Palestine Journal of Mathematics

Vol. 11(4)(2022) , 172–182

SOME NEW CONGRUENCES FOR (j, k)-REGULAR BIPARTITIONS INTO DISTINCT PARTS

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Communicated by Ayman Badawi

MSC 2010 Classifications:Primary 11P83; Secondary 05A17.

Keywords and phrases: (j, k)-regular bipartition, distinct parts, q-series identities.

The authors would like to thank anonymous reviewers for their valuable comments which improved the quality of the paper. The first author also thanks University Grants Commission (UGC) of India for supporting her research work through Junior Research Fellowship (JRF) vide UGC-Ref.no. 1107/CSIR-UGC NET DEC-2018.

Abstract For any relatively prime integers j and k, let $B_{j,k}(n)$ denote the number of (j, k)-regular bipartitions of a positive integer n into distinct parts. Prasad and Prasad (2020) proved infinite families of congruences modulo powers of 2 for $B_{4,5}(n)$. In this paper, we prove infinite families of congruences modulo powers of 2 for $B_{j,k}(n)$ with $(j, k) \in$ {(2,3), (3,4), (5,8)}. For example, we prove that, for integers $\alpha \geq 0$ and $1 \leq r \leq p-1$,

$$B_{3,4}\Big(4 \cdot p^{2\alpha+1}(pn+r) + \frac{7 \cdot p^{2\alpha+2} - 1}{2}\Big) \equiv 0 \pmod{8}.$$

1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers called parts, such that the sum of all the parts is equal to n. The number of partitions of a positive integer n is generally denoted by $p(n)$ (with $p(0) = 1$) and its generating function is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots, \tag{1.1}$$

where for any complex number a and $|q| < 1$,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n) \tag{1.2}$$

and one often writes, for any positive integer t,

$$f_t := (q^t;q^t)_\infty.$$

For example, $p(4) = 5$ with the partitions given by

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

Ramanujan [14] proved the following beautiful congruences satisfied by $p(n)$:

$$p(5n+4) \equiv 0 \pmod{5}, \quad p(7n+5) \equiv 0 \pmod{7}, \quad \text{and} \quad p(11n+6) \equiv 0 \pmod{11}.$$

Ramanujan's congruences for $p(n)$ inspired many mathematicians to investigate for arithmetic properties of other partition functions. One such partition function is ℓ-regular partition which counts the number of partitions of n in which no part is divisible by ℓ, where ℓ is a positive integer. If $b_\ell(n)$ denotes the number of ℓ-regular partitions of n (with $b_\ell(0) = 1$), then its generating function is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1}. \tag{1.3}$$

For example, $b_3(4) = 4$ with the relevant partitions

$$4, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

One can refer [2, 11, 15] for arithmetic properties of $b_\ell(n)$.

A bipartition (κ,ξ) of a positive integer n is a partition pair (κ,ξ) such that the sum of all the parts of κ and ξ equals n. Further, a bipartition (κ,ξ) of n is called (j,k)- regular if κ is j-regular and ξ is k-regular. If $b_{j,k}(n)$ denotes the number of (j,k)-regular bipartitions of n, then its generating function is given by

$$\sum_{n=0}^{\infty} b_{j,k}(n)q^n = \frac{f_j f_k}{f_1^2}.$$

For example, $b_{5,7}(3) = 10$ and the corresponding bipartitions are given by

$$(3,\emptyset), \quad (\emptyset,3), \quad (\emptyset,2+1), \quad (2+1,\emptyset), \quad (2,1), \quad (1,2), \quad (\emptyset,1+1+1),$$
$$(1+1+1,\emptyset), \quad (1+1,1), \quad (1,1+1).$$

For arithmetic properties of (j,k)-regular bipartitions, one can see [4, 9].

Prasad and Prasad [12] investigated the partition function $a_{j,k}(n)$ which counts the number of (j,k)-regular partitions of a positive integer of n with distinct parts, where

$$\sum_{n=0}^{\infty} a_{j,k}(n)q^n = \frac{(-q;q)_\infty(-q^{jk};q^{jk})_\infty}{(-q^j;q^j)_\infty(-q^k;q^k)_\infty} = \frac{f_2 f_{2jk} f_j f_k}{f_{2j} f_{2k} f_1 f_{jk}}. \tag{1.4}$$

For example, $(2,5)$-regular partitions of 11 into distinct parts are 11 and 7+3+1 (that is $a_{2,5}(11) = 2$). Prasad and Prasad [12] also proved some infinite families of congruences modulo 2 for $a_{3,5}(n)$. Drema and Saikia [5] proved infinite families of congruences modulo 2 and 4 for $a_{2,5}(n)$, $a_{2,7}(n)$, $a_{4,5}(n)$ and $a_{4,9}(n)$.

