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SOLVABILITY OF HAMMERSTEIN INTEGRAL EQUATION IN BANACH SPACE ℓ_p

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Abstract In this paper our main aim is to study the solvability of non-linear Hammerstein integral equations in Banach space ℓ_p, $1 < p < \infty$. Our proof is based on Darbo's fixed point theorem using the idea of the measure of noncompactness. An example is presented in order to illustrate the obtained result.

1 Introduction

In 1895 Le Roux [17] introduced integral equations as a powerful tool for investigating partial differential equations. This theory has many applicabilities like in population dynamics, economic theory, feedback systems, stability of nuclear reactors [11, 28, 29]. Various issues of different parts of science lead to the need of exploring the solvability of nonlinear Hammerstein integral equations, such equtions have been studied in [3, 10, 21] and the references therein. The principal aim of the paper is to study the infinite system of Hammerstein type integral equations in two variables of the form:

$$v_n(t) = \mathsf{r}_n(t) + \int_a^b K_n(s,t) f_n(s, v(s)) ds \tag{1.1}$$

where $t \in [a,b]$, and $f_j(t,v)$, $j = 1,2,3,\dots$ is continuously differentiable function on $[a,b]$ and $K_j(s,t)$ is the kernel function. We denote the interval $[a,b]$ by J. The solvability of (1.1) is studied using the idea of measure of noncompactness.

2 Preliminaries

In 1930 Kuratowski [16] introduced the concept of measure of noncompactness which was further extendend to general Banach space by Banaś and Goebel [6]. In 1955 Darbo [12] proved a fixed point theorem for condensing operators using the concept of measure of noncompactness, which generalized the classical Schauder fixed point theorem and Banach contraction principle. The method of fixed point arguments has been widely used to study the existence of solutions of functional equations, like Banach contraction principle in [1, 26] and Schauder's fixed point theorem in [15, 18]. But if compactness and Lipschitz condition are not satisfied these results can not be used. Measure of noncompactness comes handy in such situations.

There are many measures of noncompactness among all of them Hausdorff measure of noncompactness is most important. It has been used to find the condition for the existence of solution of system of integral equations [5, 8, 13, 14] and the references therein. Different properties of Hausdorff measure of noncompactness χ can be found in [2, 6, 8].

Definition 2.1. [6] Let (Ω, d) be a metric space and A be a bounded subset of Ω. Then the *Hausdorff measure of noncompactness* (the ball-measure of noncompactness) of the set A, denoted by $\chi(A)$ is defined to be the infimum of the set of all $\epsilon > 0$ such that A can be covered by a finite number of balls of radii $< \epsilon$, that is

$$\chi(A) = \inf \left\{ \epsilon > 0 : A \subset \bigcup_{i=1}^{n} B_{r_i}(x_i), x_i \in \Omega, r_i < \epsilon \ (i = 1, \ldots, n) \ , n \in \mathbb{N} \right\}$$

where $B_{r_i}(x_i)$ denotes ball of radius r_i centered at x_i.

Let $(X, \|\cdot\|)$ be a Banach space, for any $\mathsf{E} \subset X$, $\bar{\mathsf{E}}$ denotes closure of E and $conv(\mathsf{E})$ denotes the closed convex hull of E. We denote the family of non-empty bounded subsets of X by $\mathfrak{M}_X$ and family of non-empty and relatively compact subsets of X by $\mathfrak{N}_X$. Let $\mathbb{R}_+ = [0, \infty)$ the axiomatic definition of measure of noncompactness is

Definition 2.2. [6] A mapping $\mu : \mathfrak{M}_X \to \mathbb{R}_+$ is said to be the measure of noncompactness in X if the following conditions hold:

(i) The family Ker $\mu = \{\mathsf{E} \in \mathfrak{M}_X : \mu(\mathsf{E}) = 0\}$ is non-empty and Ker $\mu \subset \mathfrak{N}_X$;

(ii) $\mathsf{E}_1 \subset \mathsf{E}_2 \Rightarrow \mu(\mathsf{E}_1) \le \mu(\mathsf{E}_2)$;

(iii) $\mu(\bar{\mathsf{E}}) = \mu(\mathsf{E})$;

(iv) $\mu(conv\mathsf{E}) = \mu(\mathsf{E})$;

(v) $\mu\left[\lambda\mathsf{E}_1 + (1-\lambda)\mathsf{E}_2\right] \le \lambda\mu(\mathsf{E}_1) + (1-\lambda)\mu(\mathsf{E}_2)$ for $0 \le \lambda \le 1$;

(vi) If (E_n) is a sequence of closed sets from $\mathfrak{M}_X$ such that $\mathsf{E}_{n+1} \subset \mathsf{E}_n$ and $\lim_{n\to\infty} \mu(\mathsf{E}_n) = 0$ then the intersection set $\mathsf{E}_\infty = \bigcap_{n=1}^{\infty} \mathsf{E}_n$ is non-empty.

