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SUBTRACTIVE EXTENSION OF IDEALS IN (m, n)-SEMIRINGS

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Abstract The concept of a subtractive extension of an ideal in a commutative semiring with identity element was introduced by Chaudhari and Bonde [3]. In this paper, we extend this concept of subtractive extension of an ideal to a commutative (m, n)-semiring R with identity element. Let R be a commutative (m, n)-semiring with identity element and $I \subseteq A$ be ideals of R. Then (i) We obtain the smallest subtractive extension of I containing A; (ii) If I is a partitioning ideal (i.e.Q-ideal), then characterizations of subtractive extensions of I are obtained; (iii) We prove that a subtractive extension P of a Q-ideal I is a prime (semiprime) ideal if and only if $P/I_{(Q\cap P)}$ is a prime (semiprime) ideal of the quotient (m, n)-semiring $R/I_{(Q)}$.

1 Introduction

Generalizing the notion of an (m, n)-ring introduced by Crombez [8], Pop [9] introduced the notion of an (m, n)-semiring. A non-empty set R with an m-ary operation f (i.e.$f : R^m \to R$ where $R^m = \{(x_1, x_2, ..., x_m) : x_i \in R, 1 \leq i \leq m\}$) called addition and an n-ary operation g (i.e. $g : R^n \to R$) called multiplication is called an (m, n)-semiring if it satisfies the following conditions :

1) (R, f) is an m-ary commutative semigroup ;
2) (R, g) is an n-ary semigroup;
3) There exists $0 \in R$ (called the zero element of R) such that:
   a) 0 is an f-identity of R (i.e. $f\big(x, 0, 0, ..., 0\big) = x$ for all $x \in R$);
   b) 0 is a g-zero (i.e. $x_1, x_2, x_3, ..., x_n \in R$ and $x_i = 0$ for some i $\Rightarrow g(x_1, x_2, x_3, ..., x_n) = 0$);
4) g is distributive with respect to f.

We denote this (m, n)-semiring by (R, f, g) or just R.
Clearly, every semiring is an (m, n)-semiring. The set of all non-positive integers $(\mathbb{Z}_0^-)$ is a $(2, 3)$-semiring, (i.e. a ternary semiring) under usual addition and multiplication of non-positive integers but it is neither a ring nor a semiring. If $R = \{1, 3, 5, 7, ..., 69\} = \{2n - 1 : 1 \leq n \leq 35\}$, then R is a $(3, 2)$-semiring under addition modulo 70 $(+_{70})$ and multiplication modulo 70 $(\times_{70})$ with zero element 35. But it is neither a semiring nor a ternary semiring.

Zero-divisor free, zero-sum free, additively cancellative and multiplicatively cancellative (m, n)-semirings are studied by Alam, Rao and Davvaz [1]. Jacobson radical of an (m, n)-semiring is studied by Zhu [12], properties of ideals in (m, n)-semirings, subtractive ideals in (m, n)-semirings are studied respectively by Pop and Pop [11], Pop and Lauran [10]. Further, subtractive ideals in (m, n)-semirings are studied by Chaudhari and Nemade [7]. Allen [2] has

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introduced the notion of a partitioning ideal in semirings and prove the fundamental theorem of homomorphism for semirings. Chaudhari and Nemade [6] have extended this concept to (m,n)-semirings. Further, the fundamental theorem of homomorphism for (m,n)-semirings is proved by Chaudhari, Nemade and Davvaz [6]. The concept of subtractive extension of an ideal in a commutative semiring with identity element is introduced by Chaudhari and Bonde [3] and then it is studied by Chaudhari, Davvaz and Ingale [4]. Further, it is extended to commutative ternary semiring with identity element by Chaudhari and Ingale [5].

Let R be a commutative (m,n)-semiring with identity element and $I \subseteq A$ be ideals of R. In section 2, we extend the concept of subtractive extension of an ideal of a semiring to an (m,n)-semiring R and obtain the smallest subtractive extension of I containing A. In section 3, we obtain the characterizations of a subtractive extensions of a Q-ideal I in R. Also, we prove that a subtractive extension P of a Q-ideal I in R is a prime (semiprime) ideal if and only if $P/I_{(Q\cap P)}$ is a prime (semiprime) ideal of the quotient (m,n)-semiring $R/I_{(Q)}$.

The following notations, definitions and results will be used in this paper to prove our results.

Notations:

1) If $x_1, x_2, ..., x_m \in R$, then we write this as $x_1^m \in R$.

2) For $1 \leq i \leq j \leq m$, if $x_{i+1}, x_{i+2}, ..., x_j \in R$, then we write $x_{i+1}^j \in R$. We denote x_i^j as empty symbol if $i > j$.

