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On FI-Projective Modules and Dimensions

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Abstract In this paper, we introduce the FI-projective module and the FI-projective dimensions of modules and rings. An R-module M is called FI-projective if $Ext_R^1(M, F) = 0$ for any FP-projective module F. We show that a module is FI-projective if and only if it is a cokernel of an FP-projective preenvelope $f : A \to B$ with B is projective. Also we see that rings over which all modules are FI-projective are exactly the semisimple rings. Furthermore, we study the FI-projective dimension of modules over short exact sequences and we characterize the hereditary ring using the FI-projective dimension of rings. Finally, we see the relation between these dimensions and other homological dimensions.

1 Introduction

Throughout this paper all rings are commutative with identity element and all modules are unital. For an R-module M, we denote by $pd_R(M)$, $id_R(M)$ and $fd_R(M)$, the usual projective, injective and flat dimension of M respectively, and $gdim(R)$ $wdim(R)$ are the classical global and weak (global) dimension of R.

We first recall some known notions and facts needed in the sequel. Let R be a ring, an R-module F is called FP-projective [6], if $Ext_R^1(F, B) = 0$ for any FP-injective R-module B. The FP-projective dimension of F, denoted by $FP - pd_R(F)$, is defined to be the smallest integer $n \geq 0$ such that $Ext_R^{n+1}(F, B) = 0$ for any FP-injective R-module B (if no such that n exists, set $FP - pd_R(F) = \infty$) and the FP-projective dimension of the ring R, denoted $FP - dim(R)$, is defined by $FP - dim(R) = \sup\{FP - pd_R(M)/\text{M is an } R\text{-module}\}$.

The class of FI-injective (resp. FI-flat) modules was introduced in (2007) by L. Mao and N. Ding in [7] as a generalization of injective (resp. flat) modules. To complete this concept, in this paper, we introduce the class of FI-projective modules and dimension as a generalization of projective modules and dimension. Also we we'll try to give similar results as in [1, 3, 4, 7] .

In section 2, we give a characterization of the class of FI-projective modules, and we prove that this class is closed under direct sum. Also, we discuss the relationship between FI-projective modules and projective modules. Furthermore, in Theorem 2.8 we prove that every R-module is FI-projective if and only if R is a semisimple ring. In section 3 we introduce the FI-projective dimension of modules and we give its characterization and we study the FI-projective dimension over short exact sequences Theorem 3.4. In section 4 we study the FI-projective dimension of rings, in Proposition 4.5 we prove that R is hereditary if and only if FI-projective dimension of ring less than or equal to 1 and every FI-projective ideal is projective. Furthermore, we show that over an hereditary ring R every FI-projective R-module is projective.

2 FI-projective modules.

In this section we will introduce the class FI-projective modules and we study their properties and we give their characterization.

Definition 2.1. A module M is called FI-projective if $Ext_R^1(M, F) = 0$ for any FP-projective module F.

The following theorem gives a characterization of FI-projective modules.

Theorem 2.2. *Let M be a module the following statements are equivalent:*

1) *M is FI-projective.*

2) *$Ext^n_R(M,F) = 0$ for all $n \geq 1$ and all FP-projective modules F.*

3) *For every exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$, where P is FP-projective, $K \longrightarrow P$ is an FP-projective preenvelope of K.*

4) *M is a cokernel of an FP-projective preenvelope $f : A \longrightarrow B$ of a module A with B is projective.*

5) *M is projective with respect to every exact sequence $0 \longrightarrow N \longrightarrow L \longrightarrow C \longrightarrow 0$ where N is an FP-projective module.*

Proof. $1) \Rightarrow 2)$ Let M be an FI-projective module, if we apply the functor $Hom_R(.,F)$ to the sequence $0 \to M \to M \to 0$, where F is FP-projective, we obtain $Ext^i_R(M,F) \cong Ext^{i+1}_R(M,F)$ for all $i \geq 1$. Then by indiction $Ext^i_R(M,F) = 0$ for all $i \geq 1$.

$2) \Rightarrow 1)$ Obvious.

