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ANNIHILATOR IDEAL GRAPH OF A LATTICE

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The article is dedicated to Late Sopan Kawade, school teacher of the second author.

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Abstract For a lattice L we associate a graph called annihilator ideal graph of L, it is denoted by $AnnIG(L)$, is a graph with vertex set all ideals of L having nontrivial annihilators, and two ideal I and J are adjacent if either annihilator of I contains a nonzero element of J or annihilator of J contains a nonzero element of I. For any atomic lattice L, it is shown that $AnnIG(L)$ is a complete multipartite graph. We characterize all lattices whose annihilator ideal graph is planar or complete. For any positive integers $m_1, m_2, \cdots, m_t$, we have constructed a lattice L such that $AnnIG(L) = K_{m_1,m_2,\cdots,m_t}$.

1 Introduction

A lattice is an algebraic structure $L = \langle L, \wedge, \vee \rangle$ such that both $\wedge, \vee$ operations are commutative, associative and $x \wedge x = x \vee x = x,\ (x \wedge y) \vee z = (x \vee y) \wedge z$ for all $x, y, z \in L$. A nonempty subset K of L which is closed under both operations, $\wedge,\ \vee$ is called the sublattice of L. A sublattice I of L is called an ideal of L, if $x \wedge i \in I$ for all $x \in L,\ i \in I$. The set $(x] = \{y \in L \colon y \leq x\}$ is the principal ideal generated by x in a lattice L. A lattice L is said to be a bounded lattice if there exist elements, say 0 and 1 such that $0 \leq x \leq 1$, for all $x \in L$. For any two ideals I and J of lattice L, we have $I \wedge J = \{a \wedge b : a \in I, b \in J\}$. For an ideal I of a lattice L, $Ann(I)$=$\{x \in L : x \wedge i = 0, \forall i \in I\}$. Let $a_1 \in L$ then $[a_1]^u = \{x \in L \mid x \geq a_1\}$. Let L be a lattice and $x, y \in L$ we say that y is covered by x or x covers y (denoted by $y \prec x$ or $x \succ y$), if $y < x$ and there does not exist $z \in L$ such that, $y < z < x$. An element $x \in L$ is said to be an atom or co-atom if $0 \prec x$ or $x \prec 1$ respectively. Let $A(L)$ be the set of all atoms in a lattice L then $A(I) = I \cap A(L)$ is the set of all atoms in an ideal I. If $a, b \in L$ such that $a \nleq b$ and $b \nleq a$ then we say that a and b are incomparable, we denote it by $a||b$. A lattice L is said to be atomic if for every element $y \in L$ there is an atom $a_y \in A(L)$ such that $a_y \leq y$. Atomic lattice L is said to be atomistic if every $x \in L$ is either an atom or join of atoms in L. For concepts in lattice theory please refer to ([6],[3]). In ([7]), authors have given various constructions of new lattices from two lattices.

If V is a nonempty set(called vertices) and E is a set of 2-subsets of V (called edges), then $G =< V, E >$ is called a graph on V with the edge set E. The 2-subset $e = \{u, v\} \in E$ is called an edge between u and v, in this case, we say that u and v are adjacent in G. If there exist subsets $V_1, V_2, \cdots, V_n$ of V such that $V = \bigcup_{i=1}^{n} V_i,\quad V_i \cap V_j = \phi$ for all $i \neq j$ and no two vertices in the same set V_i are adjacent and any two vertices from different sets $V_i's$ are adjacent, then the graph $G =< V, E >$ is called a complete multipartite graph. It is denoted by $K_{\langle |V_1|, |V_2|, \cdots, |V_n| \rangle}$. Let $G(V, E)$ be a graph. If there is a path joining any two vertices of graph G, then G is called a connected graph otherwise, we say it is a disconnected graph. For any vertices, a and b of a graph G, the length of a shortest path from a to b is denoted by $d(a, b)$, and $diam(G)$=$sup\,\{d(a, b) \mid a, b \in V(G)\}$ is called a diameter of graph G. The girth of a graph G is the length of shortest cycle in G and it is denoted by $gr(G)$. A graph is said to be a null graph if it is a graph without edges. In ([12]) the authors given a good introduction to the graph theory. Throughout this paper, L denotes an atomic lattice with the smallest element 0.

In ([9]), Avinash Patil characterized all finite Rickart $*$-rings whose zero-divisor graphs are planar. In ([8]), the authors examine preservation of diameter and girth of the zero-divisor graph under extension to Laurent polynomial and Laurent power series rings. In ([1]), D. F. Anderson et al. explored the construction of graphs from commutative rings. In ([2]), the authors introduced the graph $\Gamma_{Ann}(R)$ of a commutative ring R in which two nontrivial ideals (vertices) are adjacent if and only if $I \cap Ann(J) \neq \{0\}$ or $Ann(I) \cap J \neq 0$. Motivated from this, we define the annihilator ideal graph $AnnIG(L)$ of a lattice L as follows.

Definition 1.1. Let L be a lattice and $N(L) = \{I \text{ is ideal in } L\colon Ann(I) \neq \{0\}\}$. The annihilator ideal graph $AnnIG(L)$ of a lattice L is the graph with vertex set $N(L)$ and two vertices I and J in $N(L)$ are adjacent if and only if $I \cap Ann(J) \neq \{0\}$ or $J \cap Ann(I) \neq \{0\}$.

In ([4], [5], [10]) the authors studies zero-divisor graph of lattices. In ([11]) the authors studies zero-divisor graph of a ring with respect to an automorphism. In this paper, we study various properties of $AnnIG(L)$ such as diameter, girth, planarity, the existence of Hamiltonian cycle, the existence of Eulerian tour, we give a characteristic polynomial for adjacency matrix of $AnnIG(L)$. In fact, we prove that $AnnIG(L)$ of a lattice L is a complete multipartite graph. For any positive integers $m_1, m_2, \cdots, m_t$, we have constructed a lattice L such that $AnnIG(L) = K_{m_1, m_2, \cdots, m_t}$. Also, we have given the relation between the zero-divisor graph and the annihilator ideal graph of a lattice.

2 $AnnIG(L)$ is a complete multipartite graph

In this section, we prove that $AnnIG(L)$ is a complete multipartite graph. We begin with the following lemma which gives a characterization of adjacency of vertices in $AnnIG(L)$.

