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Characterizations of Artinian and Noetherian Gamma Rings in Terms of Homogeneous Complex Fuzzy Ideals

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Abstract The aim of this paper is to characterise Noetherian and Artinian Gamma rings by complex fuzzy ideals.

1 Introduction

The Γ-rings which are generalizations of classical rings was constructed by Barnes [6] in 1966. Since then many mathematicians have been trying to extend known results in the setting of rings to the setting of Γ-ring (see e.g. [5, 8, 10, 11, 12, 16, 20, 24]).
The notion of complex fuzzy sets which is a generalization of the notion of the traditional fuzzy sets (due to Zadeh [25] in 1965) was introduced by Ramot and Milo [18] in 2002. Further for the work concerning complex fuzzy sets and their applications we refer to [18, 17, 23]. Fuzzy and complex fuzzy set theories have been applied in various algebra structures by many authors (see e.g. [1, 2, 3, 4, 7, 19, 22] and references therein).
We introduced the concept of complex fuzzy sets to Gamma rings in [22]. In this paper we examine some properties of complex fuzzy ideals of Gamma Rings. We use these to characterize Gamma rings with chain conditions in terms of complex fuzzy Gamma rings.

2 PRELIMINARIES

Suppose that Γ and R are additive abelian groups. Then R is a Γ-ring (due to Barnes) if there is a map

$$R \times \Gamma \times R \to R : (x, \alpha, y) \mapsto x\alpha y$$

such that

(i) $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)z = x\alpha z + x\beta z$, $x\alpha(y+z) = x\alpha y + x\alpha z$,

(ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

for every $x, y, z \in R$ and $\alpha, \beta \in \Gamma$.
It is clear that the notion above generalizes the notion of a ring. Let R be a Γ-ring. An element $e \in R$ (unique) is called a unity of R corresponding to $\alpha_0 \in \Gamma$ if $a\alpha_0 e = e\alpha_0 a = a$ for all $a \in R$. In this case, we say R is a Γ-ring with unity.
A nonempty subset S of a Γ-ring R is a sub-Γ-ring of R if $a - b \in S$ and $a\alpha b \in S$ for any $a, b \in S$ and $\alpha \in \Gamma$. A subset I of the Γ-ring R is a left (right) ideal of R if I is an additive subgroup of R and $R\Gamma I = \{r\alpha a \mid a \in R, \alpha \in \Gamma, a \in I\}$ ($I\Gamma R$) is contained in I. If I is both a left and a right ideal of I, then we say that I is an ideal (or two-sided ideal) of R. For more details on Gamma Rings we refer the reader to [5, 6, 8, 10, 11, 12, 16, 20, 24].
Let $\mathbb{E}^2$ denotes the set $\{(a, b) \mid a^2 + b^2 \leq 1\}$ ($\subseteq \mathbb{R}^2$). A complex fuzzy set on a nonempty set X is a function

$$\mu_A : X \to \mathbb{E}^2;\ x \mapsto r_A(x)e^{iw_A(x)},$$

where $i = \sqrt{-1}, r_A(x) \in [0,1]$, and $w_A(x) \in [0, 2\pi]$. If $w_A(x) = 0$ for all $x \in X$, then we return back to the traditional fuzzy set (due to Zadeh [25]).
For $t_1 = r_1 e^{iw_1}, t_2 = r_2 e^{iw_2} \in \mathbb{E}^2$ ($0 \leq r_1, r_2 \leq 1$ and $0 \leq w_1, w_2 \leq 2\pi$), we say that $t_1 \preceq t_2$ if and only if $r_1 \leq r_2$ and $w_1 \leq w_2$.
A complex fuzzy set μ_A on X is homogeneous if for all $x_1, x_2 \in X$ we have $r(x_1) \leq r(x_2)$ if and only if $w(x_1) \leq w(x_2)$. Throughout this paper all complex fuzzy sets are homogeneous. Thus if μ_A is a complex fuzzy set on X, then for each $x_1, x_2 \in X$ we either have $\mu_A(x_1) \preceq \mu_A(x_2)$ or $\mu_A(x_2) \preceq \mu_A(x_1)$.
A complex fuzzy set μ_A on a Γ-ring R is a complex fuzzy sub-Γ-ring if $\mu_A(x_1 - x_2) \succeq \mu_A(x_1) \wedge \mu_A(x_2)$ (=min$\{\mu_A(x_1), \mu_A(x_2)\}$) and $\mu_A(x_1 \alpha x_2) \succeq \mu_A(x_1) \wedge \mu_A(x_2)$, for all $x_1, x_2 \in R$ and $\alpha \in \Gamma$. If the second identity is replaced by $\mu_A(x_1 \alpha x_2) \succeq \mu_A(x_1) \vee \mu_A(x_2)$ (= max$\{\mu_A(x_1), \mu_A(x_2)\}$), then μ_A is called a complex fuzzy ideal of R ([22]).