Analogous to ℓ-regular bipartition, one can define (j,k)-regular bipartitions of a positive integer n into distinct parts. Let $B_{j,k}(n)$ count the number of (j,k)-regular bipartitions of n into distinct parts with $B_{j,k}(0) = 1$, then its generating function is given by

$$\sum_{n=0}^{\infty} B_{j,k}(n)q^n = \frac{(-q;q)_\infty^2(-q^{jk};q^{jk})_\infty^2}{(-q^j;q^j)_\infty^2(-q^k;q^k)_\infty^2} = \frac{f_2^2 f_{2jk}^2 f_j^2 f_k^2}{f_{2j}^2 f_{2k}^2 f_1^2 f_{jk}^2}. \tag{1.5}$$

For example, $B_{3,4}(9) = 12$, with the relevant partitions

$$(7,2), \quad (2,7), \quad (7+2,\emptyset), \quad (\emptyset,7+2), \quad (7+1,1), \quad (1,7+1), \quad (5+2,2),$$
$$(2,5+2), \quad (5+2+1,1), \quad (1,5+2+1), \quad (5+1,2+1), \quad (2+1,5+1).$$

Recently, Prasad and Prasad [13] proved infinite families of congruences modulo powers of 2 for $B_{4,5}(n)$. In this paper, we prove infinite families of congruences modulo powers of 2 for $B_{j,k}(n)$ with $(j,k) \in \{(2,3), (3,4), (5,8)\}$. In particular, we prove congruences modulo 4 for $B_{2,3}(n)$ and congruences modulo 4 and 8 for $B_{3,4}(n)$ and $B_{5,8}(n)$. To prove our results, we employ some q-series identities which are enlisted in Section 2. In Section 3, we prove our congruences.

2 Preliminaries

In this section, we present the q-series identities that are used in our proofs. Ramanujan's general theta-function $f(a,b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \ |ab| < 1. \tag{2.1}$$

One of the important special cases of $f(a,b)$ [1, p. 36, Entry 22 (ii)] is the theta-function $\psi(q)$ given by

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{{f_2}^2}{f_1}. \tag{2.2}$$

Lemma 2.1. *[3, Theorem 2.1] For any prime $p > 2$, we have:*

$$\psi(q) = \sum_{j=0}^{(p-3)/2} q^{(j^2+j)/2} f\left(q^{(p^2+(2j+1)p)/2}, q^{(p^2-(2j+1)p)/2}\right) + q^{(p^2-1)/8}\psi(q^{p^2}). \tag{2.3}$$

Furthermore, $(j^2+j)/2 \not\equiv (p^2-1)/8 \pmod p$ *for* $0 \le j \le (p-3)/2$.

Lemma 2.2. *[3, Theorem 2.2] For any prime $p \ge 5$, we have:*

$$f_1 = \sum_{\substack{k=-(p-1)/2 \\ k \neq (\pm p-1)/6}}^{(p-1)/2} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{(\pm p-1)/6} q^{\frac{p^2-1}{24}} f_{p^2}, \tag{2.4}$$

where

$$\frac{\pm p-1}{6} = \begin{cases} \dfrac{(p-1)}{6}, & if\ p \equiv 1 \pmod 6 \\ \dfrac{(-p-1)}{6}, & if\ p \equiv -1 \pmod 6). \end{cases}$$

Furthermore, if $\dfrac{-(p-1)}{2} \le k \le \dfrac{(p-1)}{2}$ *and* $k \neq \dfrac{(\pm p-1)}{6}$, *then*

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p.$$

Lemma 2.3. *[1, p. 303, Entry 17(v)] We have:*

$$f_1 = f_{49}\left(\frac{E(q^7)}{C(q^7)} - q\frac{D(q^7)}{E(q^7)} - q^2 + q^5\frac{C(q^7)}{D(q^7)}\right), \tag{2.5}$$

where $D(q) = f(-q^3, -q^4), E(q) = f(-q^2, -q^5)$ *and* $C(q) = f(-q, -q^6)$.

