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# IDEALS OF A MULTIPLICATIVE SEMIGROUP IN PICTURE FUZZY ENVIRONMENT

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**Abstract**. In this paper, the concept of picture fuzzy ideal of a multiplicative semigroup is introduced. Also, the notions of different types of ideals under a picture fuzzy environment are initiated and relationships between them are established. It has been shown that a picture fuzzy prime ideal of a multiplicative semigroup is a picture fuzzy primary ideal provided that the picture fuzzy prime ideal satisfies the condition of picture fuzzy subsemigroup. Also, it has been highlighted that a picture fuzzy primary semiprime ideal is a picture fuzzy prime ideal. It has been proved that $(\theta, \phi, \psi)$-cut of a picture fuzzy prime ideal is a prime ideal and similar type of concept is also valid in case of picture fuzzy primary ideal and picture fuzzy semiprime ideal.

## 1 Introduction

Fuzzy set was propounded by Zadeh [25] to tackle vagueness in practical life situation. After the initiation of fuzzy set, fuzzy set theory was applied in different fields of research. Notions of fuzzy ideals in case of different kinds of algebraic structures were studied by several researchers [2, 4, 17, 19]. In the case of a fuzzy set, each element of the universal set has only the grade of membership. When an information is clear enough then the grade of membership and the grade of non-membership are complementary measurements to each other. In case of doubtful information, this concept does not work. For this case, individual measurements of both types of membership are needed. Based on this idea, Atanassov [1] initiated an intuitionistic fuzzy set as an extension of fuzzy sets. Ideals of semigroups under intuitionistic fuzzy environments were proposed by Kim and Jun [16]. More generalized works on intuitionistic fuzzy ideal of semigroup were done by Palaniappan [18]. Intuitionistic fuzzy set was applied in case of different types of algebraic structure by several researchers [20, 21, 22]. Later on it was observed that two components are not sufficient to represent some special type of information. In medical science, a disease may have three types of effects ('positive effect', 'neutral effect' and 'negative effect') on a particular symptom. As for example, dengue fever may have a positive effect on the symptom 'pain like breakbone pain' whereas normal fever generally has neutral effect on the symptom 'pain like breakbone pain'. In case of voting, voting results can be classified into three parts : 'vote for' (yes), 'vote against' (no) and 'abstain' (neither 'vote for' nor 'vote against' i.e. white ballot paper is given by the voter). Considering the necessity of neutral component in the fields of medical science, voting results etc., Cuong [3] initiated the concept of picture fuzzy set by generalizing the concept of intuitionistic fuzzy set.

After the initiation of picture fuzzy set, research works on some important aspects of picture fuzzy set and application of picture fuzzy set in multicriteria decision making problem were carried out by different researchers [10, 14, 15]. Algebraic structures in different types of set environment were investigated by [11, 12, 13, 23, 24]. Different kinds of algebraic structures in picture fuzzy setting were studied by Dogra and Pal [5, 6, 7, 8, 9].

In this paper, we are going to introduce a picture fuzzy ideal of a multiplicative semigroup. Multiplicative semigroup means semigroup with ordinary multiplication as its composition. Here, we define different types of picture fuzzy ideals and investigate some important results connected to these. An attempt has been made to establish relationships between different types of ideals.

**List of Abbreviations:**

Intuitionistic Fuzzy Set - IFS
Intuitionistic fuzzy subsemigroup - IFSSEG
Picture fuzzy set - PFS
Measure of Membership - MMS
Measure of non-membership - MNonMS
Measure of positive membership - MPMS
Measure of neutral membership - MNeuMS
Measure of negative membership - MNegMS
Picture fuzzy subsemigroup - PFSSEG
Left Ideal - LI
Right Ideal - RI
Picture fuzzy left ideal - PFLI
Picture fuzzy right ideal - PFRI
Picture fuzzy ideal - PFI
Picture fuzzy prime ideal - PFPREI
Picture fuzzy semi prime ideal - PFSPREI
Picture fuzzy primary ideal - PFPRYI
Prime Ideal - PREI
Primary Ideal - PRYI
Semiprime Ideal - SPREI
Algebraic structure - AS
Picture fuzzy algebraic structure - PFAS

## 2 Preliminaries

In this section, we will recapitulate some basic concepts about IFS, IFSSEG, PFS and different types of operations on PFSs.

**Definition 2.1.** [1] An IFS $P$ over the set of universe $A$ is defined as $P = \{(a, \tau_P^{(1)}, \tau_P^{(2)}) : a \in A\}$, where $\tau_P^{(1)}(a) \in [0,1]$ is MMS and $\tau_P^{(2)}(a) \in [0,1]$ is MNonMS such that $0 \leqslant \tau_P^{(1)}(a) + \tau_P^{(2)}(a) \leqslant 1$ for all $a \in A$.

Based on the notion of IFS, Kim [16] defined IFSSEG.

**Definition 2.2.** [16] Let $P = \{(a, \tau_P^{(1)}(a), \tau_P^{(2)}(a)) : a \in G\}$ be an IFS of a multiplicative semigroup $G$. Then $P$ is said to be IFSSEG of $G$ if $\tau_P^{(1)}(a \cdot b) \geqslant \tau_P^{(1)}(a) \wedge \tau_P^{(1)}(b)$ and $\tau_P^{(2)}(a \cdot b) \leqslant \tau_P^{(2)}(a) \vee \tau_P^{(2)}(b)$ for all $a, b \in G$.