Lemma 2.1. *Let L be a lattice. Then for any $I, J \in N(L)$, I and J are adjacent in $AnnIG(L)$ if and only if $Ann(I) \neq Ann(J)$.*

Proof. If I and J are adjacent in $AnnIG(L)$, then $I \cap Ann(J) \neq 0$ *or* $J \cap ann(I) \neq 0$. Suppose $I \cap Ann(J) \neq 0$. Then there exist $i \in I - \{0\}$ such that $i \wedge j = 0$ for all $j \in J$. If $i \in Ann(I)$ then $i \wedge i = 0$, a contradiction, as $i \neq 0$. Therefore $Ann(I) \neq Ann(J)$. Similarly, if $J \cap Ann(I) \neq \{0\}$, then $Ann(I) \neq Ann(J)$. Conversely, suppose $Ann(I) \neq Ann(J)$. We have to prove I and J are adjacent. On the contrary, suppose I and J are not adjacent in $AnnIG(L)$. Therefore $I \cap Ann(J) = \{0\}$ and $J \cap Ann(I) = \{0\}$. Let $x \in Ann(I)$. If $x \notin Ann(J)$ then $x \in Ann(I) \setminus Ann(J)$. Hence $x \wedge i = 0$ for all $i \in I$ and $x \wedge j_0 \neq 0$ for some $j_0 \in J \setminus \{0\}$. Since J is an ideal, $x \wedge j_o \in J$. Since $J \cap Ann(I) = \{0\}$, $0 < (x \wedge j_0) \wedge i$ for all $i \in I,\ i \neq 0$. But for all $i \in I$, $(x \wedge j_0) \wedge i = j_0 \wedge (x \wedge i) = j_0 \wedge 0 = 0$, a contradiction. Thus, we get $Ann(I) \subseteq Ann(J)$. Similarly, $Ann(J) \subseteq Ann(I)$. Therefore $Ann(I) = Ann(J)$. This is a contradiction to the assumption that $Ann(I) \neq Ann(J)$. Therefore I and J must be adjacent in $AnnIG(L)$. □

Note that, if ideal $I \in N(L)$ then $A(I) \neq \{0\}$. In the following lemma, we characterize the adjacency of vertices I and J in $AnnIG(L)$ in terms of $A(I)$ and $A(J)$.

Lemma 2.2. *Let L be a lattice and $I, J \in N(L)$. Then I and J are adjacent in $AnnIG(L)$ if and only if $A(I) \neq A(J)$.*

Proof. Assume that $A(I) \neq A(J)$. Let $a_1 \in A(I) \setminus A(J)$. Since $a_1 = a_1 \wedge a_1 \neq 0$, $a_1 \notin Ann(I)$. If $a_1 \notin Ann(J)$ then there is $j_0 \in J \setminus \{0\}$ such that $a_1 \wedge j_0 \neq 0$. Since a_1 is an atom, $a_1 = a_1 \wedge j_0 \in J \setminus \{0\}$. Therefore $a_1 \in A(J)$, a contradiction. Hence $a_1 \in Ann(J)$. Thus $a_1 \in Ann(J) \setminus Ann(I)$. This gives $Ann(I) \neq Ann(J)$. Therefore by Lemma 2.1, I and J are adjacent in $AnnIG(L)$. Conversely, suppose $I, J \in N(L)$ are adjacent in $AnnIG(L)$. Therefore by Lemma 2.1, $Ann(I) \neq Ann(J)$. Hence $Ann(I) \setminus Ann(J) \neq \phi$ or $Ann(J) \setminus Ann(I) \neq \phi$. Assume that $Ann(I) \setminus Ann(J) \neq \phi$. Let $x \in Ann(I) \setminus Ann(J)$, then $x \wedge i = 0$ for all $i \in I$ and $x \wedge j_0 \neq 0$ for some $j_0 \in J \setminus \{0\}$. Let $y = x \wedge j_0$. Therefore $y \wedge i = (x \wedge j_0) \wedge i \leq x \wedge i = 0$ for all $i \in I$. That is $y \in Ann(I)$. If y is an atom then $y \in A(J) \setminus A(I)$ and hence $A(I) \neq A(J)$. If y is not an atom in L then choose $a \in A(J)$ such that $a < y$, it is possible because lattice L is an atomic lattice with the smallest element 0. Since $y \in Ann(I)$, we have $a \in Ann(I)$. Therefore $a \notin A(I)$. That is $a \in A(J) \setminus A(I)$, thus $A(I) \neq A(J)$. Similarly if $Ann(J) \setminus Ann(I) \neq \phi$ then $A(I) \neq A(J)$. Thus in any case $A(I) \neq A(J)$. □

The multiset is denoted by $\langle x_1, x_2, \cdots \rangle$, where $x_i's$ need not be distinct. For $S \subset A(L)$, $\overline{S}$ denotes the smallest ideal of L containing S. Moreover $\overline{S} = \{s_1 \vee s_2 \vee \cdots \vee s_n \mid s_1, s_2, \cdots, s_n \in S,\ n \in \mathbb{N}\}$. Define a relation $\sim$ on $N(L)$ as follows. For $I, J \in N(L)$ we write, $I \sim J$ if and only if $A(I) = A(J)$. Note that $\sim$ is an equivalence relation. In the following theorem, we show that $AnnIG(L)$ is a complete multipartite graph.

Theorem 2.3. *Let L be a lattice. Then $AnnIG(L)$ is a complete multipartite graph, moreover $AnnIG(L) = K_{\langle |[\overline{S}]| \mid \phi \neq S \subsetneq A(L) \rangle}$.*

Proof. Let $I, J \in N(L)$. By Lemma 2.2, I and J are adjacent if and only if $A(I) \neq A(J)$. That is I and J are adjacent if and only if $I \nsim J$. For any $I \in N(L)$, $[I] = \{J \in N(L) \mid I \sim J\}$ is an equivalence class of I. Any two elements in $[I]$ are not adjacent. If $[I]$ and $[J]$ are distinct equivalence classes, then for any $I_1 \in [I]$ and $J_1 \in [J]$, $I_1 \nsim J_1$. Therefore I_1 and J_1 are adjacent. Hence $AnnIG(L)$ is a complete multipartite graph. Note that for an ideal $I \in N(L)$ there exists unique $\phi \neq S \subsetneq A(L)$, $S = A(I)$. Hence $[I] = [\overline{S}]$. Also, for any $\phi \neq S \subsetneq A(L)$, there exists a unique ideal $\overline{S}$ such that $S = A(\overline{S})$. Therefore $AnnIG(L) = K_{\langle |[\overline{S}]| \mid \phi \neq S \subsetneq A(L) \rangle}$. □

Corollary 2.4. *Let L be a lattice then $AnnIG(L)$ is connected and $diam(AnnIG(L)) \leq 2$.*

Proof. By Theorem 2.3, $AnnIG(L)$ is the complete multipartite graph. Hence $diam(AnnIG(L)) \leq 2$. □

Corollary 2.5. *Let L is a lattice, then $gr(AnnIG(L)) \in \{3, 4, \infty\}$. Moreover,*

(i) $gr(AnnIG(L)) = 3$ if and only if $|A(L)| \geq 3$.