Lemma 2.4. *[6] We have:*

$$f_1 = f_{25}(Z(q^5) - q - q^2 Z(q^5)^{-1}), \tag{2.6}$$

where

$$Z(q) = \frac{(q^2;q^5)_\infty (q^3;q^5)_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty}.$$

Lemma 2.5. *[8] The following 2-dissections hold:*

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{2.7}$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}. \tag{2.8}$$

Lemma 2.6. *[10] We have:*

$$f_1 f_5^3 = f_2^3 f_{10} - q\frac{f_2^2 f_{10}^2 f_{20}}{f_4} + 2q^2 f_4 f_{20}^3 - 2q^3 \frac{f_4^4 f_{10} f_{40}^2}{f_2 f_8^2}, \tag{2.9}$$

$$\frac{1}{f_1^3 f_5} = \frac{f_4^4}{f_2^7 f_{10}} - 2q\frac{f_4^6 f_{20}^2}{f_2^9 f_{10}^3} + 5q\frac{f_4^3 f_{20}}{f_2^8} + 2q^2\frac{f_4^9 f_{40}^2}{f_2^{10} f_8^2 f_{10}^2 f_{20}}. \tag{2.10}$$

Lemma 2.7. *[7] We have:*

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}} \tag{2.11}$$

Lemma 2.8. *[16] We have:*

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q \frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}. \tag{2.12}$$

Lemma 2.9. *[1, Page 345] The following 3-dissection hold:*

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3 \frac{f_3^2 f_{18}^6}{f_6^2 f_9^3}. \tag{2.13}$$

To end this section, we record the following congruences which can be easily proved using the binomial theorem: For any positive integers s and t,

$$f_s^{2t} \equiv f_{2s}{}^t \pmod 2 \tag{2.14}$$

and

$$f_s^{4t} \equiv f_{2s}{}^{2t} \pmod 4. \tag{2.15}$$

3 Congruences for $B_{2,3}(n)$

Theorem 3.1. *Let $p \geq 5$ be a prime with $\left(\frac{-6}{p}\right) = -1$ and r any integer with $1 \leq r \leq p-1$. Then for all integers $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} B_{2,3}\Big(4 \cdot p^{2\alpha} n + \frac{7 \cdot p^{2\alpha} - 1}{6}\Big) q^n \equiv f_1 f_6 \pmod 4 \tag{3.1}$$

and

$$B_{2,3}\Big(4 \cdot p^{2\alpha+1}(pn+r) + \frac{7 \cdot p^{2(\alpha+1)} - 1}{6}\Big) \equiv 0 \pmod 4. \tag{3.2}$$

Proof. Setting $j=2$ and $k=3$ in (1.5), we obtain

$$\sum_{n=0}^{\infty} B_{2,3}(n) q^n = \frac{f_2^4 f_{12}^2 f_3^2}{f_6^4 f_4^2 f_1^2} = \frac{f_2^4 f_{12}^2}{f_6^4 f_4^2} \Big(\frac{f_3}{f_1}\Big)^2. \tag{3.3}$$

Employing (2.12) in (3.3) and then extracting the terms involving odd powers of q, we obtain

$$\sum_{n=0}^{\infty} B_{2,3}(2n+1) q^n = 2 \frac{f_4 f_{12} f_6}{f_2} \Big(\frac{1}{f_3^2}\Big). \tag{3.4}$$

Employing (2.8) in (3.4) and then extracting the terms involving even powers of q, we have

$$\sum_{n=0}^{\infty} B_{2,3}(4n+1) q^n = 2 \frac{f_2 f_6 f_{12}^5}{f_1 f_{24}^2 f_3^4}. \tag{3.5}$$

With the help of (2.14), (3.5) can be written as

$$\sum_{n=0}^{\infty} B_{2,3}(4n+1) q^n \equiv 2 f_1 f_6 \pmod 4, \tag{3.6}$$

which is the $\alpha = 0$ case of (3.1). Assume that (3.1) is true for some integer $\alpha \geq 0$. Using (2.4) in (3.1), we find that

$$\sum_{n\geq 0} B_{2,3}\Big(4 \cdot p^{2\alpha} n + \frac{7 \cdot p^{2\alpha} - 1}{6}\Big) q^n$$

$$\equiv 2\Bigg[\sum_{\substack{k=-(p-1)/2 \\ k \neq (\pm p-1)/6}}^{(p-1)/2} q^{(3k^2+k)/2} f\big(q^{(3p^2+(6k+1)p)/2}, q^{(3p^2-(6k+1)p)/2}\big) + q^{(p^2-1)/24} f_{p^2}\Bigg] \times$$

$$\left[\sum_{\substack{m=-(p-1)/2 \\ m\neq(\pm p-1)/6}}^{(p-1)/2} q^{3(3m^2+m)} f\left(q^{3(3p^2+(6m+1)p)}, q^{3(3p^2-(6m+1)p)}\right) + q^{(p^2-1)/4} f_{6p^2}\right] \pmod 4. \quad (3.7)$$

