \times \big[k_{\mathbf{M}}(v_1) \cup k_{\mathbf{M}}(v_2)\big] \\
&= \big[g_{\mathbf{N}}(u_1) \times k_{\mathbf{M}}(v_1)\big] \cup \big[g_{\mathbf{N}}(u_2) \times k_{\mathbf{M}}(v_2)\big] \\
&= h_{\mathbf{N}\times\mathbf{M}}(u_1, v_1) \cup h_{\mathbf{N}\times\mathbf{M}}(u_2, v_2).
\end{aligned}$$

Hence, $h_{\mathbf{N}\times\mathbf{M}}$ is an S.U. Γ-hypernear-ring of $\mathbf{N} \times \mathbf{M}$ over $\mathcal{U} \times \mathcal{U}$. □

Definition 2.17. Let $\mathbf{N}$, $\mathbf{M}$ be two Γ-hypernear-rings. Let $g_{\mathbf{N}}$ be an S.U. Γ-hyperideal of $\mathbf{N}$ over $\mathcal{U}$ and $k_{\mathbf{M}}$ an S.U. Γ-hyperideal of $\mathbf{M}$ over $\mathcal{U}$. Then the cross product of $g_{\mathbf{N}}$ and $k_{\mathbf{M}}$ is defined as $h_{\mathbf{N}\times\mathbf{M}} = g_{\mathbf{N}} \times k_{\mathbf{M}}$, where $h_{\mathbf{N}\times\mathbf{M}}(u,v) = g_{\mathbf{N}}(u) \times k_{\mathbf{M}}(v)$ for all $(u,v) \in \mathbf{N} \times \mathbf{M}$.

Theorem 2.18. *If $g_{\mathbf{N}}$ is an S.U. Γ-hyperideal of $\mathbf{N}$ over $\mathcal{U}$ and $k_{\mathbf{M}}$ is an S.U. Γ-hyperideal of $\mathbf{M}$ over $\mathcal{U}$. Then the cross product $h_{\mathbf{N}\times\mathbf{M}}$ is an S.U. Γ-hyperideal of $\mathbf{N}\times\mathbf{M}$ over $\mathcal{U}\times\mathcal{U}$.*

Proof. Let $g_{\mathbf{N}}$ be an S.U. Γ-hyperideal of $\mathbf{N}$ over $\mathcal{U}$ and $k_{\mathbf{M}}$ be an S.U. Γ-hyperideal of $\mathbf{M}$ over $\mathcal{U}$. By Theorem 2.16, the cross product $h_{\mathbf{N}\times\mathbf{M}}$ is an S.U. Γ-hypernear-ring of $\mathbf{N}\times\mathbf{M}$ over $\mathcal{U}\times\mathcal{U}$. Now suppose (u_1,v_1), (u_2,v_2), $(x_3,y_3) \in \mathbf{N}\times\mathbf{M}$ and $\alpha_1 \in \Gamma_1$, $\alpha_2 \in \Gamma_2$. Then

$$\begin{aligned}
&\bigcup_{(\vartheta_1,\vartheta_2)\in(u_1,v_1)\,\bigoplus\,(u_2,v_2)\,\ominus\,(u_1,v_1)} h_{\mathbf{N}\times\mathbf{M}}(\vartheta_1,\vartheta_2) = \\
&\bigcup_{(\vartheta_1,\vartheta_2)\in(u_1\,+_1\,u_2\,-_1\,u_1)\,\times\,(v_1\,+_2\,v_2\,-_2\,v_1)} h_{\mathbf{N}\times\mathbf{M}}(\vartheta_1,\vartheta_2) \\
&= \bigcup_{\vartheta_1\in(u_1\,+_1\,u_2\,-_1\,u_1),\vartheta_2\in(v_1\,+_2\,v_2\,-_2\,v_1)} g_{\mathbf{N}}(\vartheta_1)\times k_{\mathbf{M}}(\vartheta_2) \\
&= \bigcup_{\vartheta_1\in(u_1\,+_1\,u_2\,-_1\,u_1)} g_{\mathbf{N}}(\vartheta_1)\times \bigcup_{\vartheta_2\in(v_1\,+_2\,v_2\,-_2\,v_1)} k_{\mathbf{M}}(\vartheta_2) \\
&\subseteq g_{\mathbf{N}}(u_2)\times k_{\mathbf{M}}(v_2) \\
&= h_{\mathbf{N}\times\mathbf{M}}(u_2,v_2),
\end{aligned}$$