(ii) $gr(AnnIG(L)) = 4$ if and only if $|A(L)| = 2$ and there exist elements x_a and x_b such that $a \prec x_a,\ b||x_a$ and $b \prec x_b,\ a||x_b$, where $A(L) = \{a, b\}$.

(iii) $gr(AnnIG(L)) = \infty$ if and only if $|A(L)| \leq 2$ and there doesn't exist elements x_a and x_b such that $a \prec x_a,\ b||x_a$ and $b \prec x_b,\ a||x_b$, where $A(L) = \{a, b\}$.

Proof. If $|A(L)| \geq 3$ then K_{m_1, m_2, m_3} is subgraph of $AnnIG(L)$ for some integers $m_1, m_2, m_3 \geq 1$ and hence contains cycle of length 3.
Suppose $|A(L)| = 2$, $A(L) = \{a, b\}$ and there exist element x_a such that $a \prec x_a,\ b||x_a$ and element x_b such that $b \prec x_b,\ a||x_b$. Then $AnnIG(L) = K_{|[\overline{\{a\}}]|, |[\overline{\{b\}}]|}$. Since $a \prec x_a,\ b||x_a$ and $b \prec x_b,\ a||x_b$, hence $|[\overline{\{a\}}]| \geq 2$ and $|[\overline{\{b\}}]| \geq 2$. So $AnnIG(L)$ contains cycle of length 4. Since $AnnIG(L)$ is bipartite, it doesn't contains cycle of length 3. Thus $gr(AnnIG(L)) = 4$. Conversely, assume that $gr(AnnIG(L)) = 4$. Therefore $AnnIG(L)$ is complete bipartite graph with vertex partition $\{[\overline{\{a\}}], [\overline{\{b\}}]\}$ with $|[\overline{\{a\}}]| \geq 2$ and $|[\overline{\{b\}}]| \geq 2$. Therefore there exist elements x_a, x_b such that $a \prec x_a,\ b||x_a$ and $b \prec x_b,\ a||x_b$.
Now, if $|A(L)| = 1$ then $AnnIG(L)$ is null graph and hence $gr(AnnIG(L)) = \infty$. Assume that $A(L) = \{a, b\}$ and there doesn't exist elements x_a and x_b such that $a \prec x_a,\ b||x_a$ and $b \prec x_b,\ a||x_b$. Then $a \prec x_b$ or $b \prec x_a$, whenever $a \prec x_a$ and $b \prec x_b$. In this case $|[\overline{\{a\}}]| = 1$ or $|[\overline{\{b\}}]| = 1$ and hence $AnnIG(L)$ does not contains a cycle. Therefore $gr(AnnIG(L)) = \infty$. □

Corollary 2.6. *Let L be a lattice. Then $AnnIG(L)$ is a null graph if and only if $|A(L)| = 1$.*

Definition 2.7. An element in a lattice is said to be atomic if it is either an atom or join of atoms in the lattice. An element is said to be nonatomic if it is not atomic. For any lattice L, the set of all atomic elements in $L \setminus \{1\}$ is denoted by $\mathcal{A}(L)$. An element x in a lattice L is said to be join irreducible if for $y, z \in L$ whenever $x = y \vee z$ then $x = y$ or $x = z$.

Definition 2.8. Let L be a complete lattice, $\phi \neq S \subsetneq A(L)$ and $u_S = \bigvee\limits_{x \in S} x$.
Let $\mathcal{B}(S) = \{I \in N(L) \mid \overline{S} \subseteq I$ and I do not contain atomic element y such that $y > u_S\}$.

Now, we determine $|[\overline{S}]|$ for any $\phi \neq S \subsetneq A(L)$.

Proposition 2.9. *Let L be a complete lattice and $\phi \neq S \subsetneq A(L)$. Then $[\overline{S}] = \mathcal{B}(S)$, $AnnIG(L) = K_{\langle |\mathcal{B}(S)| \mid \phi \neq S \subsetneq A(L) \rangle}$ and number of equivalence classes of the relation $\sim$ is $|\mathcal{A}(L)|$.*

Proof. Let $I \in [\overline{S}]$. Therefore $A(I) = S$. This implies $S \subset I$. Hence $\overline{S} \subset I$. Suppose there exist atomic element $y \in I$ such that $y > u_S$. Then there exist atoms $T \subseteq A(L)$ such that $y = \bigvee_{t \in T} t$ and $S \subsetneq T$. Since $y \in I$, we have $T \subset I$, a contradiction. Therefore there does not exist an atomic element $y \in I$ such that $y > u_S$. Hence $I \in \mathcal{B}(S)$. Thus $[\overline{S}] \subseteq \mathcal{B}(S)$. Conversely, suppose $I \in \mathcal{B}(S)$. Therefore $\overline{S} \subseteq I$ and I do not contain atomic element y such that $y > u_S$. This gives $S \subseteq I$ and hence $A(\overline{S}) = S \subseteq A(I)$. Suppose there exists atom $a \in I \setminus S$. Let $y = a \vee u_S$, then $y \in I$ be an atomic element such that $y > u_S$, a contradiction. This gives $A(I) = S$ and hence $I \in [\overline{S}]$. Therefore $\mathcal{B}(S) \subseteq [\overline{S}]$. This implies that $[\overline{S}] = \mathcal{B}(S)$. Thus $AnnIG(L) = K_{\langle |\mathcal{B}(S)| \mid \phi \neq S \subsetneq A(L) \rangle}$

Let $\mathcal{E} = \{[\overline{S}] \mid \phi \neq S \subsetneq A(L)\}$. Note that $\mathcal{E}$ is a set of all equivalence classes of relation $\sim$. Let $x \in \mathcal{A}(L)$. There exist a nonempty proper subset S of $A(L)$ such that $x = u_S$. If there exists another nonempty proper subset T of $A(L)$ such that $x = u_T$. Let $s \in S$ then $x \vee s = x$. This gives $s \leq x$, that is $s \leq u_T$. Therefore $s \in T$ and hence $S \subseteq T$. Similarly $T \subseteq S$. Thus $S = T$. This implies that $[\overline{S}] = [\overline{T}]$. Define a map $f : \mathcal{A}(L) \to \mathcal{E}$, $f(x) = [\overline{S}]$, where $x = u_S$. We prove that f is a bijection. Let $x = u_S,\ y = u_T \in \mathcal{A}(L)$, where $S, T \subset A(L)$. Suppose $f(x) = f(y)$, that is $[\overline{S}] = [\overline{T}]$. This implies $S = T$, therefore $u_S = u_T$. Hence $x = y$. Therefore f is injective. Let $[\overline{S}] \in \mathcal{E}$. Then $x = u_S \in \mathcal{A}(L)$ be such that $f(x) = [\overline{S}]$. Hence f is surjective. Thus f is a bijection between $\mathcal{A}(L)$ and $\mathcal{E}$. Therefore $|\mathcal{A}(L)| = |\mathcal{E}|$. Thus Number of equivalence classes of the relation $\sim$ is $|\mathcal{A}(L)|$. □