Consider the congruence

$$\frac{(3k^2+k)}{2} + 3(3m^2+m) \equiv \frac{7(p^2-1)}{24} \pmod p,$$

which is equivalent to

$$(6k+1)^2 + 6(6m+1)^2 \equiv 0 \pmod p.$$

Since $\left(\frac{-6}{p}\right) = -1$, the above congruence has only solution $k = m = \frac{(\pm p-1)}{6}$. Therefore, extracting the terms involving $q^{pn+7(p^2-1)/24}$ from both sides of (3.7), dividing throughout by $q^{7(p^2-1)/24}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} B_{2,3}\Big(4\cdot p^{2\alpha+1}n + \frac{7\cdot p^{2\alpha+2}-1}{6}\Big)q^n \equiv 2f_p f_{6p} \pmod 4. \quad (3.8)$$

Extracting the terms involving q^{pn} from (3.8) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} B_{2,3}\Big(4\cdot p^{2(\alpha+1)}n + \frac{7\cdot p^{2\alpha+2}-1}{6}\Big)q^n \equiv 2f_1 f_6 \pmod 4, \quad (3.9)$$

which is the $\alpha+1$ case of (3.1). Thus, by the principle of mathematical induction, we complete the proof of (3.1).

Extracting the coefficients of terms involving q^{pn+r}, for $1 \le r \le p-1$, from both sides of (3.8), we arrive at (3.2). □

4 Congruences for $B_{3,4}(n)$

Theorem 4.1. *For any prime $p \ge 5$ with $\left(\frac{-3}{p}\right) = -1$ and r any integer with $1 \le r \le p-1$, then for any integers $\alpha \ge 0$, we have*

$$\sum_{n=0}^{\infty} B_{3,4}\Big(4\cdot p^{2\alpha}n + \frac{7\cdot p^{2\alpha}-1}{2}\Big)q^n \equiv 4\psi(q^4)\psi(q^3) \pmod 8 \quad (4.1)$$

and

$$B_{3,4}\Big(4\cdot p^{2\alpha+1}(pn+r) + \frac{7\cdot p^{2\alpha+2}-1}{2}\Big) \equiv 0 \pmod 8. \quad (4.2)$$

Proof. Setting $j=3$ and $k=4$ in (1.5), we obtain

$$\sum_{n=0}^{\infty} B_{3,4}(n)q^n = \frac{f_2^2 f_{24}^2 f_3^2 f_4^2}{f_6^2 f_8^2 f_{12}^2 f_1^2} = \frac{f_2^2 f_{24}^2 f_4^2}{f_6^2 f_8^2 f_{12}^2}\left(\frac{f_3}{f_1}\right)^2. \quad (4.3)$$

Employing (2.12) in (4.3) and then extracting the terms involving odd powers of q, we obtain

$$\sum_{n=0}^{\infty} B_{3,4}(2n+1)q^n = 2\frac{f_2^3 f_{12}^3}{f_4 f_6^3}\left(\frac{1}{f_1^2}\right). \quad (4.4)$$

Employing (2.8) in (4.4) and then extracting the terms involving odd powers of q, we have

$$\sum_{n=0}^{\infty} B_{3,4}(4n+3)q^n = 4\frac{f_2 f_8^2 f_6^3}{f_4 f_3^3 f_1^2}. \quad (4.5)$$

With the help of (2.2) and (2.14), (4.5) can be rewritten as

$$\sum_{n=0}^{\infty} B_{3,4}(4n+3)q^n \equiv 4\psi(q^3)\psi(q^4) \pmod 8, \tag{4.6}$$

which is the $\alpha = 0$ case of (4.1). Assume that (4.1) is true for some integer $\alpha \geq 0$. Using (2.3) in (4.1), we obtain

$$\sum_{n=0}^{\infty} B_{3,4}\Big(4\cdot p^{2\alpha} n + \frac{7\cdot p^{2\alpha}-1}{2}\Big)q^n$$

$$\equiv 4\left[\sum_{k=0}^{(p-3)/2} q^{3(k^2+k)/2} f\left(q^{3(p^2+(2k+1)p)/2}, q^{3(p^2-(2k+1)p)/2}\right) + q^{3(p^2-1)/8}\psi(q^{3p^2})\right]\times$$

$$\left[\sum_{j=0}^{(p-3)/2} q^{2(j^2+j)} f\left(q^{2(p^2+(2j+1)p)}, q^{2(p^2-(2j+1)p)}\right) + q^{(p^2-1)/2}\psi(q^{4p^2})\right] \pmod 8. \tag{4.7}$$