3) For all $1 \leq i \leq j \leq m$ the term, $f(x_1, x_2, ..., x_i, y_{i+1}, y_{i+2}, ..., y_j, z_{j+1}, z_{j+2}, ..., z_m)$, is represented as, $f(x_1^i, y_{i+1}^j, z_{j+1}^m)$.
If $y_{i+1} = ... = y_j = y$, then, $f(x_1, x_2, ..., x_i, y_{i+1}, y_{i+2}, ..., y_j, z_{j+1}, z_{j+2}, ..., z_m)$ can be written as $f\left(x_1^i, \overset{(j\text{-}i)}{y}, z_{j+1}^m\right)$.

4) If $x_1 = ... = x_i = y_{i+1} = ... = y_j = z_{j+1} = ... = z_m = a$ (say) then, $f(x_1, x_2, ..., x_i, y_{i+1}, y_{i+2}, ..., y_j, z_{j+1}, z_{j+2}, ..., z_m)$ can be written as $f\left(\overset{(m)}{a}\right)$.

Definition 1.1. Let R be a non-empty set.

1) Let $x_1^{2m-1} \in R$. Then the associativity for the m-ary operation f on R is defined as follows,

$$f\left(x_1^{i-1}, f\left(x_i^{m+i-1}\right), x_{m+i}^{2m-1}\right) = f\left(x_1^{j-1}, f\left(x_j^{m+j-1}\right), x_{m+j}^{2m-1}\right)$$

for all $1 \leq i \leq j \leq m$.

2) Let x_1^n, $a_1^m \in R$. Then the n-ary operation g is distributive with respect to the m-ary operation f if $g\left(x_1^{i-1}, f\left(a_1^m\right), x_{i+1}^n\right) =$
$f\left(g\left(x_1^{i-1}, a_1, x_{i+1}^n\right), g\left(x_1^{i-1}, a_2, x_{i+1}^n\right), ..., g\left(x_1^{i-1}, a_m, x_{i+1}^n\right)\right)$.

3) An element 1 of R is called an identity element if $g\left(\overset{(i)}{1}, x, \overset{(n\text{-}i\text{-}1)}{1}\right) = x$ for all $x \in R$ and for all $1 \leq i \leq n$.

4) An n-ary operation $g : R^n \to R$ is commutative if $g(x_1, x_2, ..., x_n) = g(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)})$ for every permutation σ of $\{1, 2, ..., n\}$ and for every $x_1^n \in R$.

5) An (m,n)-semiring (R, f, g) is called commutative if g is a commutative operation.

Throughout this paper by an (m,n)-semiring we mean a commutative (m,n)-semiring with identity element 1. For the definition of an ideal, partitioning ideal, subtractive ideal in (m,n)-semirings and quotient (m,n)-semirings we refer [6] and [7].

Notation: If I is an ideal of an (m,n)-semiring R and $x \in R$,

then $f(x, \underbrace{I, \cdots, I}_{m-1})$ is denoted by $f\left(x, \overset{(m\text{-}1)}{I}\right)$.

2 Subtractive Extension of an ideal

Let R be a commutative (m, n)-semiring with identity element and $I \subseteq A$ be ideals of R. In this section, we extend the concept of subtractive extension of an ideal of a semiring to an (m, n)-semiring R and obtain the smallest subtractive extension of I containing A. We begin with the key definition of this paper:

Definition 2.1. Let I be an ideal of an (m, n)-semiring R. An ideal A of R with $I \subseteq A$ is said to be a subtractive extension of I if $i_2^m \in I$, $f\big(x, i_2^m\big) \in A$, $x \in R$, then $x \in A$.

Clearly, every ideal of an (m, n)-semiring R is a subtractive extension of $\{0\}$. Also, every subtractive ideal of an (m, n)-semiring R containing an ideal I of R is a subtractive extension of I. The following example shows that the subtractive extension of an ideal of an (m, n)-semiring is not a subtractive ideal.

Example 2.2. Let $R = \{0, a, b, c\}$. Define a commutative ternary operation $f : R^3 \to R$ by $f(a, x, y) = a$ for all $x, y \in R$; $f(b, x, y) = b$ for all $x, y \in R - \{a\}$; $f(c, c, c) = f(c, c, 0) = f(c, 0, 0) = c$ and $f(0, 0, 0) = 0$. Also, define a binary commutative operation $g : R^2 \to R$ by $g(a, b) = g(a, a) = a, g(b, b) = b, g(a, c) = g(b, c) = g(c, c) = c$ and $g(x, y) = 0$ if $x = 0$ or $y = 0$. Then $\big(R, f, g\big)$ forms a $(3, 2)$-semiring with zero element 0. Let $I = \{0, c\}$ and $A = \{0, a, c\}$ be ideals in R. Clearly, A is a subtractive extension of I but not a subtractive ideal.