Let $S \subset A(L)$. If a lattice L satisfies acc, then for any ideal I there exist $x = u_I$ such that $I = (u_I]$, principal ideal generated by u_I. In this case, $\mathcal{B}(S)$ becomes
$\mathcal{B}(S) = \{(x] \in N(L) \mid u_S \leq x$ and there does not exist $z \in \mathcal{A}(L)$ such that $u_S < z \leq x\}$. Let $\mathcal{D}(S) = \{x \in L \mid u_S \leq x$ and there does not exist $z \in \mathcal{A}(L)$ such that $u_S < z \leq x\}$. Clearly $|\mathcal{B}(S)| = |\mathcal{D}(S)|$.

Corollary 2.10. *If L is lattice satisfying acc, then $AnnIG(L)$ is a complete multipartite graph $K_{\langle |\mathcal{D}(S)| \mid \phi \subsetneq S \subsetneq A(L) \rangle}$.*

Proof. Follows from proposition 2.9. □

3 Lattices whose annihilator ideal graphs are complete or planar

In this section, we determine lattices whose annihilator ideal graphs are complete or planar.

Theorem 3.1. *Let L be a complete lattice, then $AnnIG(L)$ is complete if and only if L is atomistic. Moreover, if L is an atomistic lattice then $AnnIG(L) = K_m$, where $m = |L \setminus \{0, 1\}|$.*

Proof. Let L be a complete lattice. Suppose $AnnIG(L)$ is complete. By theorem (2.3), $AnnIG(L)$ is complete multipartite graph. Since $AnnIG(L)$ is complete, $|[\overline{S}]| \leq 1$ for any $S \subsetneq A(L)$. Let $x \neq 0$ be any element in L and $A((x]) = B$. Then $(x]$ and $\big(\bigvee_{b \in B} b\big]$ are in $[\overline{B}]$. Since $|[\overline{B}]| \leq 1$, $(x] = \big(\bigvee_{b \in B} b\big]$. Hence $x = \bigvee_{b \in B} b$. Therefore the lattice L is atomistic. Conversely, let L be an atomistic lattice. Let $S \subsetneq A(L)$. To prove $AnnIG(L)$ is complete we prove that $|[\overline{S}]| = 1$. On the contrary, suppose $|[\overline{S}]| \geq 2$. Then there exists an ideal I of L such that $\overline{S} \subsetneq I$. Let $x \in I \setminus \overline{S}$. Clearly, x is not atomic, otherwise $x \in \overline{S}$. This is a contradiction. Therefore $|[\overline{S}]| = 1$. Thus $AnnIG(L)$ is complete. If L is an atomistic lattice then $\mathcal{A}(L) = L \setminus \{0, 1\}$. Therefore by Proposition 2.9, $AnnIG(L) = K_m$, where $m = |L \setminus \{0, 1\}|$. □

Corollary 3.2. *If L is Boolean algebra $(\mathcal{P}(\{1,2,3,\cdots,n\},\cup,\cap))$ then $AnnIG(L) = K_{2^n-2}$.*

Proof. The lattice $(\mathcal{P}(\{1,2,3,\cdots,n\},\cup,\cap))$ is atomistic. Since any nonzero element of this lattice is join of atoms, therefore $\mathcal{A}(L) = \mathcal{P}(\{1,2,3...,n\}) \setminus \{\phi,\{1,2,3,\cdots,n\}\}$. Hence $|\mathcal{A}(L)| = 2^n - 2$. Thus by Theorem 3.1, $AnnIG(L) = K_{2^n-2}$. □

Remark 3.3. Let L_1 and L_2 be two lattices with smallest elements 0_{L_1} and 0_{L_2} respectively. In [7], new lattices are constructed as below. Let $(a,b) \in L_1 \times L_2$, $a\Box b = \{(x,y) \in L_1 \times L_2 \mid x \leq a \text{ or } y \leq b\}$,$a \circ b = \{(x,y) \in L_1 \times L_2 \mid x \leq a \text{ and } y \leq b\}$ and $a\Delta b = (a\Box\, 0_{L_2}) \cap (0_{L_1}\Box\, b)$. Set of all finite subsets H of $L_1 \times L_2$ of the form $H = \cap\{a_i\Box b_i \mid i < n\}$ for some positive integer n is lattice under set inclusion. It is denoted by $L_1\Box L_2$. Set of all finite subsets J of $L_1 \times L_2$ of the form $J = \bigcup\{a_i\Box b_i \mid i < m\} \cup \bigcup\{a_i \circ b_i \colon i < n\}$ for some positive integer m,n is a lattice under set inclusion. It is denoted by $L_1\Diamond L_2$. Note that for any atom $(a,b) \in L_1 \times L_2$, $a\Box b$ is an atom in $L_1\Box L_2$ and $a \circ b$ is an atom in $L_1\Diamond L_2$. Also, it is clear that every element in $L_1\Box L_2$ and $L_1\Diamond L_2$ is the join of finite number of atoms. Therefore $L_1\Box L_2$ and $L_1\Diamond L_2$ are atomistic lattices and hence their annihilator ideal graphs are complete.

Note that if G' is a graph obtained by deleting vertices of degree less than or equal to 1 from the graph G, then G is planar if and only if G' is planar. In the following theorem, we characterize lattices whose annihilator ideal graph is planar.

Theorem 3.4. *Let L be a complete lattice. Then $AnnIG(L)$ is planar if and only if the following conditions hold.*
(i) $|A(L)| \leq 4$.
(ii) There is subset $\{a,b\}$ of $A(L)$ such that every element in $A(L) \setminus \{a,b\}$ is not covered by nonatomic element and a or b is covered by atmost one nonatomic element in L.

Proof. A graph is planar if and only if it has no minor isomorphic to $K_{3,3}$ or K_5. Hence a complete multipartite graph $K_{r_1,r_2,\cdots,r_n}$ is planar if and only if it is one of the following form.

(i) $K_{r_1,2}$ or $K_{r_1,1}$.

(ii) $K_{r_1,1,1}$ or $K_{2,2,1}$ or $K_{2,2,2}$.