Consider the congruence

$$\frac{3(k^2+k)}{2} + 2(j^2+j) \equiv \frac{7(p^2-1)}{8} \pmod p,$$

which is equivalent to

$$(4j+2)^2 + 3(2k+1)^2 \equiv 0 \pmod p.$$

For $\left(\frac{-3}{p}\right) = -1$, the above congruence has only solution $k = j = \frac{(p-1)}{2}$. Therefore, extracting the terms involving $q^{pn+7(p^2-1)/8}$ from both sides of (4.7), dividing throughout by $q^{7(p^2-1)/8}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} B_{3,4}\Big(4\cdot p^{2\alpha+1} n + \frac{7\cdot p^{2\alpha+2}-1}{2}\Big)q^n \equiv 4\psi(q^{3p})\psi(q^{4p}) \pmod 8. \tag{4.8}$$

Extracting the terms involving q^{pn} from (4.8) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} B_{3,4}\Big(4\cdot p^{2(\alpha+1)} n + \frac{7\cdot p^{2\alpha+2}-1}{2}\Big)q^n \equiv 4\psi(q^3)\psi(q^4) \pmod 8, \tag{4.9}$$

which is the $\alpha + 1$ case of (4.1). Hence by the method of induction, we complete the proof of (4.1).

Extracting the terms involving q^{pn+r}, for $1 \leq r \leq p-1$, from both sides of (4.8), we arrive at (4.2). □

Theorem 4.2. *Let* $t \in \{5, 9\}$, $u_1 \in \{33, 57\}$, $u_2 \in \{51, 99\}$, $u_3 \in \{39, 63, 87, 111\}$ *and* $u_4 \in \{45, 69, 93, 117, 141, 165\}$ *. Then for all integers* $\alpha, \beta, \gamma \geq 0$*, we have*

$$B_{3,4}(12n+t) \equiv 0 \pmod 4, \tag{4.10}$$

$$\sum_{n=0}^{\infty} B_{3,4}\Big(12\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}\cdot n + \frac{3\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1}{2}\Big)q^n \equiv 2f_1^3 \pmod 4, \tag{4.11}$$

$$\sum_{n=0}^{\infty} B_{3,4}\Big(12\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}\cdot n+\frac{3\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}-1}{2}\Big)q^n \equiv 2f_7^3 \pmod 4, \tag{4.12}$$

$$B_{3,4}\Big(12\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}\cdot n+\frac{3\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1}{2}\Big) \equiv \begin{cases} 2 \pmod 4, & if\ n\ is\ a\ pentagonal\ number \\ 0 \pmod 4, & otherwise, \end{cases} \tag{4.13}$$

$$\sum_{n=0}^{\infty} B_{3,4}\Big(12\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}\cdot n+\frac{3\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1}{2}\Big)q^n \equiv 2f_3^3 \pmod 4, \tag{4.14}$$

$$B_{3,4}\Big(12\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}\cdot n+\frac{17\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1}{2}\Big) \equiv 0 \pmod 4, \tag{4.15}$$

$$B_{3,4}\Big(12\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}\cdot n+\frac{u_1\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1}{2}\Big) \equiv 0 \pmod 4, \tag{4.16}$$

$$\sum_{n=0}^{\infty} B_{3,4}\Big(12\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}\cdot n+\frac{3\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}-1}{2}\Big)q^n \equiv 2f_5^3 \pmod 4, \tag{4.17}$$

$$B_{3,4}\Big(12\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}\cdot n+\frac{u_2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1}{2}\Big) \equiv 0 \pmod 4, \tag{4.18}$$

$$B_{3,4}\Big(12\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}\cdot n+\frac{u_3\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1}{2}\Big) \equiv 0 \pmod 4 \tag{4.19}$$

$$B_{3,4}\Big(12\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}\cdot n+\frac{u_4\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}-1}{2}\Big) \equiv 0 \pmod 4. \tag{4.20}$$

Proof. Employing (2.8) in (4.4) and then extracting the terms involving even powers of q, we obtain

$$\sum_{n=0}^{\infty} B_{3,4}(4n+1)q^n = 2\frac{f_6^3 f_4^5}{f_2 f_3^3 f_8^2 f_1^2}. \tag{4.21}$$