The Darbo fixed point theorem [12] that is used in present paper states that:

Lemma 2.3. *[12] Let E be a non-empty, bounded, closed, and convex subset of Banach space X with arbitrary measure of noncompactness μ and let $T : \mathsf{E} \to \mathsf{E}$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that $\mu(T(\mathsf{A})) \le k\mu(\mathsf{A})$ for any non-empty subset A of E. Then T has a fixed point in the set E.*

Definition 2.4 (Equicontinuity). Let (Ω_1, d) and (Ω_2, d) be two metric spaces, and $\mathcal{T}$ the family of functions from Ω_1 to Ω_2. The family $\mathcal{T}$ is equicontinuous at a point $m_0 \in \Omega_1$ if for every $\epsilon > 0$, there exists $\delta > 0$, such that $d(f(m), f(m_0)) < \epsilon$ for all $f \in \mathcal{T}$ and all $m \in \Omega_1$ such that $d(m, m_0) < \delta$. The family is pointwise equicontinuous if it is equicontinuous at each point of Ω_1.

For fixed p , $p \ge 1$, we denote by ℓ_p the Banach sequence space consisting of real sequences $x = (x_n)$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$ with the norm $\|\cdot\|_p$ defined as:

$$\|x\|_p = \|(x_n)\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$$

for $x = (x_n) \in \ell_p$. To apply Lemma 2.3 for a given Banach space X, a simple formula expressing the measure of noncompactness is required. Such formulas are not known for all spaces [6, 8]. For the Banach space $(\ell_p, \|\cdot\|_p)$, Hausdorff measure of noncompactness is given by

$$\chi(\mathsf{E}) = \lim_{n\to\infty} \left\{ \sup_{(e_k)\in\mathsf{E}} \left(\sum_{k\ge n} |e_k|^p \right)^{\frac{1}{p}} \right\} \tag{2.1}$$

provided $\mathsf{E} \in \mathfrak{M}_{\ell_p}$. The above formulas will be used in the sequel of the paper.

Measure of noncompactness has been used to obtain conditions under which an infinite system of differential equations has a solution in given Banach space [4, 7, 8, 9, 19, 20, 23, 24, 25].

3 Main Results

In order to find conditions under which the system (1.1) has a solution in ℓ_p, we need the following assumptions:

(4_{A_1}) The functions f_j are real valued, defined on the set $J \times \mathbb{R}^\infty$, $(j = 1, 2, 3, \ldots)$. An operator f defined on the space $J \times \ell_p$ as

$$(t, v) \mapsto (f_j(t, v)) = (f_1(t, v), f_2(t, v), f_3(t, v), \ldots)$$

transforms the space $J \times \ell_p$ into ℓ_p.

The class of all functions $\{(f_j(t, v))\}_{t\in J}$ is equicontinuous at each point of the space ℓ_p. That is, for each $v \in \ell_p$, fixed arbitrarily and given $\epsilon > 0$, there exists $\delta > 0$ such that, whenever $\|u - v\|_p < \delta$

$$\sup_{t\in J} \|f(t, u) - f(t, v)\|_p < \epsilon. \tag{3.1}$$

(4_{A_2}) The functions $K_n : J^2 \to \mathbb{R}$ are uniformly continuous on $J^2, (n = 1, 2, \ldots)$. Also, these functions $K_n(s, t)$ are equicontinuous with respect to t that is, for all $\epsilon > 0$ there exists $\delta > 0$, such that

$$|K_n(s, t_2) - K_n(s, t_1)| \le \epsilon \text{ whenever } |t_2 - t_1| \le \delta.$$

for all $s \in J$. Also, the function sequence $(K_n(s, t))$ is equibounded on the set J^2 and the constant K defined as

$$\mathsf{K} = \sup \{|K_n(s, t)| : s, t \in J, n = 1, 2, \ldots\}$$

is finite.