(iii) $K_{2,1,1,1}$ or $K_{1,1,1,1}$.

Therefore by Theorem 2.3, if $AnnIG(L)$ is planer then $|A(L)| \leq 4$.
Suppose $A(L) \subseteq \{a_1,a_2,a_3,a_4\}$.
Let $|[\overline{a_i}]| = m_i, |[\overline{\{a_i,a_j\}}]| = m_{ij}, |[\overline{\{a_i,a_j,a_k\}}]| = m_{ijk}$ for all $i < j < k$ and $i,j,k \in \{1,2,3,4\}$. Note that $m_i \geq 1$ and $m_{ij}, m_{ijk} \geq 0$. By Theorem 2.3,
$AnnIG(L) = K_{m_1,m_2,m_3,m_4,m_{12},m_{13},\cdots,m_{34},m_{123},m_{134},\cdots,m_{234}}$.
Case 1: $A(L) = \{a_1,a_2,a_3,a_4\}$.
Therefore $AnnIG(L)$ is planar if and only if $m_{ij} = m_{ijk} = 0$ for all i,j,k and $1 \leq m_i \leq 2$ for all i with exactly one of the m_i is 2. That is $AnnIG(L) = K_{m_1,m_2,m_2,m_4}$ is $K_{2,1,1,1}$ or $K_{1,1,1,1}$ if and only if there is subset $\{a,b\}$ of $A(L)$ such that every element in $A(L) \setminus \{a,b\}$ is not covered by nonatomic element and a or b is covered by atmost one nonatomic element in L.
Case 2: Suppose $A(L) = \{a_1,a_2,a_3\}$.
In this case atmost one of m_{ij} is nonzero and all $m_{ijk} = 0$. If exactly one of m_{ij} is nonzero, then as in case 1, $AnnIG(L)$ is planar if and only if $AnnIG(L) = K_{m_1,m_2,m_3,m_{ij}} = K_{2,1,1,1}$ or $K_{1,1,1,1}$. If all $m_{ij} = 0$ then, $AnnIG(L)$ is planar if and only if $AnnIG(L) = K_{m_1,m_2,m_3} = K_{r_1,1,1}$ or $K_{2,2,1}$ or $K_{2,2,2}$ if and only if there is subset $\{a,b\}$ of $A(L)$ such that every element in $A(L) \setminus \{a,b\}$ is not covered by nonatomic element and a or b is covered by atmost one nonatomic element in L.
Case 3: Suppose $A(L) = \{a_1,a_2\}$.
$AnnIG(L)$ is planar if and only if $AnnIG(L) = K_{m_1,m_2}$ is $K_{r_1,2}$ or $K_{r_1,1}$ if and only if there is subset $\{a,b\}$ of $A(L)$ such that every element in $A(L) \setminus \{a,b\}$ not covered by nonatomic element and a or b is covered by atmost one nonatomic element in L. □

4 Applications and examples

In this section, we give applications and examples of $AnnIG(L)$. Let L be a finite bounded lattice, then $G(L)$ be the zero-divisor graph of a lattice with vertices
$Z^*(L) = \{a \in L \mid a \neq 0,\ \exists\, b \neq 0\ such\ that\ a \wedge b = 0\}$. Vertices a and b are adjacent if and only if $a \wedge b = 0$.

Theorem 4.1. *Let L be a finite bounded lattice. Then $G(L)$ is a multipartite graph with partitioned sets similar to that of $AnnIG(L)$. Moreover, for any two partitioned sets U, V, either every vertex in U is adjacent to every vertex in V or no vertex in U is adjacent to any vertex in V.*

Proof. Let $V_1, V_2, \cdots, V_n$ be partitioned sets in $AnnIG(L)$ and $U_i = \{x \mid (x] \in V_i\}\}$ for $i = 1, 2, \cdots, n$. Let x, y be any two vertices in $G(L)$. If $x, y \in U_i$ for some i then $(x], (y] \in V_i$. Therefore $A((x]) = A((y])$. Let $a \in A((x])$ then $a \leq x \wedge y$. Hence x and y are not adjacent in $G(L)$. Let $p \in U_i$ and $q \in U_j$ are adjacent. Let $x_i \in U_i$ and $y_j \in U_j$. Therefore $A((p]) = A((x_i])$ and $A((q]) = A((y_j])$. Since $p \wedge q = 0$, $x_i \wedge y_j = 0$. Hence x_i and y_j are adjacent in $G(L)$. □
Note that $A((x]) \cap A((y]) = \phi$ if and only if x and y are adjacent in $G(L)$. In the following theorem, we give a relation between the diameters of $G(L)$ and $AnnIG(L)$.

Corollary 4.2. *Let L be a finite bounded lattice. Then the following statements hold.*

(i) $diam(AnnIG(L)) = 0$ *if and only if* $diam(G(L)) = 0$.

(ii) If $diam(AnnIG(L)) = 1$, *then* $diam(G(L)) = 1$ *or* 2 *or* 3.

(iii) If $diam(AnnIG(L)) = 2$, *then* $diam(G(L)) = 2$ *or* 3.

Proof. (1) It is clear.
(2) Let diam$(AnnIG(L)) = 1$. Since $AnnIG(L)$ has a subgraph isomorphic to $G(L)$, hence $diam(AnnIG(L)) \leq diam(G(L))$. Also it is known that $diam(G(L)) \leq 3$. By Example 4.4, Example 4.5, Example 4.6, $diam(G(L)) = 1$ *or* 2 *or* 3.
(3) Let diam$(AnnIG(L)) = 2$. Since $diam(AnnIG(L)) \leq diam(G(L)) \leq 3$, diam$(G(L)) = 2$ *or* 3 (see Example 4.7, Example 4.8). □

Corollary 4.3. *Let L be a finite bounded lattice then $gr(AnnIG(L)) = gr(G(L))$.*

Proof. Let $A(L) = \{a_1, a_2, \cdots, a_m\}$ be the set of all atoms in L and $I(L) = \{(a_1], (a_2], \cdots, (a_m]\}$. Observe that $A(L)$ is a clique in $G(L)$ and $I(L)$ is a clique in $AnnIG(L)$. If $m = 1$ *or* 2, then $gr(AnnIG(L)) = gr(G(L)) = \infty$ *or* 4. If $m \geq 3$ then $gr(AnnIG(L)) = gr(G(L)) = 3$. □

Example 4.4. Let $L = (D(30), |)$ be a lattice, then $diam(G(L)) = 3$ and $diam(AnnIG(L)) = 1$, see Figure 1.