Eliminating the limitation of IFS, Cuong [3] defined PFS as an extended version of IFS.

**Definition 2.3.** [3] A PFS $P$ over the set of universe $A$ is defined as $P = \{(a, \tau_P^{(1)}(a), \tau_P^{(2)}(a), \tau_P^{(3)}(a)) : a \in A\}$, where $\tau_P^{(1)}(a) \in [0,1]$ is the MPMS, $\tau_P^{(2)}(a) \in [0,1]$ is the MNueMS and $\tau_P^{(3)}(a) \in [0,1]$ is the MNegMS such that $0 \leqslant \tau_P^{(1)}(a) + \tau_P^{(2)}(a) + \tau_P^{(3)}(a) \leqslant 1$ for all $a \in A$.

**Definition 2.4.** Let $A_1$ and $A_2$ be two sets of universe. Also, let $P = \{(a_1, \tau_P^{(1)}(a_1), \tau_P^{(2)}(a_1), \tau_P^{(3)}(a_1)) : a_1 \in A_1\}$ be a PFS over $A_1$ and $h : A_1 \to A_2$ be a surjective mapping. Then the image of $P$ under the map $h$ is the PFS $h(P) = \{(a_2, \tau_{h(P)}^{(1)}(a_2), \tau_{h(P)}^{(2)}(a_2), \tau_{h(P)}^{(3)}(a_2)) : a_2 \in A_2\}$, where $\tau_{h(P)}^{(1)}(a_2) = \bigvee_{a_1 \in h^{-1}(a_2)} \tau_P^{(1)}(a_1)$, $\tau_{h(P)}^{(2)}(a_2) = \bigwedge_{a_1 \in h^{-1}(a_2)} \tau_P^{(2)}(a_1)$ and $\tau_{h(P)}^{(3)}(a_2) = \bigwedge_{a_1 \in h^{-1}(a_2)} \tau_P^{(3)}(a_1)$ for all $a_2 \in A_2$.

**Definition 2.5.** Let $A_1$ and $A_2$ be two sets of universe. Also, let $Q = \{(a_2, \tau_Q^{(1)}(a_2), \tau_Q^{(2)}(a_2), \tau_Q^{(3)}(a_2)) : a_2 \in A_2\}$ be a PFS over $A_2$ and $h : A_1 \to A_2$ be a mapping. Then the inverse image of $Q$ under the map $h$ is the PFS $h^{-1}(Q) = \{(a_1, \tau_{h^{-1}(Q)}^{(1)}(a_1), \tau_{h^{-1}(Q)}^{(2)}(a_1), \tau_{h^{-1}(Q)}^{(3)}(a_1)) : a_1 \in A_1\}$, where $\tau_{h^{-1}(Q)}^{(1)}(a_1) = \tau_Q^{(1)}(h(a_1))$, $\tau_{h^{-1}(Q)}^{(2)}(a_1) = \tau_Q^{(2)}(h(a_1))$ and $\tau_{h^{-1}(Q)}^{(3)}(a_1) = \tau_Q^{(3)}(h(a_1))$ for all $a_1 \in A_1$.

**Definition 2.6.** [10] Let $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$ be a PFS over the set of universe $A$. Then $(\theta, \phi, \psi)$-cut of $P$ is the crisp subset of $A$ defined as $C_{\theta,\phi,\psi}(P) = \{a \in A : \tau_P^{(1)}(a) \geqslant \theta, \tau_P^{(2)}(a) \geqslant \phi, \tau_P^{(3)}(a) \leqslant \psi\}$, where $\theta, \phi, \psi \in [0, 1]$ such that $0 \leqslant \theta + \phi + \psi \leqslant 1$.

Now, it is the time to define PFSSEG as a generalization of IFSSEG with respect to a multiplicative semigroup.

**Definition 2.7.** Let $P = \{(a, \tau_P^{(1)}(a), \tau_P^{(2)}(a), \tau_P^{(3)}(a)) : a \in G\}$ be a PFS of a multiplicative semigroup $G$. Then $P$ is said to be PFSSEG of $G$ if $\tau_P^{(1)}(a \cdot b) \geqslant \tau_P^{(1)}(a) \wedge \tau_P^{(1)}(b)$, $\tau_P^{(2)}(a \cdot b) \geqslant \tau_P^{(2)}(a) \wedge \tau_P^{(2)}(b)$ and $\tau_P^{(3)}(a \cdot b) \leqslant \tau_P^{(3)}(a) \vee \tau_P^{(3)}(b)$ for all $a, b \in G$.

Throughout the paper, we write PFS $P = \{(a, \tau_P^{(1)}(a), \tau_P^{(2)}(a), \tau_P^{(3)}(a)) : a \in A\}$ as $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$.