Let $I \subseteq A$ be ideals of an (m, n)-semiring R. Then we denote,

(1) $\overline{A}_I = \big\{x \in R : f\big(x, i_2^m\big) \in A \text{ for some } i_2^m \in I\big\}$ and will be called closure of A with respect to I.

(2) If I is a partitioning ideal of R, then $\tilde{A} = \Big\{x \in R :$ there exists $f\Big(q, \overset{(m\text{-}1)}{I}\Big) \in R/I_{(Q)}$ such that $x \in f\Big(q, \overset{(m\text{-}1)}{I}\Big)$ and $f\Big(q, \overset{(m\text{-}1)}{I}\Big) \bigcap A \neq \emptyset\Big\}$.

(3) $\overline{A} = \big\{x \in R : f\big(x, y_2^m\big) \in A \text{ for some } y_2^m \in A\big\}$ and will be called the k-closure of A.

We can easily show that i) $\overline{A}_A = \overline{A}$ where $I \subseteq A$ are ideals of R; ii) $I \subseteq \overline{I} \subseteq \overline{A}_I \subseteq \overline{A}$. The inequality may be strict. For, in Example 2.2, $\overline{I} = \{0, c\}$, $\overline{A}_I = \{0, a, c\}$ and $\overline{A} = \{0, a, b, c\}$. Now $\overline{I} \subsetneq \overline{A}_I \subsetneq \overline{A}$.

Theorem 2.3. *Let $I \subseteq A$ be ideals of an (m, n)-semiring R. Then $\overline{A}_I$ is the smallest subtractive extension of I containing A.*

Proof. 1) First we will show that $\overline{A}_I$ is an ideal of R. i) Let $a_1^m \in \overline{A}_I$. Then there exist $i_{12}^{1m}, i_{22}^{2m}, ..., i_{m2}^{mm} \in I$ such that $f\big(a_j, i_{j2}^{jm}\big) \in A$ for all $j = 1, 2, ..., m$. Since A is an ideal of R, $f\Big(f\big(a_1, i_{12}^{1m}\big), f\big(a_2, i_{22}^{2m}\big), ..., f\big(a_m, i_{m2}^{mm}\big)\Big) \in A$ and hence $f\Big(f\big(a_1^m\big), f\big(i_{12}, i_{22}^{2m}\big), f\big(i_{13}, i_{32}^{3m}\big),$ $..., f\big(i_{1m}, i_{m2}^{mm}\big)\Big) \in A$. Therefore, $f\big(a_1^m\big) \in \overline{A}_I$.
ii) Let $a \in \overline{A}_I$ and $r_1^n \in R$. Then there exists $i_2^m \in I$ such that $f\big(a, i_2^m\big) \in A$. Since A is an ideal of R, $g\Big(r_1^{k-1}, f\big(a, i_2^m\big), r_{k+1}^n\Big) \in A$.
Hence by distributive law, $f\Big(g\big(r_1^{k-1}, a, r_{k+1}^n\big), g\big(r_1^{k-1}, i_2, r_{k+1}^n\big), ..., g\big(r_1^{k-1}, i_m, r_{k+1}^n\big)\Big) \in A$...(∗)
Now for each $2 \leq t \leq m$, $i_t \in I$ implies $g\big(r_1^{k-1}, i_t, r_{k+1}^n\big) \in I$.
Hence by (∗), $g\big(r_1^{k-1}, a, r_{k+1}^n\big) \in \overline{A}_I$. Hence $\overline{A}_I$ is an ideal of R.
2) Clearly $A \subseteq \overline{A}_I$.
3) Let $i_2^m \in I, f\big(x, i_2^m\big) \in \overline{A}_I$ and $x \in R$. Then there exists, $j_2^m \in I$ such that $f\Big(f\big(x, i_2^m\big), j_2^m\Big) \in A$. Hence $f\Big(x, f\big(i_2^m, j_2\big), j_3^m\Big) \in A$. Now $i_2^m, j_2^m \in I$ implies $f\big(i_2^m, j_2\big)$, $j_3^m \in I$. Therefore, $x \in \overline{A}_I$. Hence $\overline{A}_I$ is a subtractive extension of I.
4) Let J be a subtractive extension of I containing A and let $x \in \overline{A}_I$. Then there exists $i_2^m \in I$ such that $f\big(x, i_2^m\big) \in A \subseteq J$. Since J is subtractive extension of I, $x \in J$. Hence $\overline{A}_I \subseteq J$. □

From Theorem 2.3, we have the following corollary.