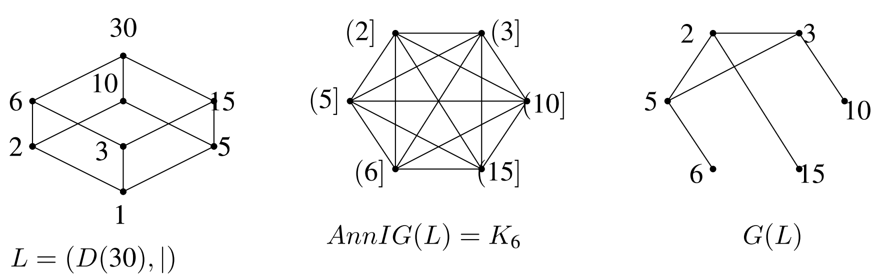

Figure 1.

Example 4.5. Let L be a lattice in Figure 2, then $diam(G(L)) = 2$ and $diam(AnnIG(L)) = 1$, see Figure 2.

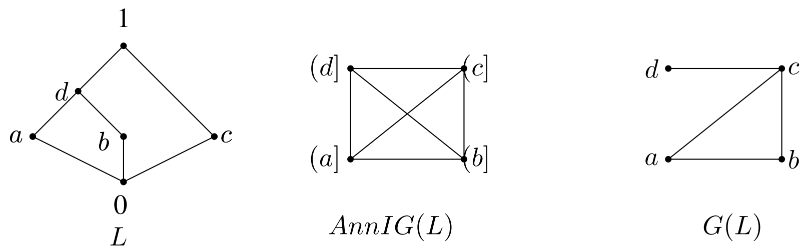

Figure 2.

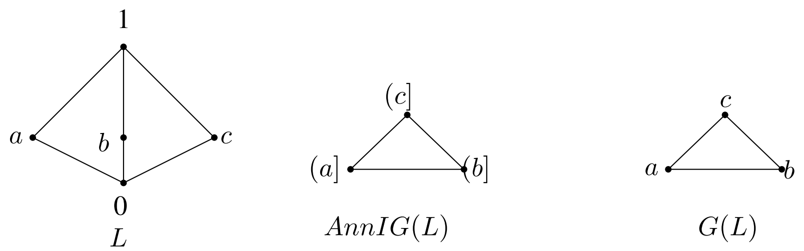

Figure 3.

Example 4.6. Let L be a lattice in Figure 3, then $diam(G(L)) = 1$ and $diam(AnnIG(L)) = 1$, see Figure 3.

Example 4.7. Let $L = (D(60), |)$ be a lattice, then $diam(G(L)) = 3$ and $diam(AnnIG(L)) = 2$, see Figure 4.

Example 4.8. Let L be a lattice as shown in Figure 5, then $diam(G(L)) = diam(AnnIG(L)) = 2$, see Figure 5.

Proposition 4.9. *Let L_1 and L_2 be two lattices. If $AnnIG(L_1) = K_{m_1,m_2,\cdots,m_t}$ and $AnnIG(L_2) = K_{n_1,n_2,\cdots,n_s}$, then $AnnIG(L_1 \times L_2) = K_{m_1,\cdots,m_t,n_1,n_2,\cdots,n_s,m_1n_1,m_1n_2,\cdots,m_tn_s}$.*

Proof. Let $A(L_1) = \{a_1, a_2, \cdots, a_m\}$ and $A(L_2) = \{b_1, b_2, \cdots, b_n\}$, then $A(L_1 \times L_2) = \{(a_1, 0,), (a_2, 0), \cdots, (a_m, 0), (0, b_1), (0, b_2), \cdots, (0, b_n)\}$. Every ideal in $L_1 \times L_2$ is of the form $I_1 \times I_2$, where I_1 and I_2 are ideals in L_1 and L_2 respectively. Also, $A(I_1 \times I_2) = A(I_1) \times A(I_2)$. For any $S \subseteq A(L_1 \times L_2)$, denote $S_1 = \{a \mid (a, 0) \in S\}$ and $S_2 = \{b \mid (0, b) \in S\}$. Let $u_1 = \bigvee_{a \in S_1} a,\ u_2 = \bigvee_{b \in S_2} b$. Suppose (x, y) is nonatomic element in $L_1 \times L_2$ such that $u = (u_1, u_2) \leq (x, y)$ and there does not exist another atomic element (l, m) satisfying $u < (l, m) \leq (x, y)$. Observe that $A((u]) = A(((x, y)]) = S$ and $((u_1, u_2)] = (u_1] \times (u_2], ((x, y)] = (x] \times (y]$. Therefore for ideal I with $A(I) = S$, we have $I = I_1 \times I_2$ and $A(I_1) = S_1,\ A(I_2) = S_2$. Therefore $|[\overline{S}]| = |[\overline{S_1 \bigsqcup S_2}]| = |[\overline{S_1}]||[\overline{S_2}]|$. Let $\phi \subsetneq S \subseteq A(L_1 \times L_2)$. If $S_2 = \phi$ then $|[\overline{S}]| = |[\overline{S_1}]| = m_i$ for some $i = 1, 2, \cdots, t$. If $S_1 = \phi$ then $|[\overline{S}]| = |[\overline{S_2}]| = n_j$ for some $j = 1, 2, \cdots, s$. If S is neither subset of $A(L_1)$ nor a subset of $A(L_2)$, then $|[\overline{S}]| = |[\overline{S_1}]||[\overline{S_2}]| = m_in_j$ for some $i = 1, 2, \cdots, t$ and $j = 1, 2, \cdots, s$. □

Example 4.10. Let $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$, where $n > 1$ and $p_1, p_2, \cdots, p_k$ are distinct primes. Let $D(n)$ be the lattice of all divisors of n with the partial order relation, $a \leq b$ if and only if a is divisor of b. Since $a \vee b = \text{lcm}(a, b)$ and $a \wedge b = \gcd(a, b)$, we have

$$A(D(n)) = \{p_1, p_2, \cdots, p_k\} \text{ and}$$
$$\mathcal{A}(D(n)) = \{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \mid \alpha_i = 0 \text{ or } 1 \text{ for all } i = 1, 2, \cdots, k\} \setminus \{1, p_1p_2 \cdots p_k\}.$$

Let $\phi \subsetneq S \subsetneq A(L)$ and $I_S = \{i \mid p_i \in S\}$. Then $u_S = \prod_{i \in I_S} p_i$ and $[\overline{S}] = \{(x] \mid x = \prod_{i \in I_S} p_i^{\beta_i},\ 1 \leq \beta_i \leq m_i\}$ and hence $|[\overline{S}]| = \prod_{i \in I_S} m_i$. Therefore $AnnIG(D(n))$ is a complete multipartite graph