With the help of (2.14), (4.21) can be written as

$$\sum_{n=0}^{\infty} B_{3,4}(4n+1)q^n \equiv 2f_3^3 \pmod 4. \tag{4.22}$$

Extracting the terms involving q^{3n+i} for $i = 1, 2$, yields (4.10). Next extracting the terms involving q^{3n} from (4.22), we obtain

$$\sum_{n=0}^{\infty} B_{3,4}(12n+1)q^n \equiv 2f_1^3 \pmod 4. \tag{4.23}$$

The equation (4.23) is the case $\alpha = \beta = \gamma = 0$ of equation (4.11). Suppose that the congruence (4.11) is true for any integer $\alpha \geq 0$ with $\beta = \gamma = 0$. Utilising (2.13) in (4.11) with $\beta = \gamma = 0$ and then extracting the coefficients of q^{3n+1}, we arrive at

$$\sum_{n=0}^{\infty} B_{3,4}\Big(12\cdot 3^{2\alpha+1}n+\frac{3\cdot 3^{2\alpha+2}-1}{2}\Big)q^n \equiv 2f_3^3 \pmod 4. \tag{4.24}$$

Extracting the coefficient of the terms involving q^{3n}, on both sides of (4.24), we obtain

$$\sum_{n=0}^{\infty} B_{3,4}\Big(12\cdot 3^{2\alpha+2}n+\frac{3\cdot 3^{2\alpha+2}-1}{2}\Big)q^n \equiv 2f_1^3 \pmod 4, \tag{4.25}$$

which implies that (4.11) is true for integers $\alpha+1$ with $\beta = \gamma = 0$. By mathematical induction, (4.11) is true for all integer α. Suppose that the congruence (4.11) holds for $\alpha, \beta \geq 0$ with $\gamma = 0$.

Utilising (2.6) in (4.11) and then extracting the terms involving q^{5n+3}, we obtain

$$\sum_{n=0}^{\infty} B_{3,4}\Big(12\cdot 3^{2\alpha}\cdot 5^{2\beta+1}n+\frac{3\cdot 3^{2\alpha}\cdot 5^{2\beta+2}-1}{2}\Big)q^n \equiv 2f_5^3 \pmod 4. \tag{4.26}$$

Extracting the coefficient of the terms involving q^{5n}, on both sides of (4.26), we obtain

$$\sum_{n=0}^{\infty} B_{3,4}\Big(12\cdot 3^{2\alpha}\cdot 5^{2\beta+2}n+\frac{3\cdot 3^{2\alpha}\cdot 5^{2\beta+2}-1}{2}\Big)q^n \equiv 2f_1^3 \pmod 4, \tag{4.27}$$

which implies that (4.11) is true for integers $\beta+1$ with $\gamma=0$. By mathematical induction, (4.11) is true for all non-negative integers α,β with $\gamma=0$. Suppose that the congruence (4.11) holds for $\alpha,\beta,\gamma\geq 0$. Utilising (2.5) in (4.11) and then extracting the terms involving q^{7n+6}, we obtain

$$\sum_{n=0}^{\infty} B_{3,4}\Big(12\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+\frac{3\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}-1}{2}\Big)q^n \equiv 2f_7^3 \pmod 4, \tag{4.28}$$

which proves (4.12). Extracting the coefficients of the terms involving q^{7n} from (4.28), we obtain

$$\sum_{n=0}^{\infty} B_{3,4}\Big(12\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+\frac{3\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}-1}{2}\Big)q^n \equiv 2f_1^3 \pmod 4, \tag{4.29}$$

which implies that (4.11) is true for all integers $\gamma+1$. By mathematical induction (4.11) is true for all non-negative integers α,β,γ.

Employing (2.13) in (4.11) and then extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2}, we arrive at (4.13), (4.14) and (4.15) respectively. Extracting the coefficients of q^{3n+i} for $i=1,2$ on both sides of (4.14), we arrive at (4.16).

Employing (2.6) in (4.11) and then extracting the terms involving q^{5n+3}, we arrive at (4.17). Again, employing (2.6) in (4.11) and then extracting the terms involving q^{5n+i} for $i=2,4$, we arrive at (4.18). Extracting the terms involving q^{5n+i} for $i=1,2,3,4$ from (4.17), yields (4.19). Finally, extracting the terms involving q^{7n+i} for $i=1,2,3,4,5,6$ from (4.12), yields (4.20).

□

5 Congruences for $B_{5,8}(n)$

Theorem 5.1. *Let $t_1\in\{166,214\}$ and $t_2\in\{62,158\}$. Then for all integers $\alpha\geq 0$, we have*

$$B_{5,8}\Big(16n+13\Big)\equiv 0 \pmod 8, \tag{5.1}$$

$$\sum_{n=0}^{\infty} B_{5,8}\Big(16\cdot 5^{2\alpha}n+\frac{22\cdot 5^{2\alpha}-7}{3}\Big)q^n \equiv 4f_1f_{10} \pmod 8, \tag{5.2}$$

$$B_{5,8}\Big(16\cdot 5^{2\alpha+1}n+\frac{t_1\cdot 5^{2\alpha}-7}{3}\Big)\equiv 0 \pmod 8, \tag{5.3}$$

$$\sum_{n=0}^{\infty} B_{5,8}\Big(16\cdot 5^{2\alpha+1}n+\frac{14\cdot 5^{2\alpha+1}-7}{3}\Big)q^n \equiv 4f_2f_5 \pmod 8 \tag{5.4}$$

and

$$B_{5,8}\Big(16\cdot 5^{2\alpha+2}n+\frac{t_2\cdot 5^{2\alpha+1}-7}{3}\Big)\equiv 0 \pmod 8. \tag{5.5}$$