Corollary 2.4. *If $I \subseteq A$ are ideals of an (m,n)-semiring R, then* $\overline{A}_I = \bigcap \{J : J \; is \; a \; subtractive \; extension \; of \; I \; containing \; A\}$.

In the following theorem, we prove the properties related to a subtractive extension of an ideal I and closure of an ideal A with respect to an ideal I in an (m,n)-semiring R.

Theorem 2.5. *Let I, A, B be ideals of an (m,n)-semiring R such that $I \subseteq A, B$.Then*

(1) A is a subtractive extension of $I \Leftrightarrow \overline{A}_I = A$.

(2) $\overline{(\overline{A}_I)}_I = \overline{A}_I$.

(3) $A \subseteq B \Rightarrow \overline{A}_I \subseteq \overline{B}_I$.

(4) $\overline{(A \cap B)}_I = \overline{A}_I \cap \overline{B}_I$.

(5) If A, B are subtractive extensions of I, then $A \cap B$ is a subtractive extension of I.

(6) If J is an ideal of R such that $I \subseteq J \subseteq A$, then $\overline{A}_I \subseteq \overline{A}_J$.

Proof. (1) Follows from Theorem 2.3.

(2) Follows from Theorem 2.3 and by using (1).

(3) Let $A \subseteq B$ and let $x \in \overline{A}_I$. Then there exists $i_2^m \in I$ such that $f(x, i_2^m) \in A \subseteq B$. Hence $x \in \overline{B}_I$. Now $\overline{A}_I \subseteq \overline{B}_I$.

(4) Since $A \cap B \subseteq A$, by using (3) we get $\overline{(A \cap B)}_I \subseteq \overline{A}_I$. Similarly, $\overline{(A \cap B)}_I \subseteq \overline{B}_I$. Now $\overline{(A \cap B)}_I \subseteq \overline{A}_I \cap \overline{B}_I$. For the other inclusion, let $x \in \overline{A}_I \cap \overline{B}_I$. Then there exist $i_2^m, j_2^m \in I$ such that $f(x, i_2^m) \in A$ and $f(x, j_2^m) \in B$. Since $j_2^m \in I \subseteq A$ and A is an ideal of R, $f\Big(f(x, i_2^m), j_2^m\Big) \in A$. Therefore $f\Big(x, f(i_2^m, j_2), j_3^m\Big) \in A$. Similarly, $f\Big(x, f(i_2^m, j_2), j_3^m\Big) \in B$. Now $f\Big(x, f(i_2^m, j_2), j_3^m\Big) \in A \cap B$. So $x \in \overline{(A \cap B)}_I$. Thus, $\overline{A}_I \cap \overline{B}_I \subseteq \overline{(A \cap B)}_I$.

(5) By (4), $\overline{(A \cap B)}_I = \overline{A}_I \cap \overline{B}_I = A \cap B$, since A, B are subtractive extension of I. Hence by (1), $A \cap B$ is a subtractive extension of I.

(6) Suppose that J is an ideal of R such that $I \subseteq J \subseteq A$. Let $x \in \overline{A}_I$. Then $f(x, i_2^m) \in A$ for some $i_2^m \in I \subseteq J$. Hence $x \in \overline{A}_J$. Now $\overline{A}_I \subseteq \overline{A}_J$.

□

Corollary 2.6. *Let A, B be ideals of an (m,n)-semiring R. Then $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.*

Proof. By using Theorem 2.5 (3) and Theorem 2.5 (6), we have $\overline{A \cap B} = \overline{(A \cap B)}_{(A \cap B)} \subseteq \overline{A}_{(A \cap B)} \subseteq \overline{A}_A = \overline{A}$. Similarly, $\overline{A \cap B} \subseteq \overline{B}$. Now $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. □

The following example shows that the equality in Corollary 2.6 may not hold.

Example 2.7. [10] Let $R = \{0, a, b, c\}$. Define a ternary operation $f : R^3 \to R$ by $f(x, 0, 0) = x$; $f(x, c, c) = c$ for all $x \in R$;

$$f(x, a, a) = \begin{cases} a, & \text{if } x \in \{0, a\} \\ c, & \text{if } x \in \{b, c\} \end{cases}$$

$$f(x, b, b) = \begin{cases} b, & \text{if } x \in \{0, b\} \\ c, & \text{if } x \in \{a, c\} \end{cases}$$

$f(0, a, b) = f(c, a, b) = f(0, c, a) = f(0, c, b) = c$ and the n-ary operation $g : R^n \to R$ by $g(x_1^n) = 0$ for all $x_1^n \in R$. Then (R, f, g) forms an $(3, n)$-semiring with zero element 0. Clearly, $A = \{0, b\}$ and $B = \{0, a, c\}$ are ideals in R. Then $A \cap B = \{0\}$ and hence $\overline{A \cap B} = \{0\}$. An inspection will show that, $\overline{A} = \{0, b\}$ and $\overline{B} = \{0, a, b, c\}$. Hence $\overline{A} \cap \overline{B} = \{0, b\}$. Now $\overline{A \cap B} \subsetneq \overline{A} \cap \overline{B}$.