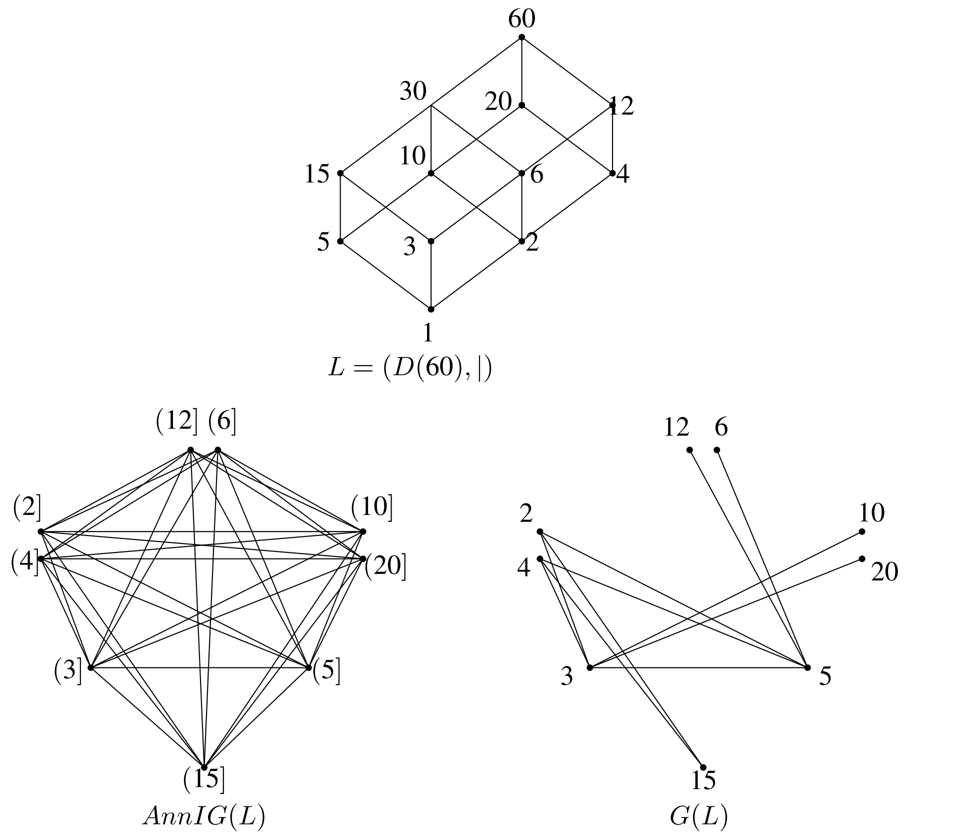

Figure 4.

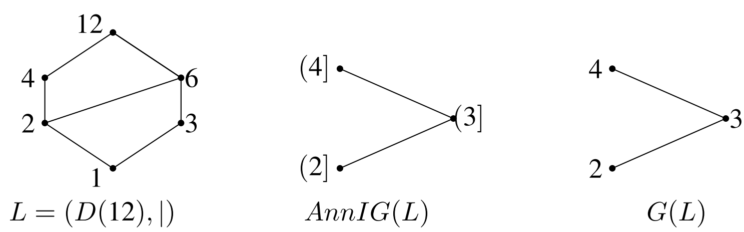

Figure 5.

$K_{\langle n \mid n \in \mathcal{M} \rangle}$, where
$\mathcal{M} = \{m_1, \cdots, m_k, m_1m_2, \cdots, m_{k-1}m_k, \cdots, (m_1m_2 \cdots m_{k-1}), \cdots, (m_2m_3 \cdots m_k)\}$ and $|\mathcal{M}| = 2^k - 2$.

Example 4.11. Let $L(G)$ be a lattice of all subgroups of a group G. Then $AnnIG(L(\mathbb{Z}_n)) = AnnIG(L(D(n)))$.

Example 4.12. Let $m_1, m_2, \cdots, m_n$ be n positive integers. Let L_1 be a lattice defined by relations, $0 \prec a_i \prec x_{i1} \prec x_{i2} \prec \cdots \prec x_{im_i} \prec 1$, for all $i = 1, 2, 3, \cdots n$ then L_1 be a lattice with n atoms whose annihilator ideal graph is $K_{m_1, m_2, \cdots, m_n}$. Let k be the least positive integer such that $n < 2^k - 2$. Let $p_1, p_2, ..., p_k$ be k distinct primes and S be set of integers given by,
$\{p_1^{m_1}, ..., p_k^{m_{\binom{k}{1}}}, (p_1p_2)^{m_{\binom{k}{1}+1}}, ..., (p_{k-1}p_k)^{m_{\binom{k}{1}+\binom{k}{2}}}, ..., (p_1...p_{k-1})^{m_{\left(\sum_{i=1}^{k-2}\binom{k}{i}\right)+1}}, \cdots, (p_2...p_k)^{m_n}\}$
with smallest element 1 and largest element l.c.m. of all integers in S, where $n = \sum_{i=1}^{k-1}\binom{k}{i} < 2^k - 2$. Let L_2 be a latiice on S under divisibility relation. Observe that $AnnIG(L_2) = K_{m_1, m_2, \cdots, m_n}$.

Remark 4.13. Let L be a lattice such that $AnnIG(L)$ is complete multipartite graph $K_{m_1, m_2, \cdots m_t}$ for some positive integers $m_1 \leq m_2 \leq m_3 \leq \cdots \leq m_t$. Hence $AnnIG(L)$ is Hamiltonian if and only if $m_t \leq \sum_{j \neq t} m_j$. If L has t atoms $a_1, a_2, \cdots a_t$ and L is defined by chains $C_i : 0 \prec a_i \prec x_{i1} \prec \cdots \prec x_{im_i} \prec 1$ for all $i = 1, 2, \cdots t$ such that no two elements from different chains are comparable. Then $AnnIG(L)$ is Hamiltonian.

Remark 4.14. Let L be a lattice and $AnnIG(L) = K_{m_1, m_2, \cdots, m_t}$ for some positive integers $m_1 \leq m_2 \leq m_3 \leq \cdots \leq m_t$. Let $m = \sum_{i=1}^{t} m_i$. Then $AnnIG(L)$ is Eulerian if and only if $m - m_j$ is even for all $j = 1, 2, 3, \cdots, t$ if and only if m, m_j have the same parity for all $j = 1, 2, 3, \cdots, t$. Hence $AnnIG(L)$ is Eulerian if and only if all m_j have same parity as that of m. If m is odd, then all $m_j' s$ are odd this implies t must be odd. In this case $AnnIG(L)$ is Eulerian if and only if $|\mathcal{A}(L)|$ is odd and $|\mathcal{B}(S)|$ is odd for any $\phi \neq S \subsetneq A(L)$. If m is even, then all $m_j' s$ are even. In this case, $AnnIG(L)$ is Eulerian if and only if $|\mathcal{B}(S)|$ is even for any $\phi \neq S \subsetneq A(L)$.