## 3 Picture fuzzy ideal

Ideals are an important matter of study for an algebraic structure. Ideals are specially characterized into two types: LI and RI. In the case of classical algebraic structure, ideal means both LI and RI. The concept is quite similar to the concept under a picture fuzzy environment. The slight difference between the classical ideal and the picture fuzzy ideal is that for defining picture fuzzy ideal, picture fuzzy membership values are taken into account although the working rule in definition remains the same as crisp ideal. Also, it is a noticeable fact that the classical subset concept is converted into an inequality concept between each type of picture fuzzy membership value for elements of universal algebraic structure. In the current section, PFI of a multiplicative semigroup as an extension of IFI is established and some properties related to it are investigated. It is a mentionable fact that multiplicative semigroup means composition of the semigroup is ordinary multiplication.

**Definition 3.1.** Let $P = (\tau_P^{(1)}, \tau_P^{(1)}, \tau_P^{(3)})$ be a PFS of a multiplicative semigroup $G$. Then $P$ is said to be PFRI of $G$ if $\tau_P^{(1)}(a \cdot b) \geqslant \tau_P^{(1)}(a)$, $\tau_P^{(2)}(a \cdot b) \geqslant \tau_P^{(2)}(a)$ and $\tau_P^{(3)}(a \cdot b) \leqslant \tau_P^{(2)}(a)$ for all $a, b \in G$.

**Definition 3.2.** Let $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$ be a PFS of a multiplicative semigroup $G$. Then $P$ is said to be PFLI of $G$ if $\tau_P^{(1)}(a \cdot b) \geqslant \tau_P^{(1)}(b)$, $\tau_P^{(2)}(a \cdot b) \geqslant \tau_P^{(2)}(b)$ and $\tau_P^{(3)}(a \cdot b) \leqslant \tau_P^{(3)}(b)$ for all $a, b \in G$.

Here, picture fuzzy ideal (PFI) means that it is both PFLI and PFRI.

**Example 3.3.** Let us consider a multiplicative semigroup $G = \mathbb{Z}_6$ (the set of integers under modulo 6) and a PFS $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$ over $G$ as follows.

$$\tau_P^{(1)}(a) = \begin{cases} 0.35, & \text{when } a = 0 \\ 0.25, & \text{when } a = 3 \\ 0.2 & \text{otherwise} \end{cases}$$

$$\tau_P^{(2)}(a) = \begin{cases} 0.25, & \text{when } a = 0 \\ 0.2, & \text{when } a = 3 \\ 0.15 & \text{otherwise} \end{cases}$$

$$\tau_P^{(3)}(a) = \begin{cases} 0.3, & \text{when } a = 0 \\ 0.35, & \text{when } a = 3 \\ 0.4, & \text{otherwise} \end{cases}$$

It is easy to show that $P$ acts as a PFI of $G$.

**Proposition 3.4.** *Let $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$ be a PFI of a multiplicative semigroup $G$. Then $C_{\theta,\phi,\psi}(P)$ is a crisp ideal of $G$.*

**Proof.** Let $r \in G$ and $a \in C_{\theta,\phi,\psi}(P)$. This implies, $\tau_P^{(1)}(a) \geqslant \theta$, $\tau_P^{(2)}(a) \geqslant \phi$ and $\tau_P^{(3)}(a) \leqslant \psi$. Then

$$\begin{aligned} \tau_P^{(1)}(r \cdot a) &\geqslant \tau_P^{(1)}(a) \text{ [because } P \text{ is a PFLI of } G] \\ &\geqslant \theta, \\ \tau_P^{(2)}(r \cdot a) &\geqslant \tau_P^{(2)}(a) \text{ [because } P \text{ is a PFLI of } G] \\ &\geqslant \phi \\ \text{and } \tau_P^{(3)}(r \cdot a) &\leqslant \tau_P^{(3)}(a) \text{ [because } P \text{ is a PFLI of } G] \\ &\leqslant \psi. \end{aligned}$$

Thus $r \in G$ and $a \in C_{\theta,\phi,\psi}(P) \Rightarrow r \cdot a \in C_{\theta,\phi,\psi}(P)$. Therefore, $C_{\theta,\phi,\psi}(P)$ is a crisp LI of $G$.

$$\begin{aligned} \text{Also, } \tau_P^{(1)}(a \cdot r) &\geqslant \tau_P^{(1)}(a) \text{ [because } P \text{ is a PFRI of } G] \\ &\geqslant \theta, \\ \tau_P^{(2)}(a \cdot r) &\geqslant \tau_P^{(2)}(a) \text{ [because } P \text{ is a PFRI of } G] \\ &\geqslant \phi \\ \text{and } \tau_P^{(3)}(a \cdot r) &\leqslant \tau_P^{(3)}(a) \text{ [because } P \text{ is a PFRI of } G] \\ &\leqslant \psi. \end{aligned}$$

Thus, $r \in G$ and $a \in C_{\theta,\phi,\psi}(P) \Rightarrow a \cdot r \in C_{\theta,\phi,\psi}(P)$. Therefore, $C_{\theta,\phi,\psi}(P)$ is a crisp RI of $G$. Consequently, $C_{\theta,\phi,\psi}(P)$ is a crisp ideal of $G$.

The following proposition is, in fact, the converse of the above proposition. The following proposition suggests a condition on $(\theta, \phi, \psi)$-cut of a PFS under which a PFS will be a PFI over a multiplicative semigroup.