$1) \Rightarrow 3)$ Suppose that M is FI-projective and let $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ be an exact sequence of modules where P is FP-projective. Applying the long exact sequence of the functor $Hom_R(.,F')$ with F' is FP-projective, we get:

$$Hom_R(P,F') \longrightarrow Hom_R(K,F') \longrightarrow Ext^1_R(M,F').$$

Since M is FI-projective we have $Ext^1_R(M,F') = 0$, then $Hom_R(P,F') \longrightarrow Hom_R(K,F')$ is surjective for any FP-projective module F'. Therefore $K \longrightarrow P$ is an FP-projective preenvelope of K.

$3) \Rightarrow 4)$ Let $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ be an exact sequence of modules with P is projective. By hypothesis $K \longrightarrow P$ is an FP-projective preenvelope.

$4) \Rightarrow 1)$ Suppose that M is the cokernel of an FP-projective preenvelope $f : A \longrightarrow B$ of a module A with B is projective. Let G be an FP-projective module, applying the long exact sequence of the functor $Hom_R(.,G)$ to the exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow M \longrightarrow 0$, we get the exact sequence:

$$Hom_R(B,G) \longrightarrow Hom_R(A,G) \longrightarrow Ext^1_R(M,G) \longrightarrow \cdots .$$

By hypothesis $Hom_R(B,G) \longrightarrow Hom_R(A,G) \longrightarrow 0$ is exact, then $Ext^1_R(M,G) = 0$ and hence M is FI-projective.

$1) \Rightarrow 5)$ Let $0 \longrightarrow N \longrightarrow L \longrightarrow C \longrightarrow 0$ be an exact sequence of modules, where N is FP-projective. By hypothesis $Ext^1_R(M,N) = 0$, Thus $Hom_R(M,L) \longrightarrow Hom_R(M,C) \longrightarrow 0$ is exact. Therefore M is projective with respect to the exact sequence $0 \longrightarrow N \longrightarrow L \longrightarrow C \longrightarrow 0$.

$5) \Rightarrow 1)$ Let G be an FP-projective module, there exists an exact sequence $0 \longrightarrow G \longrightarrow E \longrightarrow L \longrightarrow 0$ with E is injective. Applying the long exact sequence of the functor $Hom_R(M,.)$ we get:

$$Hom_R(M,E) \longrightarrow Hom_R(M,L) \longrightarrow Ext^1_R(M,G) \longrightarrow 0.$$

By hypothesis, $Hom_R(M,E) \longrightarrow Hom_R(M,L) \longrightarrow 0$ is exact. Thus $Ext^1_R(M,G) = 0$ for any FP-projective module G, so M is FI-projective. □

In the following proposition we show that FI-projective is stable under direct sum and we study the FI-projectivity over short exact sequence.

Proposition 2.3. *(i) Let I be an index set and let $(M_i)_{i\in I}$ be a family of modules. Then $\oplus M_i$ is FI-projective if and only if M_i is FI-projective for any $i \in I$.*

(ii) Let $0 \longrightarrow N \longrightarrow B \longrightarrow L \longrightarrow 0$ be an exact sequence of modules where L is FI-projective. Then N is FI-projective if and only if B is also FI-projective.

Proof. (i) Follows from [8, Theorem 7.13] since $Ext^1_R(\oplus_{i\in I}M_i,F) \cong \oplus_{i\in I}Ext^1_R(M_i,F)$.

(ii) Let $0 \longrightarrow N \longrightarrow B \longrightarrow L \longrightarrow 0$ be an exact sequence and let F be FP-projective. Applying the long exact sequence of the functor $Hom_R(.,F)$, we get:

$$0 = Ext^1_R(L,F) \longrightarrow Ext^1_R(B,F) \longrightarrow Ext^1_R(N,F) \longrightarrow Ext^2_R(L,F) = 0,$$

where the first term and the last term are zero, since L is FI-projective. Therefore $Ext^1_R(B,F) \cong Ext^1_R(N,F)$, so B is FI-projective if and only if N is also FI-projective.

□

It is easy to see that every projective module is FI-projective, result we see when we have equivalence.

Proposition 2.4. *Let R be a coherent ring, then the following statements are equivalent:*

1) *M is projective.*

2) *M is FI-projective and $FP - pd_R(M) \leq 1$.*

Proof. 1) $\Rightarrow$ 2) Obvious.