(4_{A_3}) For each $t \in J,\ v(t) = (v_j(t)) \in \ell_p$, the following inequality holds

$$|f_j(t, v(t))|^p \le g_j(t) + h_j(t)|v_j|^p \quad , \quad j \in \mathbb{N} \tag{3.2}$$

where $h_j(t)$ and $g_j(t)$ are real valued continuous functions on J and the function series $\sum_{k\ge1} g_k(t)$ is uniformly convergent. Also the function sequence $(h_j(t))_{j\in\mathbb{N}}$ is equibounded on J.

(4_{A_4}) The functions r_j $(j = 1, 2, \ldots)$ are continuous on J and the function series $\sum_{k\ge1} |\mathsf{r}_k(t)|^p$ is uniformly convergent.

Let $\mathsf{r}(t) = \sum_{j=1}^{\infty} |\mathsf{r}_j(t)|^p$, $g(t) = \sum_{j=1}^{\infty} g_j(t)$. To prove the general result we set the following constants

$$R = \max \{\mathsf{r}(t) : t \in J\},$$
$$G = \max \{g(t) : t \in J\},$$
$$H = \sup \{h_j(t) : t \in J, j \in \mathbb{N}\}.$$

which are finite because of assumptions (4_{A_3}) and (4_{A_4}).

Remark 3.1. Let $C(J, Y)$ be the space of real continuous functions on a Banach space Y, and χ_Y be the Hausdorff measure of noncompactness defined on Y, then Hausdorff measure of noncompactness of a subset E of $C(J, Y)$, is given by [6, 22]

$$\chi(\mathsf{E}) = \sup \{\chi_Y(\mathsf{E}(t)) : t \in J\}.$$

provided E is equicontinuous on the interval $J = [a, b]$ and $\mathsf{E}(t) = \{e(t) : e \in \mathsf{E}\}$.

Theorem 3.2. *Under the assumptions $(4_{A_1}) - (4_{A_4})$, the infinite system of integral equations (1.1) has atleast one solution $v = (v_j)$ in ℓ_p, $p > 1$ space for each $t \in J$ if $\mathsf{K}(H)^{\frac{1}{p}}(b - a) < 1$.*

Proof. Consider the space $C(J, \ell_p)$ with supremum norm,

$$\|v\| = \sup_{t\in J} \{\|v(t)\|_p\}.$$

Define, the operator $\mathcal{F}$ on the space $C(J, \ell_p)$, by

$$\begin{aligned}(\mathcal{F}v)(t) &= ((\mathcal{F}v)_j(t)) \\ &= \left(\mathsf{r}_j(t) + \int_a^b K_j(s,t)f_j(s,v(s))ds\right) \\ &= \left(\mathsf{r}_1(t) + \int_a^b K_1(s,t)f_1(s,v(s))ds, \mathsf{r}_2(t) + \int_a^b K_2(s,t)f_2(s,v(s))ds, \dots\right).\end{aligned} \tag{3.3}$$

We will show that the operator $\mathcal{F}$ as defined in (3.3) transforms the space $C(J, \ell_p)$ into itself.

Fix $v = v(t) = (v_j(t))$ in $C(J, \ell_p)$, then for arbitrary $t \in J$ using assumption (4_{A_3}), Minkowski's inequality and Hölder's inequality, we have

$$\begin{aligned}\|(\mathcal{F}v)(t)\|_p &= \left(\sum_{j=1}^{\infty} \left|\mathsf{r}_j(t) + \int_a^b K_j(s,t)f_j(s,v(s))ds\right|^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_{j=1}^{\infty} |\mathsf{r}_j(t)|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{\infty} \left\{\int_a^b |K_j(s,t)|^p |f_j(s,v(s))|^p ds\right\} \left(\int_a^b ds\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \\ &\le R^{\frac{1}{p}} + \mathsf{K}(b-a)^{\frac{1}{q}} \left(\sum_{j=1}^{\infty} \left\{\int_a^b [g_j(s) + h_j(s)|v_j(s)|^p]\, ds\right\}\right)^{\frac{1}{p}},\end{aligned}$$

where, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Now, by using Lebesgue Dominated Convergence Theorem [27], we get

$$\begin{aligned}\|(\mathcal{F}(v)(t)\|_p &\le R^{\frac{1}{p}} + \mathsf{K}(b-a)^{\frac{1}{q}} \left(\int_a^b g(s)ds + H\int_a^b \sum_{j=1}^{\infty} |v_j(s)|^p ds\right)^{\frac{1}{p}} \\ &\le R^{\frac{1}{p}} + \mathsf{K}(b-a)^{\frac{1}{q}} \left(G(b-a) + H(b-a)\left(\|v\|_p\right)^p\right)^{\frac{1}{p}}.\end{aligned}$$