**Proposition 3.5.** *Let $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$ be a PFS of a multiplicative semigroup $G$. Then $P$ is a PFI if all $(\theta, \phi, \psi)$-cuts of $P$ are crisp ideals.*

**Proof.** Let $\theta = \tau_P^{(1)}(a)$, $\phi = \tau_P^{(2)}(a)$ and $\psi = \tau_P^{(3)}(a)$. Then $\theta \in [0, 1]$, $\phi \in [0, 1]$ and $\psi \in [0, 1]$ with the condition $0 \leqslant \tau_P^{(1)}(a) + \tau_P^{(2)}(a) + \tau_P^{(3)} \leqslant 1$ for all $a \in G$.

Since $C_{\theta,\phi,\psi}(P)$ is crisp LI of $G$ therefore $r \in G$ and $a \in C_{\theta,\phi,\psi}(P)$ implies $r \cdot a \in C_{\theta,\phi,\psi}(P)$. So, $\tau_P^{(1)}(r \cdot a) \geqslant \theta = \tau_P^{(1)}(a)$, $\tau_P^{(2)}(r \cdot a) \geqslant \phi = \tau_P^{(2)}(a)$ and $\tau_P^{(3)}(r \cdot a) \leqslant \psi = \tau_P^{(3)}(a)$ for all $r, a \in G$. Thus, $P$ is a PFLI of $G$.

Similarly, when $C_{\theta,\phi,\psi}(P)$ is a crisp RI of $G$ then we can prove that $P$ is a PFRI of $G$. Thus, $P$ is a PFI of $G$.

Below, a proposition is stated which provides a set of inequality conditions on each type of picture fuzzy membership components under which a PFS will be a PFI over a multiplicative semigroup.

**Proposition 3.6.** *Let $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$ be a PFS of a multiplicative semigroup $G$. Then $P$ is said to be PFI of $G$ if $\tau_P^{(1)}(a \cdot b) \geqslant \tau_P^{(1)}(a) \vee \tau_P^{(1)}(b)$, $\tau_P^{(2)}(a \cdot b) \geqslant \tau_P^{(2)}(a) \vee \tau_P^{(2)}(b)$ and $\tau_P^{(3)}(a \cdot b) \leqslant \tau_P^{(3)}(a) \wedge \tau_P^{(3)}(b)$ for all $a, b \in G$.*

**Proof.** Here, the given conditions are

$$\begin{aligned} \tau_P^{(1)}(a \cdot b) &\geqslant \tau_P^{(1)}(a) \vee \tau_P^{(1)}(b) \geqslant \tau_P^{(1)}(a), \\ \tau_P^{(2)}(a \cdot b) &\geqslant \tau_P^{(2)}(a) \vee \tau_P^{(2)}(b) \geqslant \tau_P^{(2)}(a) \\ \text{and } \tau_P^{(3)}(a \cdot b) &\leqslant \tau_P^{(3)}(a) \wedge \tau_P^{(3)}(b) \leqslant \tau_P^{(3)}(a) \text{ for all } a, b \in G. \end{aligned}$$

Thus, $\tau_P^{(1)}(a \cdot b) \geqslant \tau_P^{(1)}(a)$, $\tau_P^{(2)}(a \cdot b) \geqslant \tau_P^{(2)}(a)$ and $\tau_P^{(3)}(a \cdot b) \leqslant \tau_P^{(3)}(a)$ for all $a, b \in G$. This shows that $P$ is a PFRI of $G$.
Also,

$$\tau_P^{(1)}(a \cdot b) \geqslant \tau_P^{(1)}(a) \vee \tau_P^{(1)}(b) \geqslant \tau_P^{(1)}(b),$$
$$\tau_P^{(2)}(a \cdot b) \geqslant \tau_P^{(2)}(a) \vee \tau_P^{(2)}(b) \geqslant \tau_P^{(2)}(b)$$
$$\text{and } \tau_P^{(3)}(a \cdot b) \leqslant \tau_P^{(3)}(a) \wedge \tau_P^{(3)}(b) \leqslant \tau_P^{(3)}(b) \text{ for all } a, b \in G.$$

Thus, $\tau_P^{(1)}(a \cdot b) \geqslant \tau_P^{(1)}(b)$, $\tau_P^{(2)}(a \cdot b) \geqslant \tau_P^{(2)}(b)$ and $\tau_P^{(3)}(a \cdot b) \leqslant \tau_P^{(3)}(b)$ for all $a, b \in G$. This shows that $P$ is a PFLI of $G$. As a result, $P$ is a PFI of $G$.

## 4 PFI under multiplicative semigroup homomorphism

In this section, it arises the matter of interest to discuss the effect of classical homomorphism upon PFI of a multiplicative semigroup. This has been shown by following two propositions. The first proposition says that the image of a PFI under bijective semigroup homomorphism is a PFI and the second proposition says that the inverse image of a PFI under semigroup homomorphism is a PFI.