Proof. Setting $j=5$ and $k=8$ in (1.5), we obtain

$$\sum_{n=0}^{\infty} B_{5,8}(n)q^n=\frac{f_2^2f_{80}^2f_5^2f_8^2}{f_{10}^2f_{16}^2f_1^2f_{40}^2}=\frac{f_2^2f_{80}^2f_8^2}{f_{10}^2f_{16}^2f_{40}^2}\left(\frac{f_5}{f_1}\right)^2. \tag{5.6}$$

Employing (2.11) in (5.6) and then extracting the terms involving odd powers of q, we obtain

$$\sum_{n=0}^{\infty} B_{5,8}(2n+1)q^n = 2\frac{f_2^3 f_{40}^2 f_4^2 f_{10}}{f_{20}^2 f_8^2}\Big(\frac{1}{f_1^3 f_5}\Big). \tag{5.7}$$

Employing (2.10) in (5.7) and extracting the terms involving even powers of q, we obtain

$$\sum_{n=0}^{\infty} B_{5,8}(4n+1)q^n = 2\frac{f_2^6 f_{20}^2}{f_1^4 f_4^2 f_{10}^2} + 4q\frac{f_2^{11} f_{20}^4}{f_2^2 f_4^4 f_{10}^3 f_5 f_1^3}. \tag{5.8}$$

With the help of (2.14) in second terms of (5.8), we obtain

$$\sum_{n=0}^{\infty} B_{5,8}(4n+1)q^n \equiv 2\frac{f_2^6 f_{20}^2}{f_4^2 f_{10}^2}\Big(\frac{1}{f_1^4}\Big) + 4qf_{10}^4\Big(\frac{f_5}{f_1}\Big) \pmod 8. \tag{5.9}$$

Now employing (2.7) and (2.11) in (5.9) and then extracting the terms involving odd powers of q, we obtain

$$\sum_{n=0}^{\infty} B_{5,8}(8n+5)q^n \equiv 4f_2 f_{20} \pmod 8. \tag{5.10}$$

Extracting the terms involving odd powers of q on both sides of (5.10), yields (5.1). Next, extracting the terms involving even powers of q on both sides of (5.10), we obtain

$$\sum_{n=0}^{\infty} B_{5,8}(16n+5)q^n \equiv 4f_1 f_{10} \pmod 8, \tag{5.11}$$

which is the case $\alpha = 0$ of (5.2). Assume that (5.2) is true for some integers $\alpha \geq 0$. Using (2.6) in (5.2) and then extracting the terms involving q^{5n+1}, we obtain

$$\sum_{n=0}^{\infty} B_{5,8}\Big(16\cdot 5^{2\alpha+1}n + \frac{14\cdot 5^{2\alpha+1}-7}{3}\Big)q^n \equiv 4f_2 f_5 \pmod 8. \tag{5.12}$$

Again, using (2.6) in (5.12) and then extracting the terms involving q^{5n+2}, we obtain

$$\sum_{n=0}^{\infty} B_{5,8}\Big(16\cdot 5^{2\alpha+2}n + \frac{22\cdot 5^{2\alpha+2}-7}{3}\Big)q^n \equiv 4f_1 f_{10} \pmod 8, \tag{5.13}$$

which is the case $\alpha+1$ of (5.2). Hence, by using the method of inductions, we complete the proof of (5.2). Employing (2.6) in (5.2) and then extracting the terms involving q^{5n+i} for $i = 3, 4$ and q^{5n+1}, we arrive at the result (5.3) and (5.4) respectively. Finally employing (2.6) in (5.4) and extracting the terms involving q^{5n+i} for $i = 1, 3$, we arrive at the result (5.5). □