3 Subtractive extension of a partitioning ideal

Let R be a commutative (m,n)-semiring with identity element and $I \subseteq A$ be ideals of R. In this section, we obtain the characterizations of a subtractive extensions of a Q-ideal I in R. Also, we prove that a subtractive extension P of a Q-ideal I in R is a prime (semiprime) ideal if and only if $P/I_{(Q\cap P)}$ is a prime (semiprime) ideal of the quotient (m,n)-semiring $R/I_{(Q)}$. Now we begin with the next lemma which gives the relation between $\overline{A}_I$ and $\tilde{A}$.

Lemma 3.1. *Let I be a Q-ideal of an (m,n)-semiring R and A be an ideal of R with $I \subseteq A$. Then $\overline{A}_I = \tilde{A}$.*

Proof. Let $x \in \overline{A}_I$. Then there exists $i_2^m \in I$ such that $f(x, i_2^m) \in A$. By ([7], Lemma 2.4), there exists a unique $q \in Q$ such that $x \in f\Big(x, \overset{(m\text{-}1)}{I}\Big) \subseteq f\Big(q, \overset{(m\text{-}1)}{I}\Big)$. Hence $f(x, i_2^m) \in f\Big(q, \overset{(m\text{-}1)}{I}\Big)$. Thus, $f\Big(q, \overset{(m\text{-}1)}{I}\Big) \bigcap A \neq \emptyset$. Hence $x \in \tilde{A}_I$. Now $\overline{A}_I \subseteq \tilde{A}$. On the other hand, let $z \in \tilde{A}$. Then there exists $f\Big(q, \overset{(m\text{-}1)}{I}\Big) \in R/I_{(Q)}$ such that $z \in f\Big(q, \overset{(m\text{-}1)}{I}\Big)$ and $f\Big(q, \overset{(m\text{-}1)}{I}\Big) \bigcap A \neq \emptyset$. Hence $z = f(q, i_2^m)$ for some $i_2^m \in I$. Let $y = f(q, j_2^m) \in f\Big(q, \overset{(m\text{-}1)}{I}\Big) \bigcap A$ where $j_2^m \in I$. Then $f\Big(z, j_2^m\Big) = f\Big(f(q, i_2^m), j_2^m\Big) = f\Big(f(q, j_2^m), i_2^m\Big) \in A$, since A is an ideal of R. Hence $z \in \overline{A}_I$. Now $\tilde{A} \subseteq \overline{A}_I$. □

Theorem 3.2. *Let I be a Q-ideal of an (m,n)-semiring R and A be an ideal of R with $I \subseteq A$. Then $\tilde{A}$ is the smallest subtractive extension of I containing A.*

Proof. Follows from Lemma 3.1 and Theorem 2.3. □

Now we give the following lemma which will be used in the subsequent theorem.

Lemma 3.3. *Let I be a Q-ideal of an (m,n)-semiring R and A be an ideal of R with $I \subseteq A$. Then A is a subtractive extension of I if and only if I is a $Q \cap A$-ideal of A.*

Proof. Let A be a subtractive extension of I and let $a \in A$. By ([7], Lemma 2.4), there exists a unique $q \in Q$ such that $a \in f\Big(q, \overset{(m\text{-}1)}{I}\Big)$. So $a = f(q, i_2^m)$ for some $i_2^m \in I \subseteq A$. Since A is a subtractive extension of I, $q \in A$. Therefore $q \in Q \cap A$. If $f\Big(q_1, \overset{(m\text{-}1)}{I}\Big) \bigcap f\Big(q_2, \overset{(m\text{-}1)}{I}\Big) \neq \emptyset$ for some $q_1, q_2 \in Q \cap A$, then $q_1 = q_2$, since I is a Q-ideal of R. Thus I is a $Q \cap A$-ideal of A. Conversely, suppose that I is a $Q \cap A$-ideal of A. Let $f(x, i_2^m) \in A$, $i_2^m \in I$ and $x \in R$. Since I is a $Q \cap A$-ideal of A, there exists a unique $q_1 \in Q \cap A$ such that $f(x, i_2^m) \in f\Big(q_1, \overset{(m\text{-}1)}{I}\Big)$. Also, since I is a Q-ideal of R, by ([7], Lemma 2.4), we get there exists a unique $q_2 \in Q$ such that $f\Big(x, \overset{(m\text{-}1)}{I}\Big) \subseteq f\Big(q_2, \overset{(m\text{-}1)}{I}\Big)$...$(*)$. So $f(x, i_2^m) \in f\Big(q_2, \overset{(m\text{-}1)}{I}\Big)$. Now $f\Big(q_1, \overset{(m\text{-}1)}{I}\Big) \bigcap f\Big(q_2, \overset{(m\text{-}1)}{I}\Big) \neq \emptyset$. Therefore, $q_2 = q_1 \in A$. Hence by $(*)$, $x \in f\Big(x, \overset{(m\text{-}1)}{I}\Big) \subseteq f\Big(q_2, \overset{(m\text{-}1)}{I}\Big) \subseteq A$. □