**Proposition 4.1.** *Let $G_1$ and $G_2$ be two multiplicative semigroups and $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$ be a PFI of $G_1$. Also, let $h : G_1 \to G_2$ is a bijective semigroup homomorphism. Then $h(P)$ is a PFI of $G_2$.*

**Proof.** Let $h(P) = (\tau_{h(P)}^{(1)}, \tau_{h(P)}^{(2)}, \tau_{h(P)}^{(3)})$.
It is known that for $b_1 \in G_2$, $\tau_{h(P)}^{(1)}(b_1) = \bigvee_{a_1 \in h^{-1}(b_1)} \tau_P^{(1)}(a_1)$, $\tau_{h(P)}^{(2)}(b_1) = \bigwedge_{a_1 \in h^{-1}(b_1)} \tau_P^{(2)}(a_1)$ and $\tau_{h(P)}^{(3)}(b_1) = \bigwedge_{a_1 \in h^{-1}(b_1)} \tau_P^{(3)}(a_1)$. Since $h$ is bijective therefore $h^{-1}(b_1)$ is singleton set. So, it can be written as $a_1 = h^{-1}(b_1)$ i.e. $h(a_1) = b_1$ for unique $a_1 \in G_1$.
Therefore, $\tau_{h(P)}^{(1)}(b_1) = \tau_{h(P)}^{(1)}(h(a_1)) = \tau_P^{(1)}(a_1)$, $\tau_{h(P)}^{(2)}(b_1) = \tau_{h(P)}^{(2)}(h(a_1)) = \tau_P^{(2)}(a_1)$ and $\tau_{h(P)}^{(3)}(b_1) = \tau_{h(P)}^{(3)}(h(a_1)) = \tau_P^{(3)}(a_1)$ for unique $a_1 \in G_1$.
Now,

$$\begin{aligned}
\tau_{h(P)}^{(1)}(b_1 \cdot b_2) &= \tau_{h(P)}^{(1)}(h(a_1) \cdot h(a_2)) \\
&\quad \text{[because } b_1 = h(a_1) \text{ and } b_2 = h(a_2) \text{ for unique } a_1 \text{ and } a_2 \in G_1] \\
&= \tau_{h(P)}^{(1)}(h(a_1 \cdot a_2)) \text{ [as } h \text{ is a semigroup homomorphism]} \\
&= \tau_P^{(1)}(a_1 \cdot a_2) \\
&\geqslant \tau_P^{(1)}(a_1) \text{ [as } P \text{ is a PFRI of } G_1] \\
&= \tau_{h(P)}^{(1)}(h(a_1)) \\
&= \tau_{h(P)}^{(1)}(b_1)
\end{aligned}$$

$$\begin{aligned}
\tau_{h(P)}^{(2)}(b_1 \cdot b_2) &= \tau_{h(P)}^{(2)}(h(a_1) \cdot h(a_2)) \\
&= \tau_{h(P)}^{(2)}(h(a_1 \cdot a_2)) \text{ [as } h \text{ is a semigroup homomorphism]} \\
&= \tau_P^{(2)}(a_1 \cdot a_2) \\
&\geqslant \tau_P^{(2)}(a_1) \text{ [as } P \text{ is a PFRI of } G_1] \\
&= \tau_{h(P)}^{(2)}(h(a_1)) \\
&= \tau_{h(P)}^{(2)}(b_1)
\end{aligned}$$

$$\begin{aligned}
\text{and } \tau^{(3)}_{h(P)}(b_1 \cdot b_2) &= \tau^{(3)}_{h(P)}(h(a_1) \cdot h(a_2)) \\
&= \tau^{(3)}_{h(P)}(h(a_1 \cdot a_2)) \text{ [as } h \text{ is a semigroup homomorphism]} \\
&= \tau^{(3)}_{P}(a_1 \cdot a_2) \\
&\leqslant \tau^{(3)}_{P}(a_1) \text{ [as } P \text{ is a PFRI of } G_1] \\
&= \tau^{(3)}_{h(P)}(h(a_1)) \\
&= \tau^{(3)}_{h(P)}(b_1) \text{ for all } b_1, b_2 \in G_2.
\end{aligned}$$

Thus, $h(P)$ is a PFRI of $G_2$. It is easy to show in the same way that $h(P)$ is a PFLI of $G_2$. As a result, $h(P)$ is a PFI of $G_2$.

**Proposition 4.2.** *Let $G_1$ and $G_2$ be two multiplicative semigroups and $Q = (\tau^{(1)}_Q, \tau^{(2)}_Q, \tau^{(3)}_Q)$ be a PFI of $G_2$. Also, let $h : G_1 \to G_2$ is a semigroup homomorphism. Then $h^{-1}(Q)$ is a PFI of $G_1$.*