3 Characterizations of Artinian Gamma Rings in Terms of Complex Fuzzy Ideals

Lemma 3.1. *([22]) Let μ_A be a complex fuzzy subset of a Γ-ring R. Then μ_A is a complex fuzzy ideal of R if and only if for each $s + ti \in$ Im(μ_A) the upper level $U(\mu_A, s+ti) = \{x \in R \mid \mu_A(x) \succeq s + ti\}$ is an ideal of R.*

In 2002, Özturk, Uckun, and Jun proved that If every fuzzy ideal of a Γ-ring has finite number of values, then it is Artinian ([14]) Next, in 2003, they proved the converse ([15]). The following theorem extends their results to complex fuzzy sets. There is analogous result for rings and Lie algebras in terms of fuzzy ideals.

Theorem 3.2. *Every homogeneous complex fuzzy sub-Γ-ring (ideal) μ_A defined on Γ-ring R has a finite number of values if and only if every descending chain of sub-Γ-rings (ideals) of R terminates at finite step.*

Proof. Suppose that every complex fuzzy ideal of a Γ-ring R has a finite number of values, then, in particular, every fuzzy ideal of R has a finite number of values. Hence R is Artinian; that is every descending chain of ideals of R terminates at finite step.
Conversely suppose that every descending chain of ideals of a Γ-ring R terminates at finite step. Let μ_A be a complex fuzzy ideal of R. Suppose to the contrary that Im(μ_A) is infinite. Because μ_A is homogeneous, we have every infinite subset of Im$(\mu_A) \subset \mathbb{E}^2$ contains either a strictly increasing or strictly decreasing infinite sequence. Let

$$s_1 + t_1 i \prec s_2 + t_2 i \prec \cdots$$

be a strictly increasing sequence in Im(μ_A). According to Lemma 3.1 we have the following descending chain of ideals of R:

$$U(\mu_A, s_1 + t_1 i) \supseteq U(\mu_A, s_2 + t_2 i) \supseteq \cdots .$$

Since R is Artinian, there is $n_0 \in \mathbb{N}$ such that $U(\mu_A, s_{n_0} + t_{n_0} i) = U(\mu_A, s_{n_0+j} + t_{n_0+j})$ for each $j \geq 1$. Note that since $s_{n_0} + t_{n_0} i \in$ Im(μ_A), there is $x \in R$ with $\mu_A(x) = s_{n_0} + t_{n_0} i$. Then for each $j \geq 1$ we have $x \in U(\mu_A, s_{n_0+j} + y_{n_0+j} i)$. Hence $s_{n_0} + t_{n_0} i = \mu_A(x) \succeq s_{n_0+j} + t_{n_0+j} i$. Similarly we can find $s_{n_0+j} + t_{n_0+j} i \succeq s_{n_0} + t_{n_0} i$ for each $j \geq 1$. Therefore for each $j \geq 1$ we have $s_{n_0+j} + t_{n_0+j} i = s_{n_0} + t_{n_0} i$. Contradiction (since all $(s_j + t_j i)$'s are distinct). If

$$s_1 + t_1 i_2 + t_2 i \succ \cdots$$

is a strictly decreasing sequence in Im(μ_A), then

$$U(\mu_A, s_1 + t_1 i) \subseteq U(\mu_A, s_2 + t_2 i) \subseteq \cdots$$

is an ascending chain of ideals of R. It is known that every Arinian Γ-ring is also Noetherian, so there is $n_0 \in \mathbb{N}$ such that $U(\mu_A, s_{n_0} + t_{n_0} i) = U(\mu_A, s_{n_0+j} + t_{n_0+j})$ for each $j \geq 1$. Hence $s_{n_0+j} + t_{n_0+j} i = s_{n_0} + t_{n_0} i$ for each $j \geq 1$ (since $s_k + t_k i \in$ Im(μ_A) for each $k \in \mathbb{N}$). Contradiction. □

Let μ be a complex fuzzy ideal of a Γ-ring R. We denote by Ω_μ the family of all upper levels of R with respect to μ. Also, for a set X, we denote by $|X|$ the cardinality of X. The following two corollaries was obtained in [13] and [15] in the setting of fuzzy rings and fuzzy Γ-rings. The theorem above helps us to extend them to the case of complex fuzzy Γ-rings. The proofs are very similar to the proofs of analogous results for fuzzy (Γ-)rings, so we omit them.