The following theorem gives the characterization of subtractive extension of a partitioning ideal in an (m,n)-semiring R.

Theorem 3.4. *Let $I \subseteq A$ be ideals of an (m,n)-semiring R and I a Q-ideal of R. Then following statements are equivalent:*

(1) A is a subtractive extension of I;

(2) I is a $Q \cap A$-ideal of A;

(3) $A/I_{(Q\cap A)}$ is an ideal of the quotient (m,n)-semiring $R/I_{(Q)}$;

(4) $A/I_{(Q\cap A)} \subseteq R/I_{(Q)}$.

Proof. (1) $\Rightarrow$ (2) Follows from Lemma 3.3.
(2) $\Rightarrow$ (3) Since A is an ideal of R, $A/I_{(Q\cap A)}$ is an ideal of $R/I_{(Q)}$.
(3) $\Rightarrow$ (4) Trivial.
(4) $\Rightarrow$ (1) Let $f(x, i_2^m) \in A$, $i_2^m \in I$ and $x \in R$. Then by ([7], Lemma 2.5), we have $i_2^m \in I = f\Big(q_0, \overset{(m\text{-}1)}{I}\Big)$ where $f\Big(q_0, \overset{(m\text{-}1)}{I}\Big)$ is a zero element of $R/I_{(Q)}$. Now by definition of quotient (m,n)-semiring, there exist a unique $f\Big(q_1, \overset{(m\text{-}1)}{I}\Big) \in A/I_{(Q\cap A)} \subseteq R/I_{(Q)}$ and a unique $f\Big(q_2, \overset{(m\text{-}1)}{I}\Big) \in R/I_{(Q)}$ such that $f(x, i_2^m) \in f\Big(q_1, \overset{(m\text{-}1)}{I}\Big)$ and $x \in f\Big(q_2, \overset{(m\text{-}1)}{I}\Big)$. Here $f(x, i_2^m) \in \tilde{f}\Big(f\big(q_2, \overset{(m\text{-}1)}{I}\big), f\big(q_0, \overset{(m\text{-}1)}{I}\big), ..., f\big(q_0, \overset{(m\text{-}1)}{I}\big)\Big) = f\Big(q_2, \overset{(m\text{-}1)}{I}\Big)$. Therefore, $f\Big(q_1, \overset{(m\text{-}1)}{I}\Big) \bigcap f\Big(q_2, \overset{(m\text{-}1)}{I}\Big) \neq \emptyset$. Hence $q_2 = q_1 \in A$. Now $x \in f\Big(q_1, \overset{(m\text{-}1)}{I}\Big) \subseteq A$. □

The following theorem gives the relation between subtractive extensions of a Q-ideal I in an (m,n)-semiring R and the ideals in the quotient (m,n)-semiring $R/I_{(Q)}$.

Theorem 3.5. *Let I be a Q-ideal of an (m,n)-semiring R. Then L is an ideal of $R/I_{(Q)}$ if and only if there exists an ideal A of R such that A is a subtractive extension of I and $A/I_{(Q\cap A)} = L$.*