$$\begin{aligned}
h_{\mathbf{N}\times\mathbf{M}}((u_1,v_1)\bigodot(u_2,v_2)) &= h_{\mathbf{N}\times\mathbf{M}}(u_1\alpha_1u_2, v_1\alpha_2v_2) \\
&= g_{\mathbf{N}}(u_1\alpha_1u_2)\times k_{\mathbf{M}}(v_1\alpha_2v_2) \\
&\subseteq g_{\mathbf{N}}(u_1)\times k_{\mathbf{M}}(v_1) \\
&= h_{\mathbf{N}\times\mathbf{M}}(u_1,v_1).
\end{aligned}$$

and

$$\begin{aligned}
&\bigcup_{(\vartheta_1,\vartheta_2)\in((u_1,v_1)\,\bigodot\,((u_2,v_2)\,\bigoplus\,(x_3,y_3))\,\ominus\,(u_1,v_1)\,\bigodot\,(u_2,v_2))} h_{\mathbf{N}\times\mathbf{M}}(\vartheta_1,\vartheta_2) = \\
&\bigcup_{(\vartheta_1,\vartheta_2)\in(u_1\,\alpha_1\,(u_2\,+_1\,x_3)\,-_1\,u_1\cdot_1u_2)\,\times\,(v_1\,\alpha_2\,(v_2\,+_2\,y_3)\,-_2\,v_1\alpha_2v_2)} h_{\mathbf{N}\times\mathbf{M}}(\vartheta_1,\vartheta_2) \\
&= \bigcup_{\vartheta_1\in(u_1\,\alpha_1\,(u_2\,+_1\,x_3)\,-_1\,u_1\alpha_1u_2),\vartheta_2\in(v_1\,\alpha_2\,(v_2\,+_2\,y_3)\,-_2\,v_1\alpha_2v_2)} g_{\mathbf{N}}(\vartheta_1)\times k_{\mathbf{M}}(\vartheta_2) \\
&= \bigcup_{\vartheta_1\in(u_1\,\alpha_1\,(u_2\,+_1\,x_3)\,-_1\,u_1\alpha_1u_2)} g_{\mathbf{N}}(\vartheta_1)\times \bigcup_{\vartheta_2\in(v_1\,\alpha_2\,(v_2\,+_2\,y_3)\,-_2\,v_1\alpha_2v_2)} k_{\mathbf{M}}(\vartheta_2) \\
&\subseteq g_{\mathbf{N}}(x_3)\times k_{\mathbf{M}}(y_3) \\
&= h_{\mathbf{N}\times\mathbf{M}}(x_3,y_3).
\end{aligned}$$

Hence, $h_{\mathbf{N}\times\mathbf{M}}$ is an S.U. Γ-hyperideal of $\mathbf{N}\times\mathbf{M}$ over $\mathcal{U}\times\mathcal{U}$. □

Conclusion: We have studied S.U. Γ-hypernear-ring and defined some properties using Γ-hypernear-ring theoretic concepts for soft sets. Moreover, we have introduced cross product of two S.U. Γ-hypernear-rings and proved that the cross product of two S.U. Γ-hypernear-rings is an S.U. Γ-hypernear-ring. By using the results of this paper, some properties on the Γ-hypernear-ring using fuzzy set theory and soft set theory can be investigated.

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Author information

Sabahat Ali Khan, Department of Mathematics, Jamia Millia Islamia, New Delhi-110025, India.
E-mail: khansabahat361@gmail.com

Mohammad Yahya Abbasi, Department of Mathematics, Jamia Millia Islamia, New Delhi-110025, India.
E-mail: mabbasi@jmi.ac.in

Ahmad Raza, Department of Mathematics, Jamia Millia Islamia, New Delhi-110025, India.
E-mail: arhanraza@gmail.com

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