Therefore,

$$\|\mathcal{F}(v)(t)\|_p \le R^{\frac{1}{p}} + \mathsf{K}(b-a)\left(G + H\left(\|v\|_p\right)^p\right)^{\frac{1}{p}}. \tag{3.4}$$

Hence, $\mathcal{F}v$ is bounded on the interval J. Thus, $\mathcal{F}$ transforms the space $C(J, \ell_p)$ into itself. Following the above procedure, we have

$$\|v\|_p \le R^{\frac{1}{p}} + \mathsf{K}(b-a)\left(G^{\frac{1}{p}} + H^{\frac{1}{p}}\|v\|_p\right).$$

Hence, $\|v\| \le \dfrac{R^{\frac{1}{p}} + (b-a)\mathsf{K}G^{\frac{1}{p}}}{1-(b-a)\mathsf{K}H^{\frac{1}{p}}}$.
Thus, by assumption (4_{A_3}), we can find a positive number

$$r = \frac{R^{\frac{1}{p}} + (b-a)\mathsf{K}G^{\frac{1}{p}}}{1-(b-a)\mathsf{K}H^{\frac{1}{p}}}$$

as the optimal solution for

$$R^{\frac{1}{p}} + \mathsf{K}(b-a)\left(G + HL^p\right)^{\frac{1}{p}} \le r.$$

Hence, by (3.4) the operator $\mathcal{F}$ transforms the ball $\bar{B}_r \subset C(J, \ell_p)$ into itself.

We now show that $\mathcal{F}$ is continuous on $\bar{B}_r$. Let $\epsilon > 0$ be arbitrarily fixed and $v = (v(t)) \in \bar{B}_r$ be any arbitrarily fixed function, then if $u = (u(t)) \in \bar{B}_r$ is any function such that $\|u - v\|_p < \epsilon$, then for any $t \in J$, we have

$$\begin{aligned}
(\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\|_p)^p &= \sum_{j=1}^{\infty} \left| \int_a^b K_j(t,s) \left[f_j(s,u(s)) - f_j(s,v(s)) \right] ds \right|^p \\
&\leq \sum_{j=1}^{\infty} \int_a^b |K_j(t,s)|^p \, |f_j(s,u(s)) - f_j(s,v(s))|^p \, ds \left(\int_a^b ds \right)^{\frac{p}{q}} \\
&\leq (b-a)^{\frac{p}{q}} \sum_{j=1}^{\infty} \int_a^b |K_j(t,s)|^p \, |f_j(s,u(s)) - f_j(s,v(s))|^p \, ds \\
&\leq (b-a)^{\frac{p}{q}} \, \mathsf{K} \lim_{m\to\infty} \sum_{j=1}^{m} \int_a^b |f_j(s,u(s)) - f_j(s,v(s))|^p \, ds.
\end{aligned} \tag{3.5}$$

Now, by using the assumption (4_{A_1}) of equicontinuity, define the function $\delta(\epsilon)$ as

$$\delta(\epsilon) = \sup \left\{ |f_j(s,u(s)) - f_j(s,v(s))| : u, v \in \ell_p, \|u - v\|_p \leq \epsilon, \; t \in J, j \in \mathbb{N} \right\}.$$

Then, clearly $\delta(\epsilon) \to 0$ as $\epsilon \to 0$, since the family $\{(fv)(t) : t \in J\}$ is equicontinuous at every point $v \in \ell_p$.
Therefore, by (3.5) and using Lebesgue Dominanted Convergence Theorem, we have

$$\begin{aligned}
(\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\|_p)^p &\leq (b-a)^{\frac{p}{q}} \, \mathsf{K} \int_a^b [\delta(\epsilon)]^p \, ds \\
&= (b-a)^p \, \mathsf{K} \, [\delta(\epsilon)]^p
\end{aligned}$$

This implies that the operator $\mathcal{F}$ is continuous on the ball $\bar{B}_r$.

Since $K_j(s,t)$, $j = 1, 2, \ldots$ are uniformly continuous on J^2 by assumption (4_{A_2}), so by definition of operator $\mathcal{F}$, it is easy to show that $\left\{ \mathcal{F}u : u \in \bar{B}_r \right\}$ is equicontinuous on J. Let $B_{r_1} = conv(\mathcal{F}\bar{B}_r)$, then $B_{r_1} \subset \bar{B}_r$ and the functions from the set B_{r_1} are equicontinuous on J.