**Proof.** Let $h^{-1}(Q) = (\tau^{(1)}_{h^{-1}(Q)}, \tau^{(2)}_{h^{-1}(Q)}, \tau^{(3)}_{h^{-1}(Q)})$.
Here,

$$\begin{aligned}
\tau^{(1)}_{h^{-1}(Q)}(a \cdot b) &= \tau^{(1)}_{Q}(h(a \cdot b)) \\
&= \tau^{(1)}_{Q}(h(a) \cdot h(b)) \text{ [because } h \text{ is a semigroup homomorphism]} \\
&\geqslant \tau^{(1)}_{Q}(h(a)) \text{ [because } Q \text{ is a PFRI of } G_2] \\
&= \tau^{(1)}_{h^{-1}(Q)}(a), \\
\tau^{(2)}_{h^{-1}(Q)}(a \cdot b) &= \tau^{(2)}_{Q}(h(a \cdot b)) \\
&= \tau^{(2)}_{Q}(h(a) \cdot h(b)) \text{ [because } h \text{ is a semigroup homomorphism]} \\
&\geqslant \tau^{(2)}_{Q}(h(a)) \text{ [because } Q \text{ is a PFRI of } G_2] \\
&= \tau^{(2)}_{h^{-1}(Q)}(a) \\
\text{and } \tau^{(3)}_{h^{-1}(Q)}(a \cdot b) &= \tau^{(3)}_{Q}(h(a \cdot b)) \\
&= \tau^{(3)}_{Q}(h(a) \cdot h(b)) \text{ [because } h \text{ is a semigroup homomorphism]} \\
&\leqslant \tau^{(3)}_{Q}(h(a)) \text{ [because } Q \text{ is a PFRI of } G_2] \\
&= \tau^{(3)}_{h^{-1}(Q)}(a) \text{ for all } a, b \in G_1.
\end{aligned}$$

Thus, it is obtained that $\tau^{(1)}_{h^{-1}(Q)}(a \cdot b) \geqslant \tau^{(1)}_{h^{-1}(Q)}(a)$, $\tau^{(2)}_{h^{-1}(Q)}(a \cdot b) \geqslant \tau^{(2)}_{h^{-1}(Q)}(a)$ and $\tau^{(3)}_{h^{-1}(Q)}(a \cdot b) \leqslant \tau^{(3)}_{h^{-1}(Q)}(a)$ for $a, b \in G_1$. Consequently, $h^{-1}(Q)$ is a PFRI of $G_1$. It is easy to show in the same way that $h^{-1}(Q)$ is a PFLI of $G_1$. Thus, $h^{-1}(Q)$ is a PFI of $G_1$.

## 5 Different types of PFI

In this section, the notions of different types of PFI of a multiplicative semigroup is initiated and some properties connected to these are studied.

**Definition 5.1.** Let $P = (\tau^{(1)}_P, \tau^{(2)}_P, \tau^{(3)}_P)$ be a PFS of a multiplicative semigroup $G$. Then $P$ is said to be PFPREI of $G$ if $\tau^{(1)}_P(a \cdot b) \leqslant \tau^{(1)}_P(a) \vee \tau^{(1)}_P(b)$, $\tau^{(2)}_P(a \cdot b) \leqslant \tau^{(2)}_P(a) \vee \tau^{(2)}_P(b)$ and $\tau^{(3)}_P(a \cdot b) \geqslant \tau^{(3)}_P(a) \wedge \tau^{(3)}_P(b)$ for all $a, b \in G$.

**Example 5.2.** Let us consider a PFS $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$ over the set of universe $A = \mathbb{N}$ as follows.

$$\tau_P^{(1)}(a) = \begin{cases} 0.45, & \text{when } a \in 3\mathbb{N} \\ 0.25 & \text{otherwise} \end{cases}$$

$$\tau_P^{(2)}(a) = \begin{cases} 0.43, & \text{when } a \in 3\mathbb{N} \\ 0.27 & \text{otherwise} \end{cases}$$

$$\tau_P^{(3)}(a) = \begin{cases} 0.12, & \text{when } a \in 3\mathbb{N} \\ 0.4 & \text{otherwise} \end{cases}$$

It can be proved that $P$ acts as a PFPREI of $A$.

**Definition 5.3.** Let $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$ be a PFS of a multiplicative semigroup $G$. Then $P$ is said to be PFPRYI of $G$ if $\tau_P^{(1)}(a \cdot b) \leqslant \tau_P^{(1)}(a) \vee \tau_P^{(1)}(b^2)$, $\tau_P^{(2)}(a \cdot b) \leqslant \tau_P^{(2)}(a) \vee \tau_P^{(2)}(b^2)$ and $\tau_P^{(3)}(a \cdot b) \geqslant \tau_P^{(3)}(a) \wedge \tau_P^{(3)}(b^2)$ for all $a, b \in G$.

**Example 5.4.** Let us consider a PFS $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$ over the set of universe $A = \mathbb{N}$ as follows.

$$\tau_P^{(1)}(a) = \begin{cases} 0.38, & \text{when } a \in p\mathbb{N} \text{ [where } p \text{ is a prime number]} \\ 0.23 & \text{otherwise} \end{cases}$$

$$\tau_P^{(2)}(a) = \begin{cases} 0.4, & \text{when } a \in p\mathbb{N} \text{ [where } p \text{ is a prime number]} \\ 0.25 & \text{otherwise} \end{cases}$$

$$\tau_P^{(3)}(a) = \begin{cases} 0.22, & \text{when } a \in p\mathbb{N} \text{ [where } p \text{ is a prime number]} \\ 0.52 & \text{otherwise} \end{cases}$$

It can be proved that $P$ is a PFPRYI of $A$.

**Definition 5.5.** Let $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$ be a PFS of a multiplicative semigroup $G$. Then $P$ is said to be PFSPREI of $G$ if $\tau_P^{(1)}(a^2) \leqslant \tau_P^{(2)}(a)$, $\tau_P^{(2)}(a^2) \leqslant \tau_P^{(2)}(a)$ and $\tau_P^{(3)}(a^2) \geqslant \tau_P^{(3)}(a)$ for all $a \in G$.