Remark 4.15. If L is a lattice such that $AnnIG(L) = K_{m_1, m_2, \ldots, m_t}$. It is clear that the adjacency matrix of $K_{m_1, m_2, \cdots, . m_t}$ is $A = \begin{bmatrix} 0_{m_1,m_1} & 1_{m_1,m_2} & \cdots & 1_{m_1,m_t} \\ 1_{m_2,m_1} & 0_{m_2,m_2} & \cdots & 1_{m_2,m_t} \\ \vdots & \vdots & \vdots & \vdots \\ 1_{m_t,m_1} & 1_{m_t,m_2} & \cdots & 0_{m_t,m_t} \end{bmatrix}$. The order of matrix A is $m \times m$, where $m = \sum_{i=1}^{t} m_i$. Note that $0_{m_i,m_j}$ is a matrix of order $m_i \times m_j$ and with all entries 0. Also, $1_{m_i,m_j}$ is a matrix of order $m_i \times m_j$ and with all entries 1. Since first m_1 rows are identical after that next m_2 rows are identical,$\cdots$, and last m_t rows are identical, hence $nullity(A) \geq m - t$. Consider submatrix B of A of order $t \times t$, $B = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} = J - I$, where J is a matrix of order $t \times t$ with all entries 1. Observe that the eigenvalues of B are -1 and $t-1$, hence B is invertible. This implies that nullity of A is $m - t$. Therefore $rank(A) = rank(B) = t$. In fact characteristic polynomial of A is

$$\chi(\lambda) = \lambda^{m-t} \left(1 - \sum_{i=1}^{t} \frac{m_i}{\lambda + m_i}\right) \prod_{j=1}^{t} (\lambda + m_j).$$

Therefore largest eigenvalue λ_1 (spectral radius) must satisfy $1 - \sum_{i=1}^{t} \frac{m_i}{\lambda_1 + m_i} = 0$.

Remark 4.16. Consider chains $C_1 : a_1 \prec a_2 \prec a_3 \prec \cdots \prec a_{m_1}$; $C_2 : b_1 \prec b_2 \prec b_3 \prec \cdots \prec b_{m_2}$; and $C_3 : c_1 \prec c_2 \prec c_3 \prec \cdots \prec c_{m_3}$. Consider the lattices defined by the given relations.

(i) $L_1 : 0 \prec C_1 \prec 1;\ 0 \prec C_2 \prec 1$.

(ii) $L_2 : 0 \prec C_1 \prec 1;\ 0 \prec C_2 \prec b_{m_2+1} \prec 1;\ a_1 \prec b_{m_2+1}$.

(iii) $L_3 : 0 \prec C_1 \prec a_{m_1+1} \prec 1;\ 0 \prec C_2 \prec 1;\ b_1 \prec a_{m_1+1}$.

(iv) $L_4 : 0 \prec C_1 \prec 1;\ 0 \prec C_2 \prec 1;\ 0 \prec C_3 \prec 1$.

(v) $L_5 : 0 \prec C_1 \prec 1;\ 0 \prec C_2 \prec b_{m_2+1} \prec 1;\ 0 \prec C_3 \prec 1;\ a_1 \prec b_{m_2+1};\ c_1 \prec b_{m_2+1}$.

(vi) $L_6 : 0 \prec C_1 \prec a_{m_1+1} \prec 1;\ 0 \prec C_2 \prec 1;\ 0 \prec C_3 \prec 1;\ b_1 \prec a_{m_1+1};\ c_1 \prec a_{m_1+1}$.

(vii) $L_7 : 0 \prec C_1 \prec 1;\ 0 \prec C_2 \prec 1;\ 0 \prec C_3 \prec c_{m_3+1} \prec 1;\ a_1 \prec c_{m_3+1};\ b_1 \prec c_{m_3+1}$.

It is observed that the $AnnIG(L_1) = AnnIG(L_2) = AnnIG(L_3) = K_{m_1,m_2}$ and $AnnIG(L_4) = AnnIG(L_5) = AnnIG(L_6) = AnnIG(L_7) = K_{m_1,m_2,m_3}$. Also, if L is any lattice such that $AnnIG(L)$ is either K_{m_1,m_2} or K_{m_1,m_2,m_3} then L is isomorphic to one of the lattice obtained by replacing any chain C_i by any lattice with same number of elements as of C_i in any one of lattice among $L_1, L_2, L_3, L_4, L_5, L_6, L_7$.

References

[1] D. F. Anderson, T. Asir, A. Badawi and T. T. Chelvam, Graphs from rings, First ed, Springer Nature Switzerland AG (2021).

[2] A. Abolfazl, B. Moharram, N. Reza and N. M. Javad, The annihilator ideal graph of a commutative ring, *Journal of the Korean Mathematical Society* **52**(2), 417–429 (2015).

[3] G. Birkhoff, Lattice theory, *American Mathematical Soc.* **25**, (1940).

[4] T. T. Chelvam and S. Nithya, A note on the zero-divisor graph of a lattice, *Trans. Comb.* **3**, 51–59 (2014).

[5] R. Deore and P. Tayade, Zero Divisor Graph of a Lattice and Its Unique Ideal, *Journal of Mathematics*, (2019).

[6] G. Grätzer, Lattice theory: foundation, *Springer Science & Business Media* (2011).

[7] G. Grätzer and F. Wehrung, A new lattice construction: the box product, *Journal of Algebra* **221**(1), 315–344 (1999).

[8] A. Khairnar and B. N. Waphare, Zero-Divisor Graphs of Laurent Polynomials and Laurent Power Series, *Algebra and its Applications*, 345–349, Springer (2016).

[9] A. Patil, RICKART $*$-RINGS WITH PLANAR ZERO-DIVISOR GRAPHS, *Palestine Journal of Mathematics* **10**(1), (2021).

[10] A. Patil, A. Khairnar and B. N. Waphare, Zero-divisor graph of a poset with respect to an automorphism, *Discrete Applied Mathematics* **283**, 604–612 (2020).

[11] A. Patil, A. Khairnar and B. N. Waphare, Zero-divisor graph of a ring with respect to an automorphism, *Soft Computing* available online https://doi.org/10.1007/s00500-021-06680-7, 13 pages (2022).

[12] D. B. West and others, Introduction to graph theory, **2**, Prentice hall Upper Saddle River (2001).

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