Proof. Let L be an ideal of a quotient (m,n)-semiring $R/I_{(Q)}$. Denote $A = \Big\{x \in R :$ there exist a unique $q \in Q$ such that $f\Big(x, \overset{(m\text{-}1)}{I}\Big) \subseteq f\Big(q, \overset{(m\text{-}1)}{I}\Big) \in L\Big\}$. By ([7], Lemma 2.5), we have $I = f\Big(q_0, \overset{(m\text{-}1)}{I}\Big)$ is a zero element of $R/I_{(Q)}$.
(1) Let $a \in I$. Then $f\Big(a, \overset{(m\text{-}1)}{I}\Big) \subseteq I = f\Big(q_0, \overset{(m\text{-}1)}{I}\Big) \in L$. So $a \in A$. Now $I \subseteq A$.
(2) Let $a_1^m \in A$. Then there exist unique elements $q_1^m \in Q$ such that $f\Big(a_i, \overset{(m\text{-}1)}{I}\Big) \subseteq f\Big(q_i, \overset{(m\text{-}1)}{I}\Big) \in L$ for all i.
Again there exists a unique $q \in Q$ such that $\tilde{f}\Big(f\Big(q_1, \overset{(m\text{-}1)}{I}\Big), f\Big(q_2, \overset{(m\text{-}1)}{I}\Big), ..., f\Big(q_m, \overset{(m\text{-}1)}{I}\Big)\Big) = f\Big(q, \overset{(m\text{-}1)}{I}\Big) \in L$ where $f\Big(f(q_1^m), \overset{(m\text{-}1)}{I}\Big) \subseteq f\Big(q, \overset{(m\text{-}1)}{I}\Big)$. Since $f\Big(a_i, \overset{(m\text{-}1)}{I}\Big) \subseteq f\Big(q_i, \overset{(m\text{-}1)}{I}\Big)$ for all i, $f\Big(f(a_1^m), \overset{(m\text{-}1)}{I}\Big) \subseteq f\Big(f(q_1^m), \overset{(m\text{-}1)}{I}\Big) \subseteq f\Big(q, \overset{(m\text{-}1)}{I}\Big) \in L$. Now $f\big(a_1^m\big) \in A$. Similarly, if $a \in A$ and $r_1^n \in R$, then $g\big(r_1^{i-1}, a, r_{i+1}^n\big) \in A$. Hence A is an ideal of R.
(3) Let $a_2^m \in I$, $f\big(a, a_2^m\big) \in A$ and $a \in R$. Then there exists a unique $q \in Q$ such that $f\big(a, a_2^m\big) \in f\Big(q, \overset{(m\text{-}1)}{I}\Big) \in L$. Since I is a Q-ideal of R, by ([7], Lemma 2.4) we get, there exists a unique $q' \in Q$ such that $f\Big(a, \overset{(m\text{-}1)}{I}\Big) \subseteq f\Big(q', \overset{(m\text{-}1)}{I}\Big)$. So, $f\big(a, a_2^m\big) \in f\Big(q', \overset{(m\text{-}1)}{I}\Big)$. Now $f\Big(q, \overset{(m\text{-}1)}{I}\Big) \bigcap f\Big(q', \overset{(m\text{-}1)}{I}\Big) \neq \emptyset$. Therefore, $q = q'$. Hence $f\Big(a, \overset{(m\text{-}1)}{I}\Big) \subseteq f\Big(q', \overset{(m\text{-}1)}{I}\Big) = f\Big(q, \overset{(m\text{-}1)}{I}\Big) \in L$. Thus, A is a subtractive extension of I.
(4) Clearly, $A/I_{(Q\cap A)} \subseteq L$. If $f\Big(q, \overset{(m\text{-}1)}{I}\Big) \in L$, then $q \in A$. Now $L = A/I_{(Q\cap A)}$. Converse follows from Theorem 3.4. □

Definition 3.6. A proper ideal P of an (m,n)-semiring R is called a prime ideal of R if $g(a_1^n) \in P$, $a_1^n \in R$ implies that $a_i \in P$ for some i.

The following theorem gives that a subtractive extension P of a Q-ideal I in an (m,n)-semiring R is a prime ideal if and only if $P/I_{(Q\cap P)}$ is a prime ideal of the quotient (m,n)-semiring $R/I_{(Q)}$.

Theorem 3.7. *Let I be a Q-ideal of an (m, n)-semiring R and P a subtractive extension of I. Then P is a prime ideal of R if and only if $P/I_{(Q\cap P)}$ is a prime ideal of $R/I_{(Q)}$.*

Proof. Let P be a prime ideal of R. Suppose that $f\left(q_i, \overset{(m\text{-}1)}{I}\right) \in R/I_{(Q)}$ for all $i, 1 \leq i \leq n$ and $\tilde{g}\left(f\left(q_1, \overset{(m\text{-}1)}{I}\right), f\left(q_2, \overset{(m\text{-}1)}{I}\right), ..., f\left(q_n, \overset{(m\text{-}1)}{I}\right)\right) = f\left(q, \overset{(m\text{-}1)}{I}\right) \in P/I_{(Q\cap P)}$ where $q \in Q \cap P$ is a unique element such that $f\left(g(q_1^n), \overset{(m\text{-}1)}{I}\right) \subseteq f\left(q, \overset{(m\text{-}1)}{I}\right)$. Since P is a subtractive extension of I and $q \in P$, we get $g(q_1^n) \in P$. So $q_i \in P$ for some i, as P is a prime ideal. Hence $f\left(q_i, \overset{(m\text{-}1)}{I}\right) \in P/I_{(Q\cap P)}$ for some i.