Let $\mathsf{E} \subset B_{r_1}$, then E is equicontinuous on J. For $v \in \mathsf{E}$ fix arbitrarily $t \in J$, then by assumption (4_{A_3}) and Minkowski inequality, we have

$$\begin{aligned}
\left(\sum_{j=k}^{\infty} \left| (\mathcal{F}v)_j(t) \right|^p \right)^{\frac{1}{p}} &= \left(\sum_{j=k}^{\infty} \left| \mathsf{r}_j(t) + \int_a^b K_j(s,t) f_j(s,v(s)) ds \right|^p \right)^{\frac{1}{p}} \\
&\leq \left(\sum_{j=k}^{\infty} |\mathsf{r}_j(t)|^p \right)^{\frac{1}{p}} + \left(\sum_{j=k}^{\infty} \left| \int_a^b K_j(s,t) f_j(s,v(s)) ds \right|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Using, Hölder's inequality we get,

$$\begin{aligned}
\left(\sum_{j=k}^{\infty} \left| (\mathcal{F}v)_j(t) \right|^p \right)^{\frac{1}{p}} &\leq \left(\sum_{j=k}^{\infty} |\mathsf{r}_j(t)|^p \right)^{\frac{1}{p}} + \left(\sum_{j=k}^{\infty} \left(\int_a^b |K_j(s,t)|^p |f_j(s,v(s))|^p ds \right) \left(\int_1^T ds \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \\
&\leq \left(\sum_{j=k}^{\infty} |\mathsf{r}_j(t)|^p \right)^{\frac{1}{p}} + (b-a)^{\frac{1}{q}} \mathsf{K} \left(\sum_{j=k}^{\infty} \int_a^b |f_j(s,v(s))|^p ds \right)^{\frac{1}{p}}
\end{aligned}$$

Now, by using Lebesgue dominated Convergence Theorem and the assumption (4_{A_3}), we get

$$\left(\sum_{j=k}^{\infty}\left|(\mathcal{F}v)_j(t)\right|^p\right)^{\frac{1}{p}} \leq \left(\sum_{j=k}^{\infty}|\mathsf{r}_j(t)|^p\right)^{\frac{1}{p}} + (b-a)^{\frac{1}{q}}\mathsf{K}\left(\int_a^b\sum_{j=k}^{\infty}[g_j(s)+h_j(s)|v_j(s)|^p]ds\right)^{\frac{1}{p}}$$

$$= \left(\sum_{j=k}^{\infty}|\mathsf{r}_j(t)|^p\right)^{\frac{1}{p}} + (b-a)^{\frac{1}{q}}\mathsf{K}\left\{\int_a^b\left(\sum_{j=k}^{\infty}g_j(s)\right)ds + \int_a^b\left(\sum_{j=k}^{\infty}h_j(s)|v_j(s)|^p\right)ds\right\}^{\frac{1}{p}}$$

$$\leq \left(\sum_{j=k}^{\infty}|\mathsf{r}_j(t)|^p\right)^{\frac{1}{p}} + (b-a)^{\frac{1}{q}}\mathsf{K}\left\{\int_a^b\left(\sum_{j=k}^{\infty}g_j(s)\right)^{\frac{1}{p}}ds + H^{\frac{1}{p}}\left(\int_a^b\sum_{j=k}^{\infty}|v_j(s)|^p ds\right)^{\frac{1}{p}}\right\}$$

Taking supremum over all $v \in \mathsf{E}$, we obtain

$$\sup_{v\in\mathsf{E}}\left(\sum_{j=k}^{\infty}\left|(\mathcal{F}v)_j(t)\right|^p\right)^{\frac{1}{p}} \leq \left(\sum_{j=k}^{\infty}|\mathsf{r}_j(t)|^p\right)^{\frac{1}{p}} + (b-a)^{\frac{1}{q}}\mathsf{K}\Big\{\int_a^b\left(\sum_{j=k}^{\infty}g_j(s)\right)^{\frac{1}{p}}ds+ H^{\frac{1}{p}}\sup_{v\in\mathsf{E}}\left(\int_a^b\sum_{j=k}^{\infty}|v_j(s)|^p ds\right)^{\frac{1}{p}}\Big\}.$$

Using, the definition of Hausdorff measure of noncompactness in ℓ_p space and noting that E is the set of equicontinuous functions on J, then by using Remark 3.1, we get

$$\chi(\mathcal{F}\mathsf{E}) \leq \mathsf{K}H^{\frac{1}{p}}(b-a)\chi(\mathsf{E}).$$

Therefore, if $\mathsf{K}H^{\frac{1}{p}}(b-a) < 1$, then by Lemma 2.3, the operator $\mathcal{F}$ on the set B_{r_1} has a fixed point, which completes the proof of the theorem.
The above result is illustrated by the following example.