2) $\Rightarrow$ 1) Suppose that M is an FI-projective module and $FP - pd_R(M) \leq 1$, then there exists an exact sequence of modules $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ where P is projective. Since R is a coherent ring and by [6, Proposition 3.1] we have K is FP-projective. Applying the long exact sequence of the functor $Hom_R(.,K)$ we obtain:

$$Hom_R(P,K) \longrightarrow Hom_R(K,K) \longrightarrow Ext^1_R(M,K) \longrightarrow \cdots .$$

Since M is FI-projective we have $Ext^1_R(M,K) = 0$, and hence the exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ splits, then $P \cong K \oplus M$, so M is projective as direct summand of a projective module. □

Proposition 2.5. *Let M is FI-projective module, then $Hom_R(P,M)$ and $P \otimes_R M$ are FI-projective for any finitely generated projective module P.*

Proof. Let N be an FP-projective module, for any finitely generated projective module P, the result follows from the following isomorphism: $P \otimes_R Ext^1_R(M,N) \cong Ext^1_R(Hom_R(P,M),N)$ and $Ext^1_R(P \otimes_R M, N) \cong Hom_R(P, Ext^1_R(M,N))$. □

In the following proposition we show when the k^{th} syzygy is FI-projective.

Proposition 2.6. *Let $n \geq 0$ be an integer and let M be a module such that $Ext^i_R(M,F) = 0$ for any FP-projective module F and $1 \leq i \leq n+1$. Then the k^{th} syzygy of M is FI-projective for any $0 \leq k \leq n$.*

Proof. Follows since $Ext^1_R(L_k,F) \cong Ext^{k+1}_R(M,F) = 0$ where F is an FP-projective module and L_k is the k^{th} syzygy of M $0 \leq k \leq n$ [8, Corollary 6.19]. □

In [6, Corollary 4.3] they show that a ring R is von Neumann regular if and only if it is coherent and every FP-projective module is FP-injective. The following result gives a characterization of von Neumann regular rings using FI-projective modules.

Proposition 2.7. *Let R be a coherent ring, then R is von Neumann regular if and only if every finitely presented R-module is FI-projective.*

Proof. $\Rightarrow$) Let M be a finitely presented module and let N be an FP-projective module. from [6, Corollary 4.3] N is FP-injective, then $Ext^1_R(M,N) = 0$, therefore M is FI-projective.

$\Leftarrow$) Conversely, let M be a finitely presented module, by hypothesis M is FI-projective. Then for any FP-projective module N we have $Ext^1_R(M,N) = 0$, since M is arbitrary it follows that N is FP-injective. So since R is coherent and from [6, Corollary 4.3], R is von Neumann regular as desired. □

The following theorem gives a characterization of semisimple rings using FI-projective modules.

Theorem 2.8. *Let R be a ring. Then the following are equivalent:*

1) *R is semisimple.*

2) *Every FP-projective is injective.*

3) *Every R-module is FI-projective.*

Proof. 1) $\Rightarrow$ 2) It is obvious by [8, Proposition 4.13].

2) $\Rightarrow$ 1) Let P be a projective module, by hypothesis P is injective. Then from [2, Theorem 31.9] R is a Quasi-Frobenius ring which means that R is Notherian. From [6, Proposition 2.6] every module of R is FP-projective and by hypothesis every module is injective and R is semisimple.

1) $\Rightarrow$ 3) Obvious.

3) $\Rightarrow$ 2) Let F be an FP-projective module, by hypothesis for any R-module M we have $Ext_R^1(M,F) = 0$, and so F is injective. □

Corollary 2.9. *Let R be a Notherian ring, then every FI-projective module is projective.*

Proof. From [6, Proposition 2.6], R is Notherian if and only if every module is FP-projective. And this complete the proof. □

3 The FI-projective dimension of modules.

In this section we introduce the FI-projective dimension of modules which is a generalization of the projective dimension of modules.

Definition 3.1. Let R be a ring. The FI-projective dimension of a module M is denoted $FI - pd_R(M)$, and defined to be the smallest integer $n \geq 0$ such that $Ext_R^{n+1}(M,F) = 0$ for any FP-projective R-module F.

The following result gives a characterization of the FI-projective dimension of modules.