**Example 5.6.** Let us consider a PFS $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$ over the set of universe $A = \mathbb{N}$ as follows.

$$\tau_P^{(1)}(a) = \begin{cases} 0.5, & \text{when } a = 0 \\ 0.3, & \text{when } a \in 2\mathbb{N} \\ 0.2 & \text{otherwise} \end{cases}$$

$$\tau_P^{(2)}(a) = \begin{cases} 0.4, & \text{when } a = 0 \\ 0.3, & \text{when } a \in 2\mathbb{N} \\ 0.2 & \text{otherwise} \end{cases}$$

$$\tau_P^{(3)}(a) = \begin{cases} 0.1, & \text{when } a = 0 \\ 0.4, & \text{when } a \in 2\mathbb{N} \\ 0.5, & \text{otherwise} \end{cases}$$

Clearly, $P$ forms a PFSPREI of $A$.

Now, we are eager to develop a relationship between PFPREI, PFPRYI and PFSPREI.

**Proposition 5.7.** *Let $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$ be a PFPREI of a multiplicative semigroup $G$ such that $P$ is a PFSSEG of $G$. Then $P$ is a PFPRYI of $G$.*

**Proof.** Given that PFS $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$ over the multiplicative semigroup $G$ be PFPREI in $G$.
So, it can be written that $\tau_P^{(1)}(a \cdot b) \leqslant \tau_P^{(1)}(a) \vee \tau_P^{(1)}(b)$, $\tau_P^{(2)}(a \cdot b) \leqslant \tau_P^{(2)}(a) \vee \tau_P^{(2)}(b)$ and $\tau_P^{(3)}(a \cdot b) \geqslant \tau_P^{(3)}(a) \wedge \tau_P^{(3)}(b)$ for all $a, b \in G$.
Since $P$ is a PFSSEG therefore $\tau_P^{(1)}(a \cdot b) \geqslant \tau_P^{(1)}(a) \wedge \tau_P^{(1)}(b)$, $\tau_P^{(2)}(a \cdot b) \geqslant \tau_P^{(2)}(a) \wedge \tau_P^{(2)}(b)$ and $\tau_P^{(3)}(a \cdot b) \leqslant \tau_P^{(3)}(a) \vee \tau_P^{(3)}(b)$ for all $a, b \in G$. Setting $a = b$, it is obtained that

$$\tau_P^{(1)}(b^2) \geqslant \tau_P^{(1)}(b), \tau_P^{(2)}(b^2) \geqslant \tau_P^{(2)}(b) \text{ and } \tau_P^{(3)}(b^2) \leqslant \tau_P^{(3)}(b)$$
$$\text{i.e. } \tau_P^{(1)}(b) \leqslant \tau_P^{(2)}(b^2), \tau_P^{(2)}(b) \leqslant \tau_P^{(2)}(b^2) \text{ and } \tau_P^{(3)}(b) \geqslant \tau_P^{(3)}(b^2) \text{ for all } b \in G.$$

Therefore,

$$\tau_P^{(1)}(a \cdot b) \leqslant \tau_P^{(1)}(a) \vee \tau_P^{(1)}(b) \leqslant \tau_P^{(1)}(a) \vee \tau_P^{(1)}(b^2),$$
$$\tau_P^{(1)}(a \cdot b) \leqslant \tau_P^{(2)}(a) \vee \tau_P^{(2)}(b) \leqslant \tau_P^{(2)}(a) \vee \tau_P^{(2)}(b^2)$$
$$\text{and } \tau_P^{(3)}(a \cdot b) \geqslant \tau_P^{(3)}(a) \wedge \tau_P^{(3)}(b) \geqslant \tau_P^{(3)}(a) \wedge \tau_P^{(3)}(b^2) \text{ for all } a, b \in G.$$

Consequently, $P$ is a PFPRYI of $G$.

Now, we are interested in investigating the condition under which a PFPRYI is a PFPREI i.e. the condition under which the converse of the above proposition is true. The following proposition works well in this regard.

**Proposition 5.8.** *Let $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$ be a PFPRYI of a multiplicative semigroup G such that P is PFSPRE also. Then P is a PFPREI.*

**Proof.** Given that PFS $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$ over the multiplicative semigroup $G$ be both PFPRYI and PFSPREI in $G$.
Since $P$ is a PFPRYI of $G$ therefore
$\tau_P^{(1)}(a \cdot b) \leqslant \tau_P^{(1)}(a) \vee \tau_P^{(1)}(b^2)$, $\tau_P^{(2)}(a \cdot b) \leqslant \tau_P^{(2)}(a) \vee \tau_P^{(2)}(b^2)$ and $\tau_P^{(3)}(a \cdot b) \geqslant \tau_P^{(3)}(a) \wedge \tau_P^{(3)}(b^2)$ for all $a, b \in G$.
Also, since $P$ is a PFSPREI of $G$ therefore
$\tau_P^{(1)}(b^2) \leqslant \tau_P^{(1)}(b)$, $\tau_P^{(2)}(b^2) \leqslant \tau_P^{(2)}(b)$ and $\tau_P^{(3)}(b^2) \geqslant \tau_P^{(3)}(b)$ for all $b \in G$.
Therefore,