Theorem 5.2. *Let* $w_1 \in \{9, 17, 25, 33\}$, $w_2 \in \{88, 112\}$ *and* $w_3 \in \{56, 104\}$. *Then for all integers* $\alpha \geq 0$, *we have*

$$B_{5,8}\Big(40n + w_1\Big) \equiv 0 \pmod 4, \tag{5.14}$$

$$B_{5,8}\Big(80n+1\Big) \equiv \begin{cases} 2 \pmod 4, & \textit{if } n \textit{ is a pentagonal number} \\ 0 \pmod 4, & \textit{otherwise.} \end{cases} \tag{5.15}$$

$$\sum_{n=0}^{\infty} B_{5,8}\Big(8\cdot 5^{2\alpha}n + \frac{16\cdot 5^{2\alpha}-7}{3}\Big)q^n \equiv 2f_1 f_5^3 \pmod 4, \tag{5.16}$$

$$\sum_{n=0}^{\infty} B_{5,8}\Big(8\cdot 5^{2\alpha+1}n + \frac{8\cdot 5^{2\alpha+1}-7}{3}\Big)q^n \equiv 2f_1^3 f_5 \pmod 4, \tag{5.17}$$

$$B_{5,8}\Big(8\cdot 5^{2\alpha+1}n + \frac{w_2\cdot 5^{2\alpha}-7}{3}\Big) \equiv 0 \pmod 4 \tag{5.18}$$

and

$$B_{5,8}\Big(8\cdot 5^{2\alpha+2}n + \frac{w_3\cdot 5^{2\alpha+1}-7}{3}\Big) \equiv 0 \pmod 4. \tag{5.19}$$

Proof. With the help of (2.14), equation (5.9) can be rewritten as

$$\sum_{n=0}^{\infty} B_{5,8}(4n+1)q^n \equiv 2\frac{f_2^6 f_{20}^2}{f_4^2 f_{10}^2}\left(\frac{1}{f_1^4}\right) \pmod{4}. \tag{5.20}$$

Now employing (2.7) in (5.20) and then extracting the terms involving even powers of q, we obtain

$$\sum_{n=0}^{\infty} B_{5,8}(8n+1)q^n \equiv 2\frac{f_{10}^2}{f_5^2} \pmod{4}. \tag{5.21}$$

Extracting the terms involving q^{5n+i} for $i=1,2,3,4$ from both sides of (5.21), yields the result (5.14). Again, extracting the terms involving q^{5n} from both sides of (5.21), we obtain

$$\sum_{n=0}^{\infty} B_{5,8}(40n+1)q^n \equiv 2\frac{f_2^2}{f_1^2} \equiv 2f_2 \pmod{4}. \tag{5.22}$$

Finally, extracting the terms involving even powers of q from (5.22), yields result (5.15).

Employing (2.10) in (5.7) and extracting the terms involving odd powers of q, we obtain

$$\sum_{n=0}^{\infty} B_{5,8}(4n+3)q^n = 2\frac{f_2^5 f_5 f_{20}^2}{f_1^5 f_4^2 f_{10}}. \tag{5.23}$$

With the help of (2.14) in (5.23), we obtain

$$\sum_{n=0}^{\infty} B_{5,8}(4n+3)q^n \equiv 2f_{10}^4\frac{1}{f_1^3 f_5} \pmod{4}. \tag{5.24}$$

Employing (2.10) in (5.24) and then extracting the terms involving even powers of q, we obtain

$$\sum_{n=0}^{\infty} B_{5,8}(8n+3)q^n \equiv 2f_1 f_5^3 \pmod{4}, \tag{5.25}$$

which is the $\alpha=0$ case of (5.16). Assume that (5.16) is true for some integer $\alpha \geq 0$. Using (2.6) in (5.16) and then extracting the terms involving q^{5n+1}, we obtain

$$\sum_{n=0}^{\infty} B_{5,8}\left(8\cdot 5^{2\alpha+1}n + \frac{8\cdot 5^{2\alpha+1}-7}{3}\right)q^n \equiv 2f_1^3 f_5 \pmod{4}. \tag{5.26}$$

Again, using (2.6) in (5.26) and then extracting the terms involving q^{5n+3}, we obtain

$$\sum_{n=0}^{\infty} B_{5,8}\left(8\cdot 5^{2(\alpha+1)}n + \frac{16\cdot 5^{2(\alpha+1)}-7}{3}\right)q^n \equiv 2f_1 f_5^3 \pmod{4}, \tag{5.27}$$

which is the case $\alpha+1$ of (5.16). Hence, by using the method of induction, we complete the proof of (5.16). Employing (2.6) in (5.16) and then extracting the terms involving q^{5n+1} and q^{5n+i} for $i=3,4$, yields (5.17) and (5.18) respectively. Finally, employing (2.6) in (5.17) and then extracting the terms involving q^{5n+i} for $i=2,4$ yields (5.19). □

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Received: December 21, 2021.
Accepted: April 27, 2022.