Theorem 3.2. *Let M be a module and $n \geq 0$ be an integer. Then the following conditions are equivalent:*

(i) $FI - pd_R(M) \leq n$,

(ii) $Ext_R^{n+1}(M,F) = 0$ *for any FP-projective module F,*

(iii) $Ext_R^{n+j}(M,F) = 0$ *for any FP-projective module F and for any integer $j \geq 1$,*

(iv) There exists an exact sequence $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$, *where each P_i is FI-projective.*

Proof. The proof is straightforward. □

Proposition 3.3. *Let* $0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$ *be an exact sequence of modules where P is FI-projective. If M is FI-projective, then N is FI-projective, if not we have:*

$$FI - pd_R(M) = FI - pd_R(N) + 1.$$

Proof. Suppose that M is not FI-projective and let F be an FP-projective R-module. Using the long exact sequence of the functor $Hom_R(.,F)$ to the short exact sequence we obtain:

$$(*)\ Ext_R^{n+1}(P,F) = 0 \longrightarrow Ext_R^{n+1}(N,F) \longrightarrow Ext_R^{n+2}(M,F) \longrightarrow Ext_R^{n+2}(P,F) = 0$$

The first and the last terms are zero since P is FI-projective and F is FP-projective so: $Ext_R^{n+2}(M,F) \cong Ext_R^{n+1}(N,F)$, and from Theorem 3.2 $FI - pd_R(M) = FI - pd_R(N) + 1$.

Now, if M is FI-projective and for $n = 0$ in $(*)$ we obtain: $Ext_R^2(M,F) = Ext_R^1(N,F) = 0$ So N is FI-projective. □

The following theorem is a generalization of Proposition 3.3 above.

Theorem 3.4. *Let* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *be an exact sequence of modules. If two of* $FI - pd_R(A)$, $FI - pd_R(B)$ *and* $FI - pd_R(C)$ *are finite, so is the third. Moreover:*

(i) $FI - pd_R(A) \leq max\{FI - pd_R(B), FI - pd_R(C) - 1\}$.

(ii) $FI - pd_R(C) \leq max\{FI - pd_R(B), FI - pd_R(A) + 1\}$.

(iii) $FI - pd_R(B) \leq max\{FI - pd_R(A), FI - pd_R(C)\}$.

Proof. We prove only assertion (i) and the other assertions can be proved similarly. Let $k = \max\{FI - pd_R(B), FI - pd_R(C) - 1\}$ and let N be an FP-projective module, using the long exact sequence of the functor $Hom_R(., N)$ to the short exact sequence we obtain: $0 = Ext_R^{k+1}(B, N) \longrightarrow Ext_R^{k+1}(A, N) \longrightarrow Ext_R^{k+2}(C, N) = 0$, the first and the last term vanish by hypothesis, so $Ext_R^{k+1}(A, N) = 0$ and $FI - pd_R(A) \leq k = \max\{FI - pd_R(B), FI - pd_R(C) - 1\}$. □

The next corollary study the FI-projective dimension of the direct sum of modules.

Corollary 3.5. *Let I be an index set and let $(M_i)_{i \in I}$ be a family of modules. Then $FI - pd_R(\oplus M_i) = max\{FI - pd_R(M_i)/\ M_i$ R-module$\}$.*

Proof. Follows from [8, Theorem 7.13] since $Ext_R^n(\oplus_{i \in I} M_i, F) \cong \oplus_{i \in I} Ext_R^n(M_i, F)$. Then we can deduce the result using Theorem 3.2. □

4 The global FI-projective dimension of rings.

In this section we introduce the global FI-projective dimension of rings and we give its characterization. Also we characterize semisimple and hereditary rings using this dimension.

Definition 4.1. The global FI-projective dimension of R is the supremum of FI-projective dimensions of all R-modules, denoted:

$$FI - pdim(R) = sup\{FI - pd_R(M)/ \text{ M is an } R\text{-module}\}.$$

The following result gives a characterization of the FI-projective dimension of rings.

Proposition 4.2. *Let R be a ring and let $n \geq 0$ be an integer. Then the following conditions are equivalents:*

(i) $FI - pdim(R) \leq n$.