Example 3.3. Consider the infinite system of Hammerstein integral equations

$$v_n(t) = \frac{1}{n^2}\ln(t+n) + \int_1^2 \arctan(s+t+n)\left(\frac{s^2e^{-ns}}{(2n+1)} + \sum_{k=n}^{\infty}\frac{\eta\cos s}{n(2k-1)}\cdot\frac{v_k(s)[1-(\sqrt{k-n})v_k(s)]}{\sqrt{k-n+1}}\right)ds \tag{3.6}$$

where η is a real number. Our considerations are located in the space $BC([1,2],\ell_2)$.

Comparing, the given system with (1.1), we have

$$\mathsf{r}_n(t) = \frac{1}{n^2}\ln(t+n)$$
$$K_n(s,t) = \arctan(s+t+n)$$
$$f_n(s,v(s)) = \frac{s^2e^{-ns}}{(2n+1)} + \sum_{k=n}^{\infty}\frac{\eta\cos s}{n(2k-1)}\cdot\frac{v_k(s)[1-(\sqrt{k-n})v_k(s)]}{\sqrt{k-n+1}},$$

and $a = 1, b = 2$.
Clearly, $\mathsf{r}_n(t)$ is continuous on $[1,2]$, moreover for fixed $t_1, t_2 \in [1,2]$, we have

$$\begin{aligned}\|(\mathsf{r}_n)(t_1) - (\mathsf{r}_n)(t_2)\|_2^2 &= \sum_{n=1}^{\infty}|\mathsf{r}_n(t_1) - \mathsf{r}_n(t_2)|^2\\ &= \sum_{n=1}^{\infty}\frac{1}{n^4}|\ln(t_1+n) - \ln(t_2+n)|^2\\ &= \sum_{n=1}^{\infty}\frac{1}{n^4}\left|\ln\left(1+\frac{t_1-t_2}{t_2+n}\right)\right|^2\\ &\leq \sum_{n=1}^{\infty}\frac{1}{n^6}|t_1-t_2|^2\\ &= |t_1-t_2|^2\zeta(6) < \epsilon^2,\end{aligned}$$

when $|t_1 - t_2|^2 < \frac{\epsilon^2}{\zeta(6)}$. Therefore, $\|(\mathsf{r}_n)(t_1) - (\mathsf{r}_n)(t_2)\|_2 < \epsilon$.
Also for all $t \in [1,2]$, we have

$$\begin{aligned}\mathsf{r}_n(t) &\leq \frac{1}{n^2}\ln(2+n)\\ &\leq \frac{1}{n^2}2\sqrt{(2+n)}\\ &\leq \frac{4}{n^2}\left(\frac{1}{\sqrt{n}} + \frac{\sqrt{n}}{2}\right)\end{aligned}$$

Thus, the constant

$$\begin{aligned}R &= \max\left\{\sum_{n=1}^{\infty}\mathsf{r}_n(t) : t \in [1,2]\right\}\\ &= 2\zeta(2.5) + \zeta(1.5)]\end{aligned}$$

is finite, where $\zeta(a) = \sum_{n=1}^{\infty}\frac{1}{n^a}$ is Riemann zeta function. Thus, assumption (4_{A_4}) is satisfied.
Since,

$$|K_n(s,t)| = |\arctan(s+t+n)| \leq \frac{\pi}{2}. \tag{3.7}$$

So, the function sequence $(K_n(s,t))$ is equibounded. Also, for fixed $t_1, t_2 \in [1,2]$ and any $s \in [1,2]$

$$\begin{aligned}|K_n(s,t_1) - K_n(s,t_2)| &= |\arctan(s+t_1+n) - \arctan(s+t_2+n)|\\ &\leq |t_1 - t_2|.\end{aligned}$$

Therefore, the function sequence $(K_n(s,t))$ is equicontinuous. Using (3.7), the value of the constant K is

$$\mathsf{K} = \sup\{|K_n(s,t)| : s,t \in [1,2], n \in \mathbb{N}\} = \frac{\pi}{2}.$$