$$\tau_P^{(1)}(a \cdot b) \leqslant \tau_P^{(1)}(a) \vee \tau_P^{(1)}(b^2) \leqslant \tau_P^{(1)}(a) \vee \tau_P^{(1)}(b)$$
$$\tau_P^{(2)}(a \cdot b) \leqslant \tau_P^{(2)}(a) \vee \tau_P^{(2)}(b^2) \leqslant \tau_P^{(2)}(a) \vee \tau_P^{(2)}(b)$$
$$\text{and } \tau_P^{(3)}(a \cdot b) \geqslant \tau_P^{(3)}(a) \wedge \tau_P^{(3)}(b^2) \geqslant \tau_P^{(3)}(a) \wedge \tau_P^{(3)}(b) \text{ for all } a, b \in G.$$

It follows that $\tau_P^{(1)}(a \cdot b) \leqslant \tau_P^{(1)}(a) \vee \tau_P^{(1)}(b)$, $\tau_P^{(2)}(a \cdot b) \leqslant \tau_P^{(2)}(a) \vee \tau_P^{(2)}(b)$, $\tau_P^{(3)}(a \cdot b) \geqslant \tau_P^{(3)}(a) \wedge \tau_P^{(3)}(b)$ for all $a, b \in G$. As a result, $P$ is a PFPREI of $G$.

It has been observed from the definition that $(\theta, \phi, \psi)$-cut of PFS is a crisp set. The idea is very useful in case of PFAS. The $(\theta, \phi, \psi)$-cut of a PFAS is, in fact, a classical AS. This fact has been highlighted below in case of PFPREI, PFPRYI and PFSPREI.

**Proposition 5.9.** *Let $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$ be a PFPREI of a multiplicative semigroup G. Then $C_{\theta,\phi,\psi}(P)$ is a crisp PREI of G.*

**Proof.** Let $a \cdot b \in C_{\theta,\phi,\psi}(P)$.
This implies, $\tau_P^{(1)}(a \cdot b) \geqslant \theta$, $\tau_P^{(2)}(a \cdot b) \geqslant \phi$ and $\tau_P^{(3)}(a \cdot b) \leqslant \psi$. Since $P$ is a PFPREI of $G$ therefore
$\theta \leqslant \tau_P^{(1)}(a \cdot b) \leqslant \tau_P^{(1)}(a) \vee \tau_P^{(1)}(b)$, $\phi \leqslant \tau_P^{(2)}(a \cdot b) \leqslant \tau_P^{(2)}(a) \vee \tau_P^{(2)}(b)$ and $\psi \geqslant \tau_P^{(3)}(a \cdot b) \geqslant \tau_P^{(3)}(a) \wedge \tau_P^{(3)}(b)$.

Therefore, $\tau_P^{(1)}(a) \vee \tau_P^{(1)}(b) \geqslant \theta$, $\tau_P^{(2)}(a) \vee \tau_P^{(2)}(b) \geqslant \phi$ and $\tau_P^{(3)}(a) \wedge \tau_P^{(3)}(b) \leqslant \psi$
$\Rightarrow$ [either $\tau_P^{(1)}(a) \geqslant \theta$ or $\tau_P^{(1)}(b) \geqslant \theta$], [either $\tau_P^{(2)}(a) \geqslant \phi$ or $\tau_P^{(2)}(b) \geqslant \phi$] and [either $\tau_P^{(3)}(a) \leqslant \psi$ or $\tau_P^{(3)}(b) \leqslant \psi$].
Therefore, either [$\tau_P^{(1)}(a) \geqslant \theta$, $\tau_P^{(2)}(a) \geqslant \phi$ and $\tau_P^{(3)}(a) \leqslant \psi$] or [$\tau_P^{(1)}(b) \geqslant \theta$, $\tau_P^{(2)}(b) \geqslant \phi$ and $\tau_P^{(3)}(b) \leqslant \psi$].
Therefore, either $a \in C_{\theta,\phi,\psi}(P)$ or $b \in C_{\theta,\phi,\psi}(P)$. Thus, $C_{\theta,\phi,\psi}(P)$ is a crisp PREI of $G$.

**Proposition 5.10.** *Let $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$ be a PFPRYI of a multiplicative semigroup G. Then $C_{\theta,\phi,\psi}(P)$ is a crisp PRYI of G.*

**Proof.** This proposition can be proved like the Proposition 5.9.

**Proposition 5.11.** *Let $P = (\tau_P^{(1)}, \tau_P^{(2)}, \tau_P^{(3)})$ be a PFSREI of a multiplicative semigroup G. Then $C_{\theta,\phi,\psi}(P)$ is a crisp SPREI ideal of G.*

**Proof.** This proposition can be proved like the Proposition 5.9.

## 6 Conclusion

Study of PFI of a multiplicative semigroup has a significant contribution in the study of advanced fuzzy algebra. Here, we have established some basic results related to PFI of a multiplicative semigroup. Also, we have investigated some important results on PFI under the environment of multiplicative semigroup homomorphism. Also, we have studied different types of PFI and established relationships between them. We expect that exploration of some more advanced works in future on the theory of ideal will be easy as a result of this research work.

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