(ii) $Ext_R^{n+1}(M, N) = 0$ *for any R-module M and any FP-projective R-module N.*

(iii) $Ext_R^{n+j}(M, F) = 0$ *for any integer $j \geq 1$, and R-module M and for any FP-projective R-module N.*

(iv) $id_R(N) \leq n$ *for any FP-projective R-module N.*

(v) $FI - pd(M) \leq n$ *for any R-module M.*

Proof. The proof is obvious it follows from the definition and Theorem 3.2. □

Corollary 4.3. *A ring R is semisimple if and only if $FI - pdim(R) = 0$*

Proof. follows from Proposition 4.2 and Theorem 2.8. □

The following theorem gives a characterization of the ring of $FI - pdim(R) \leq 1$.

Theorem 4.4. *Let R be a ring. Then the following are equivalent:*

1) $FI - pdim(R) \leq 1$.

2) *Every submodule of an FI-projective R-module is FI-projective.*

3) *Every submodule of a projective R-module is FI-projective.*

4) *Every FP-projective R-module has an injective dimension at most 1.*

Proof. $1) \Rightarrow 2)$ Let N be a submodule of an FI-projective R-module M, applying the long exact sequence of the functor $Hom_R(., F)$, where F is an FP-projective module, to the short exact sequence $0 \to N \to M \to M/N \to 0$, we get: $Ext^1_R(M, F) \to Ext^1_R(N, F) \to Ext^2_R(M/N, F)$. The first term vanish since M is FI-projective and the last term vanish by hypothesis. Hence $Ext^1_R(N, F) = 0$ and N is FI-projective.

$2) \Rightarrow 3)$ Obvious since every projective R-module is FI-projective.

$3) \Rightarrow 4)$ Let M be an R-module and let $0 \to K \to P \to M \to 0$, be a short exact sequence of modules where P is projective. Applying the long exact sequence of the functor $Hom_R(., F)$ where F is FP-projective, we get: $Ext^1_R(K, F) \to Ext^2_R(M, F) \to Ext^2_R(P, F) = 0$. By hypothesis $Ext^1_R(K, F) = 0$ which implies that $Ext^2_R(M, F) = 0$ hence $id_R(F) \leq 1$.

$4) \Rightarrow 2)$ Let N be a submodule of an FI-projective R-module M. Applying the long exact sequence of the functor $Hom_R(., F)$, where F is an FP-projective R-module to the short exact sequence $0 \to N \to M \to M/N \to 0$, we get:

$$Ext^1_R(M, F) \longrightarrow Ext^1_R(N, F) \longrightarrow Ext^2_R(M/N, F).$$

We have $Ext^1_R(M, F) = 0$ since M is FI-projective and $Ext^2_R(M/N, F) = 0$ since $id_R(F) \leq 1$ by hypothesis. Therefore $Ext^1_R(N, F) = 0$, and hence N is FI-projective.

$3) \Rightarrow 1)$ For any R-module M, consider an exact sequence of R-modules $0 \to K \to P \to M \to 0$ with P is projective. Applying the long exact sequence of the functor $Hom_R(., F)$, where F is an FP-projective R-module we get $Ext^1_R(K, F) \to Ext^2_R(M, F) \to Ext^2_R(P, F) = 0$. The first term is zero since K is FI-projective as a submodule of the projective module P and the last term is zero since P is projective. Thus $Ext^2_R(M, F) = 0$ and hence $FI - pd_R(M) \leq 1$, so $FI - pdim(R) \leq 1$. □

In the following proposition we see a characterization of hereditary rings using FI-projective dimensions of rings.

Proposition 4.5. *Let R be a ring. Then the following are equivalent:*

1) *R is hereditary.*

2) *$FI - pdim(R) \leq 1$ and every FI-projective ideal of R is projective.*

Proof. $1) \Rightarrow 2)$ Obvious.

$2) \Rightarrow 1)$ Let I be an ideal of R and consider the exact sequence $0 \to I \to R \to R/I \to 0$. Since $FI - pdim(R) \leq 1$ so $FI - pd_R(R/I) \leq 1$ and I is FI-projective by Theorem 4.4 so I is projective by hypothesis and R is hereditary. □

Corollary 4.6. *If R is hereditary, then every FI-projective R-module is projective.*

Proof. Let M be an FI-projective R-module, for any R-module N we consider the exact sequence $0 \to K \to P \to N \to 0$, where P is projective. Since K is a submodule of P and R is hereditary, so K is projective. Applying the long exact sequence of the functor $Hom_R(M, .)$ we get the exact sequence: $Ext^1_R(M, P) \to Ext^1_R(M, N) \to Ext^2_R(M, K)$, then since M is FI-projective we have $Ext^2_R(M, K) = 0$ and since P is FP-projective, $Ext^1_R(M, P) = 0$. Therefore $Ext^1_R(M, N) = 0$ and hence M is projective. □

Recall that finitistic projective dimension of a ring R is defined as:

$$FPD(R) = sup\{pd_R(M) | \text{ M is an } R\text{-module with } pd_R(M) \leq \infty\}.$$

Proposition 4.7. *For any ring R we have:*

$$FPD(R) \leq FI - pdim(R) \leq gldim(R)$$

with equality if $wdim(R) < \infty$.