Hence, assumption (4_{A_2}) is satisfied. We now show that, the assumptions (4_{A_3}) and (4_{A_1}) are also satisfied. By Cauchy-Schwartz inequality, we have

$$\begin{aligned}|f_n(t,v)|^2 &= \left|\frac{t^2e^{-nt}}{(2n+1)} + \sum_{k=n}^{\infty}\frac{\eta\cos t}{n(2k-1)}\cdot\frac{v_k(t)[1-(\sqrt{k-n})v_k(t)]}{\sqrt{k-n+1}}\right|^2\\ &\leq 2\left[\frac{t^4e^{-2nt}}{(2n+1)^2} + \frac{\eta^2}{n^2}\left(\sum_{k=n}^{\infty}\frac{\cos t}{(2k-1)}\cdot\frac{v_k(t)[1-(\sqrt{k-n})v_k(t)]}{\sqrt{k-n+1}}\right)^2\right]\\ &\leq \frac{2t^4e^{-2nt}}{(2n+1)^2} + \frac{2\eta^2}{n^2}\sum_{k=n}^{\infty}\left(\frac{\cos^2 t}{(2k-1)^2}\right)\sum_{k=n}^{\infty}\left(\frac{v_k(t)[1-(\sqrt{k-n})v_k(t)]}{\sqrt{k-n+1}}\right)^2\end{aligned}$$

For any real ν, β, with $\beta \neq 0$, we have

$$\frac{\nu(1-\nu\beta)}{\beta} \leq \frac{1}{(2\beta)^2}\ ,\ \beta \neq 0. \tag{3.8}$$

Hence,

$$\begin{aligned}|f_n(t,v)|^2 &\leq \frac{2t^4e^{-2nt}}{(2n+1)^2} + 2\frac{\eta^2\cos^2 t}{n^2}\times\frac{\pi^2}{8}\times\left(v_n^2 + \sum_{k=n+1}^{\infty}\left\{\frac{v_k(t)[1-(\sqrt{k-n})v_k(t)]}{\sqrt{k-n+1}}\right\}^2\right)\\ &\leq \frac{2t^4e^{-2nt}}{(2n+1)^2} + \frac{\pi^2\eta^2\cos^2 t}{4n^2}(v_n^2) + \frac{\pi^2\eta^2\cos^2 t}{4n^2}\times\sum_{k=n+1}^{\infty}\left(\frac{1}{[2\sqrt{k-n}]^2}\right)^2\\ &\leq \frac{2t^4e^{-2nt}}{(2n+1)^2} + \frac{1}{16}\frac{\pi^2\eta^2\cos^2 t}{4n^2}\times\frac{\pi^2}{6} + \frac{\pi^2\eta^2\cos^2 t}{4n^2}(v_n^2).\end{aligned}$$

Hence, by taking

$$g_n(t) = 2\frac{t^4 e^{-2nt}}{(2n+1)^2} + \frac{\pi^4\eta^2\cos^2 t}{384n^2},$$
$$h_n(t) = \frac{\pi^2\eta^2\cos^2 t}{4n^2}$$

it is clear that, $g_n(t)$ and $h_n(t)$ are real valued continuous functions on $[1,2]$. Also

$$|g_n(t)| \leq 2\frac{2^4}{(2n+1)^2} + \frac{\pi^4\eta^2}{384n^2} = \frac{32}{(2n+1)^2} + \frac{\pi^4\eta^2}{384n^2}$$

for all $t \in [1,2]$.
Thus, by Weierstrass test for uniform convergence of the function series, we see that $\sum_{k\geq 1} g_k(t)$ is uniformly convergent on $[1,2]$.
Further, we have $|h_j(t)| \leq \frac{\pi^2\eta^2}{4n^2}$ for all $t \in [1,2]$.
Thus, the function sequence $(h_j(t))$ is equibounded on $[1,2]$. This shows that (3.2) is satisfied and hence the assumption (4_{A_3}) is satisfied.
Also,

$$G = \sup\left\{\sum_{k\geq 1} g_k(t) : t \in J\right\} = \left(\frac{32\pi^2}{8} + \frac{\pi^4\eta^2}{384} \times \frac{\pi^2}{6}\right) = \pi^2\left(4 + \frac{\pi^4\eta^2}{2304}\right)$$

and

$$H = \sup\{h_j(t) : t \in J\} = \frac{\pi^2\eta^2}{4}.$$

The assumption (4_{A_1}) is also satisfied as for fixed $t \in [1,2]$ and for $(v_j(t)) = (v_1(t), v_2(t), \dots) \in \ell_2$, we have

$$\sum_{j=1}^{\infty} |f_j(t,v)|^2 = \sum_{j=1}^{\infty} g_j(t) + \sum_{j=1}^{\infty} h_j(t)|v_j(t)|^2$$
$$\leq G + H\sum_{j=1}^{\infty} |v_j(t)|^2.$$