Corollary 3.3. *Let R be an Artinian Γ-ring, and let μ be a homogeneous complex fuzzy ideal of R. Then $|\Omega_\mu| = |\mathrm{Im}(\mu)|$.*

Corollary 3.4. *Let R be an Artinian Γ-ring, and let μ_A and μ_B be homogeneous complex fuzzy ideals of R. Then*

$$\Omega_{\mu_A} = \Omega_{\mu_B} \text{ and } \mathrm{Im}(\mu_A) = \mathrm{Im}(\mu_B) \text{ if and only if } \mu_A = \mu_B.$$

4 Characterizations of Noetherian Gamma Rings in Terms of Complex Fuzzy ideals

Let R be an Artinian Γ-ring. Then R is Noetherian ([9]). Hence, and according to Theorem 3.2, if every complex fuzzy ideal of a Γ-ring R is finite valued, then R is Noetherian. There is an example shows that there is a Noehterian Γ-ring with an infinite valued of complex fuzzy ideal ([13, Example 3.4]).

Lemma 4.1. *Let μ_A be a homogeneous complex fuzzy subset defined on Γ-ring R and let $\mathrm{Im}(\mu_A) = \{0, s_0 + t_0 i, s_1 + t_1 i, \ldots\}$, where*

$$e^{2\pi i} \succeq s_0 + t_0 i \succ s_1 + t_1 i \succ s_2 + t_2 i \succ \cdots \succeq 0.$$

If $X_0 \subset X_1 \subset X_2 \subset \cdots$ are sub-Γ-rings (ideals) of R such that $\mu_A(X_k \setminus X_{k-1}) = \{s_k + t_k i\}$ for $k = 0, 1, 2, \ldots$ where $X_{-1} = \Phi$ and $\mu_A(R \setminus \bigcup_k X_k) = \{0\}$, then μ_A is a complex fuzzy sub-Γ-ring (ideal) of R.

Proof. We consider the case when all X_i are subalgebras. Let $x_1, x_2 \in R$ and $\alpha \in \Gamma$. If $x_1 \alpha x_2 \in R \setminus \bigcup_k X_k$, then also at least one of x_1 or x_2 is in $R \setminus \bigcup_k X_k$ (otherwise x_1, x_2 and $x_1 \alpha x_2$ will be in some X_k). So, in this case

$$\mu_A(x_1 \alpha x_2) = 0 = \mu_A(x_1) \wedge \mu_A(x_2).$$

If $x_1 \alpha x_2 \in \bigcup_k X_k$, then there exists m_0 such that $x_1 \alpha x_2 \in X_{m_0} \setminus X_{m_0 - 1}$. Hence we have either $x_1 \notin X_{m_0-1}$ or $x_2 \notin X_{m_0-1}$. Without loss of generality we may assume $x_1 \notin X_{m_0-1}$. Now if $x_1 \notin \bigcup X_k$, then

$$\mu_A(x_1) \wedge \mu_A(x_2) = 0 \preceq s_{m_0} + t_{m_0} i = \mu_A(x_1 \alpha x_2).$$

Otherwise there is n_0 ($\geq m_0$) such that $x_1 \in X_{n_0} \setminus X_{n_0 - 1}$. In this case we find

$$\mu_A(x_1 \alpha x_2) = s_{m_0} + t_{m_0} i \succeq s_{n_0} + t_{n_0} i \succeq \mu_A(x_1) \wedge \mu_A(x_2).$$

Almost in the same way we can prove that $\mu_A(x_1 - x_2) \succeq \mu_A(x_1) \wedge \mu_A(x_2)$ for every $x_1, x_2 \in R$. Next let all X_i be ideals and let $x_1 \alpha x_2 \in X_{m_0} \setminus X_{m_0-1}$ ($\alpha \in \Gamma$). Then x_1 and x_2 are not in X_{m_0-1}. Indeed if x_1 or x_2 is in X_{m_0-1}, then $x_1 \alpha x_2 \in X_{m_0-1}$ because X_{m_0-1} is ideal. Thus

$$\mu_A(x_1) \vee \mu_A(x_2) \preceq s_{m_0} + t_{m_0} i = \mu_A(x_1 \alpha x_2).$$