Conversely, suppose that $P/I_{(Q\cap P)}$ is a prime ideal of $R/I_{(Q)}$. Let $g(a_1^n) \in P$ where $a_1^n \in R$. Since I is a Q-ideal of R, there exist unique elements $q_1^n \in Q$ such that $a_i \in f\left(q_i, \overset{(m\text{-}1)}{I}\right)$ for all i...$(*)$.

Now, $\tilde{g}\left(f\left(q_1, \overset{(m\text{-}1)}{I}\right), f\left(q_2, \overset{(m\text{-}1)}{I}\right),..., f\left(q_n, \overset{(m\text{-}1)}{I}\right)\right) = f\left(q, \overset{(m\text{-}1)}{I}\right)$

where $f\left(g(q_1^n), \overset{(m\text{-}1)}{I}\right) \subseteq \left(q, \overset{(m\text{-}1)}{I}\right)$. Since P is a subtractive extension of I, $q \in P$. So $\tilde{g}\left(f(q_1, \overset{(m\text{-}1)}{I}), f(q_2, \overset{(m\text{-}1)}{I}), ..., f(q_n, \overset{(m\text{-}1)}{I})\right) = f\left(q, \overset{(m\text{-}1)}{I}\right) \in P/I_{(Q\cap P)}$. Since $P/I_{(Q\cap P)}$ is a prime ideal, $f\left(q_i, \overset{(m\text{-}1)}{I}\right) \in P/I_{(Q\cap P)}$ for some i. Hence $q_i \in P$. Therefore, by $(*)$ we get, $a_i \in P$. Now P is a prime ideal of R. □

Definition 3.8. An ideal P of an (m, n)-semiring R is called semiprime if $g(\overset{(n)}{a}) \in P, a \in R$ implies $a \in P$.

Theorem 3.9. *Let R be an (m, n)-semiring, I a Q-ideal of R and P a subtractive extension of I. Then P is a semiprime ideal of R if and only if $P/I_{(Q\cap P)}$ is a semiprime ideal of $R/I_{(Q)}$.*

Proof. Let P be a semiprime ideal of R. Let $f\left(q, \overset{(m\text{-}1)}{I}\right) \in R/I_{(Q)}$ and $\tilde{g}\left(f\left(q, \overset{(m\text{-}1)}{I}\right), f\left(q, \overset{(m\text{-}1)}{I}\right), ..., f\left(q, \overset{(m\text{-}1)}{I}\right)\right) = f\left(q', \overset{(m\text{-}1)}{I}\right) \in P/I_{(Q\cap P)}$ where $q' \in Q \cap P$ is a unique element such that $f\left(g(\overset{(n)}{q}), \overset{(m\text{-}1)}{I}\right) \subseteq f\left(q', \overset{(m\text{-}1)}{I}\right)$. So $g(\overset{(n)}{q}) = f(q', a_2^m) \in P$ for some $a_2^m \in I$. Since P is a semiprime ideal of R, $q \in P$. Hence $f\left(q, \overset{(m\text{-}1)}{I}\right) \in P/I_{(Q\cap P)}$.

Conversely, suppose that $P/I_{(Q\cap P)}$ is a semiprime ideal of $R/I_{(Q)}$. Let $g(\overset{(n)}{a}) \in P$ where $a \in R$. Since I is a Q-ideal of R, there exist unique elements $q, q' \in Q$ such that $a \in f\left(q, \overset{(m\text{-}1)}{I}\right)$ and $g(\overset{(n)}{a}) \in \tilde{g}\left(f\left(q, \overset{(m\text{-}1)}{I}\right), f\left(q, \overset{(m\text{-}1)}{I}\right), ..., f\left(q, \overset{(m\text{-}1)}{I}\right)\right) = f\left(q', \overset{(m\text{-}1)}{I}\right)$. Then $g(\overset{(n)}{a}) = f(q', b_2^m)$ for some $b_2^m \in I$. Since P is a subtractive extension of I, $q' \in P$. So $\tilde{g}\left(f\left(q, \overset{(m\text{-}1)}{I}\right), f\left(q, \overset{(m\text{-}1)}{I}\right), ..., f\left(q, \overset{(m\text{-}1)}{I}\right)\right) = f\left(q', \overset{(m\text{-}1)}{I}\right) \in P/I_{(Q\cap P)}$. Since $P/I_{(Q\cap P)}$ is a semiprime ideal, $f\left(q, \overset{(m\text{-}1)}{I}\right) \in P/I_{(Q\cap P)}$. Now $a \in f\left(q, \overset{(m\text{-}1)}{I}\right) \Rightarrow a = f(q, c_2^m)$ for some $c_2^m \in I \Rightarrow a \in P$, as $q \in Q \cap P \subseteq P$. □

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