Hence, the operator $\mathcal{F} = (f_j)$ transforms the space (J, ℓ_2) into ℓ_2.
Also, for $\epsilon > 0$ and $u = (u_j), v = (v_j)$ in ℓ_2 with $\|u - v\|_2 < \epsilon$, we have

$$\left(\|(fu)(t) - (fv)(t)\|_2\right)^2 = \sum_{n=1}^{\infty} |f_n(t,u(t)) - f_n(t,v(t))|^2$$
$$= \sum_{n=1}^{\infty}\left\{\left|\sum_{k=n}^{\infty} \frac{\eta(\cos t)u_k(t)[1-(\sqrt{k-n})u_k(t)]}{n(\sqrt{k-n+1})(2k-1)} - \frac{\eta(\cos t)v_k(t)[1-(\sqrt{k-n})v_k(t)]}{n(\sqrt{k-n+1})(2k-1)}\right|^2\right\}$$
$$\leq \sum_{n=1}^{\infty}\left\{\left(\frac{\eta^2}{n^2}\right)\left|\sum_{k=n}^{\infty} \frac{u_k(t)[1-(\sqrt{k-n})u_k(t)] - v_k(t)[1-(\sqrt{k-n})v_k(t)]}{(\sqrt{k-n+1})(2k-1)}\right|^2\right\}$$
$$\leq \eta^2\sum_{n=1}^{\infty}\left\{\left(\frac{1}{n^2}\right)\left[\sum_{k=n}^{\infty}\left|\frac{(u_k(t)-v_k(t))[1-(\sqrt{k-n})(u_k(t)+v_k(t))]}{(2k-1)(\sqrt{k-n+1})}\right|\right]^2\right\}.$$

Using, Hölder's inequality, we get

$$\begin{aligned}
&\Big(\| (fu)(t) - (fv)(t)\|_2\Big)^2 \\
&\leq \eta^2 \sum_{n=1}^{\infty} \left\{ \left(\frac{1}{n^2}\right) \left(\sum_{k=n}^{\infty} \frac{1}{(2k-1)^2}\right) \left[\sum_{k=n}^{\infty} \left|\frac{(u_k(t)-v_k(t))[1-(\sqrt{k-n})(u_k(t)+v_k(t))]}{\sqrt{k-n+1}}\right|^2\right]\right\} \\
&\leq \frac{(\pi\eta)^2}{8} \sum_{n=1}^{\infty} \left\{ \frac{1}{n^2} \left[\sum_{k=n}^{\infty} |u_k(t)-v_k(t)|^2 \left|\frac{1-(\sqrt{k-n})(u_k(t)+v_k(t))}{(\sqrt{k-n+1})}\right|^2\right]\right\} \\
&\leq \frac{(\pi\eta)^2}{8} \sum_{n=1}^{\infty} \left\{ \frac{1}{n^2} \left[|u_n(t)-v_n(t)|^2 + \sum_{k=n+1}^{\infty} |u_k(t)-v_k(t)|^2 \left|\frac{1-(\sqrt{k-n})(u_k(t)+v_k(t))}{(\sqrt{k-n+1})}\right|^2\right]\right\}.
\end{aligned}$$

Further, using (3.8), we get

$$\begin{aligned}
\Big(\| (fu)(t) - (fv)(t)\|_2\Big)^2 &\leq \frac{(\pi\eta)^2}{8} \sum_{n=1}^{\infty} \left\{ \frac{1}{n^2} \left[|u_n(t)-v_n(t)|^2 + \frac{\pi^2}{48}\right]\right\} \\
&\leq \left(\frac{\pi^4}{48}\right) \eta^2 \epsilon^2.
\end{aligned}$$

Thus, for any $t \in J$, we have

$$\| (fu)(t) - (fv)(t)\|_2 < \frac{\pi^2 \eta \epsilon}{4\sqrt{3}}.$$

Therefore, the family $\{(fv)(t) : t \in J\}$ is equicontinuous.

Finally, we see that the condition $\mathsf{K}(H)^{\frac{1}{p}}(b-a) = \frac{\pi}{2}\sqrt{\frac{(\pi\eta)^2}{4}} < 1$, for all $\eta < \frac{1}{3}$. So, by Theorem 3.2 there exists at least one solution to the infinite system of integral equations (3.6) in ℓ_2, for fixed $s \in [1, 2]$.

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