If $x_1 \alpha x_2 \in R \setminus \bigcup_k X_k$, then also $x_1, x_2 \notin \bigcup_k X_k$, and so

$$\mu_A(x_1 \alpha x_2) = \mu_A(x_1) \vee \mu_A(x_2).$$

□

The set of ideals of a given Γ-ring is a poset with the usual inclusion $\subseteq$ as the partial order. Let $(S, \leq)$ be a poset. A subset X of S is called a chain if for all $x, y \in X$, we have either $x \leq y$ or $y \leq x$. If $(S, \leq)$ is a poset, and $A \subseteq S$, we say $x \in A$ is a smallest element of A if $x \leq y$ for all $y \in A$. A poset $(S, \leq)$ is said to be well ordered if every nonempty subset of S has a smallest element. It is well known $(S, \leq)$ is well-ordered if and only if it does not contain infinite descending chains, that is there does not exist a sequence $a_0, a_1, a_2 \dots$ in S such that $a_0 > a_1 > a_2 > \cdots$.
Öztürk, M. Uckun and Y. B. Jun ([14]) proved that A Γ-ring R is (left) Noetherian if and only if the set of values of any fuzzy (left) ideal of R is a well ordered subset of $[0, 1]$. Here we extend the result to the case of complex fuzzy Γ-rings

Theorem 4.2. *Every ascending chain of sub-Γ-rings (ideals) of a Γ-ring R terminates at finite step if and only if the set of values of any homogeneous complex fuzzy sub-Γ-ring (ideal) of R is a well-ordered subset of $\mathbb{E}^2$.*

Proof. Suppose that the set $\text{Im}(\mu_A)$ of values of a complex fuzzy sub-Γ-ring (ideal) μ_A for a Γ-ring R is not well ordered. Then there is a strictly decreasing sequence $\{s_j + t_j i\}$ such that $s_j + t_j i = \mu_A(r_j)$ for some $r_j \in R$. But in this case $U(\mu_A, s_j + t_j i)$ form a strictly ascending chain of sub-Γ-rings (ideals) of R, which is a contradiction because R is Noetherian.
Conversely suppose that every ascending chain of sub-Γ-rings (ideals) of a Γ-ring R terminates at finite step. Suppose that there exists a strictly ascending chain

$$S_1 \subset S_2 \subset S_3 \subset \cdots$$

of sub-Γ-rings (ideals) of R. Then one can easily see that $S = \bigcup_k S_k$ is a sub-Γ-ring (ideal) of R. Define a complex fuzzy subset on R such that $\mu_A(S_k \setminus S_{k-1}) = \{\frac{1}{k} + \frac{1}{k} i\}$ for $k = 1, 2, \dots$ and with $X_0 = \Phi$ and $\mu_A(R \setminus S) = \{0\}$. According to the lemma above we have μ_A is a complex fuzzy sub-Γ-ring (ideal) of R. Also since the chain $S_1 \subset S_2 \subset \cdots$ is not terminating, μ_A has a strictly descending sequence of values. This contradicts with the assumption that the set of images any complex fuzzy sub-Γ-ring (ideal) of R is well-ordered. □

Corollary 4.3. *Let*

$$e^{2\pi i} \succeq s_0 + t_0 i \succ s_1 + t_1 i \succ s_2 + t_2 i \succ \cdots \succeq 0$$

be a strictly decreasing in $\mathbb{E}^2$. Then a Γ-ring R is Noetherian if and only if for each homogeneous complex fuzzy ideal μ of R with $\text{Im}(\mu) \subseteq \{s_0 + t_0 i, s_1 + t_1 i, s_2 + t_2 i, \cdots, 0\}$ there is $m_0 \in \mathbb{N}$ such that $\text{Im}(\mu) \subseteq \{s_0 + t_0 i, s_1 + t_1 i, \dots, s_{m_0} + t_{m_0} i\} \cup \{0\}$.

Proof. If R is Noetherian, then according to the theorem above for each complex fuzzy ideal of R the set $\text{Im}(\mu)$ is a well ordered subset, and a set is well ordered if and only if it does not contain any infinite descending sequence.
Conversely suppose to the contrary that R is not Noetherian Γ-ring. Let

$$X_0 \subset X_1 \subset X_2 \subset \cdots$$

be a strictly ascending chain of ideals of R. Then, using Lemma 4.1 the complex fuzzy subset μ_A defined by $\mu_A(X_k \setminus X_{k-1}) = \{s_k + t_k i\}$ for $k = 0, 1, 2, \dots$ where $X_{-1} = \Phi$ and $\mu_A(R \setminus \bigcup_k X_k) = \{0\}$ is a complex fuzzy ideal and does not satisfy our assumption. □

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