Proof. Since every projective R-module is FP-projective it is obvious that $FI - pdim(R) \leq gldim(R)$. On the other hand, suppose that $FI - pdim(R) = n < \infty$ and let F be an R-module with $pd_R(F) = m < \infty$. We claim that $m \leq n$, otherwise, suppose that $m > n$ and for any R-module N we consider the exact sequence $0 \to K \to P \to N \to 0$ with P projective. Applying the long exact sequence of the functor $Hom_R(F, .)$ we get:

$$Ext_R^m(F, P) \longrightarrow Ext_R^m(F, N) \longrightarrow Ext_R^{m+1}(F, K).$$

The first term is zero since $m > n$, and the last term is zero since $pd_R(F) = m$. Thus $Ext_R^m(F, N) = 0$, which implies that $pd_R(F) < m$, contradiction, so $m \leq n$.

Now if $wdim(R) < \infty$, from [5, Corollary 4] we have $FPD(R) = gldim(R)$. □

References

[1] N. H. Alshehri and K. Ouarghi, SFI-Injective and SFI-Flat Modules. JP Journal of Algebra, Number Theory and Applications, 42, 205–228, (2019).

[2] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules. Springer-Verlag, New york, (1974).

[3] L. Duan and B. Ouyang, Relative FI-Injective and FI-Flat Modules. Indian Journal Pure Applied Mathematics, 42, 417–441, (2011).

[4] Z. Gao, On n-FI-Injective and n-FI-Flat Modules. Communication in Algebra., 40, 2757–2770, (2012).

[5] C. U. Jensen, On the vanishing of $\lim\limits_{\longleftarrow}^{i}$. Journal of Algebra, 15, 151–166, (1970).

[6] L. X. Mao and N. Q. Ding, FP-projective Dimensions. Communication in Algebra, 33, 1153–1170, (2005).

[7] L. Mao and N. Ding, FI-Injective and FI-Flat Modules. Journal of Algebra, 309, 367–385, (2007).

[8] J. J. Rotman, An Introduction to Homological Algebra. Academic Press, New York, (1979).

[9] K. E. Aubert, On the ideal theory of commutative semi-groups, *Math. Scand.* **1**, 39–54 (1953).

[10] G. D. Bullington, On the expected number of generators of a submodule of a free module over a finite special principal ideal ring, *Rend. Circ. Mat. Palermo* **51**, 5–50 (2002).

[11] L. Fuchs, A remark on the Jacobson radical, *Acta Sci. Math. (Szeged)* **14**, 167–168 (1952).

[12] R. Gilmer, *Commutative Semigroup Rings*, Chicago Lectures in Math., Univ. Chicago Press, Chicago (1984).

[13] K. W. Gruenberg, *Cohomological Topics in Group Theory*, Lecture Notes in Math. **143**, Springer-Verlag, Berlin (1970).

[14] M. Hall, Jr., *The Theory of Groups*, Macmillan, New York (1959).

[15] N. Jacobson, *Lectures in Abstract Algebra*, volume I, Van Nostrand, Princeton (1951).

[16] I. Kaplansky, *Commutative Rings* (Revised Edn), Univ. Chicago Press, Chicago (1974).

[17] R. L. Kruse and D. T. Price, *Nilpotent Rings*, Gordon and Breach, New York (1969).

[18] E. S. Ljapin, *Semigroups*, Translations Math. Monographs **3**, Amer. Math. Soc., Providence (1974).

[19] W. R. Scott, *Group Theory*, Prentice-Hall, Englewood Cliffs (1964).

[20] O. Zariski and P. Samuel, *Commutative Algebra*, volume I, Van Nostrand, Princeton (1958).

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