Palestine Journal of Mathematics

Vol. 11(4)(2022) , 23–25

Characterizing $\mathbb{F}_p$ via noncommutative minimal ring extensions

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 13B99, 16B99; Secondary 12E20, 16P10.

Keywords and phrases: Associative ring, minimal ring extension, field, Frobenius map, noncommutative ring

Abstract. We prove that if p is a prime number, then up to isomorphism, $\mathbb{F}_p$ is the only field of characteristic p all of whose minimal ring extensions are commutative.

1 Introduction

All rings considered in this note are associative and unital, but not necessarily commutative; all inclusions of rings, ring extensions and ring homomorphisms are unital. For distinct rings $A \subset B$, one says, extending the context of a definition from [5], that $A \subset B$ is a *minimal ring extension* if there is no ring C such that $A \subset C \subset B$. Some of the results in [5] were anticipated by arguments in [6] that studied a concept that is now regarded as a special kind of minimal ring extension. While all the rings considered in [6] and [5] were commutative (and most of those in [6] were integral domains), there has been some work on minimal ring extensions involving nontrivial zero-divisors and noncommutative rings, perhaps most notably in [4] and [1]. Some significant work in that direction began with our results in [2] that used idealizations to prove that every nonzero commutative ring has a commutative minimal ring extension. Indeed, that work was adapted and generalized by Dorsey and Mesyan [4, Lemma 2.4 and Remark 2.5] who used Dorroh extensions (where we had used idealizations) to prove that every (not necessarily commutative) ring has a minimal ring extension.

Our interest here is in the first question that Dorsey and Mesyan raised in [4, Question 6.5] in regard to minimal ring extensions and prime fields. (Recall that a field K is called a *prime field* if K does not have any proper subfields or, equivalently, if K coincides with its subfield that is generated by $\{1\}$. A field K is a prime field if and only if K is isomorphic to either $\mathbb{Q}$ or $\mathbb{F}_p$ for some prime number p.) The specific question of Dorsey and Mesyan that we address here was formulated by them as follows: "If k is a field with no noncommutative ... minimal [ring] extensions, must k be a prime field?" Some evidence for an affirmative answer to this question can be found in [4]. For instance, it was observed in [4, page 3482], as a consequence of a familiar fact about centralizers, that no prime field can have a noncommutative minimal ring extension.

It turns out that one can make significant headway on the question of Dorsey and Mesyan by using familiar facts. Indeed, the **purpose of this note** is to answer the first question raised in [4, Question 6.5] in the affirmative for fields of positive characteristic. The proof is given in Theorem 2.1. Corollary 2.2 merely states a reformulation of Theorem 2.1, while Remark 2.3 completes the note by commenting about the (incomplete) state of our knowledge about the possible validity of an analogue of Theorem 2.1 for fields of characteristic 0.

We will be using the following standard notation: $\subset$ denotes proper inclusion; and X denotes an indeterminate over any ambient coefficient ring(s).

2 Results

We move at once to Theorem 2.1, which answers the first question raised in [4, Question 6.5] in the affirmative for fields of positive characteristic.

Theorem 2.1. *Let p be a prime number and let F be a field of characteristic p. Then the following conditions are equivalent:*

(*1*) *F is isomorphic* (*as a field*) *to $\mathbb{F}_p$;*
(*2*) *F does not have a noncommutative minimal ring extension;*
(*3*) *Every minimal ring extension of F is commutative.*

Proof. As recalled in the Introduction, if $\mathcal{F}$ is a prime field and $\mathcal{F} \subset E$ is a minimal ring extension, then E is commutative. Since any ring that is isomorphic to $\mathbb{F}_p$ is a prime field, it follows that (1) $\Rightarrow$ (2). As it is clear that (2) $\Leftrightarrow$ (3), it remains only to prove that (3) $\Rightarrow$ (1). We will, in fact, prove the contrapositive of (3) $\Rightarrow$ (1). Suppose, then, that F is not isomorphic to $\mathbb{F}_p$. It will suffice to construct a minimal ring extension $F \subset B$ such that B is not commutative.

View $\mathbb{F}_p$ as a subfield of F in the usual way. Since F is not isomorphic to $\mathbb{F}_p$, the vector space dimension of F as a vector space over $\mathbb{F}_p$ is greater than 1 (with "greater than" being taken in the sense of cardinal numbers). So, there exists a basis $\mathcal{B}$ of F as a vector space over $\mathbb{F}_p$ such that $\{1\}$ is a proper subset of $\mathcal{B}$ (cf. [7, Theorem 2.4, page 183]). Pick $b \in \mathcal{B} \setminus \{1\}$. Since 1 and b are linearly independent over $\mathbb{F}_p$, we get $b \notin \mathbb{F}_p$. Next, consider the polynomial $f \in \mathbb{F}_p[X]$ defined by $f(X) := X^p - X$. Recall that each of the p elements in $\mathbb{F}_p$ is a root of f (essentially by the little Fermat theorem; cf. also [7, Proposition 5.6, page 280]). As $\deg(f) = p$, it follows that $\mathbb{F}_p$ is the set of all the roots of f in any given field extension of $\mathbb{F}_p$ (cf. [7, Theorem 6.7, page 160]). Applying this to the field extension F of $\mathbb{F}_p$, we get that b is not a root of f; that is, $b^p \neq b$.

Consider the Frobenius map $\sigma : F \to F$ (given by $\sigma(a) := a^p$ for all $a \in F$). It is clear that σ preserves multiplication and the multiplicative identity element; it is also well known (cf. [7, Exercise 11, page 121]) that σ preserves addition. Thus, since $\sigma|_{\mathbb{F}_p}$ is the identity map on $\mathbb{F}_p$, we get that σ is an $\mathbb{F}_p$-algebra homomorphism. Also by the above comments, $\sigma(b) \neq b$.

Next, consider a two-dimensional vector space $B = F + Fx$ over F (so that $\{1, x\}$ is a basis of B as a vector space over F). Induce a multiplication on B by requiring that $xa = \sigma(a)x$ for all $a \in F$ and $x^2 = 0$. More precisely, define the binary operation of multiplication on B as follows: if $\{\alpha, \beta, \gamma, \delta\} \subseteq F$, then

$$(\alpha + \beta x)(\gamma + \delta x) := \alpha\gamma + (\alpha\delta + \beta\sigma(\gamma))x.$$

As σ preserves addition and multiplication in F, it is straightforward to verify that the above multiplication on B is both left-and right distributive over addition; and that this multiplication is associative. As the multplicative identity element of the ring $\mathbb{F}_p$ (which is the same as the multplicative identity element of the ring F) also serves as a/the multplicative identity element for B, it follows that B is a ring. Thus, $\mathbb{F}_p \subset F \subset B$ is a chain of (unital) ring extensions. However, while B is thereby an $\mathbb{F}_p$-algebra, B is *not* an F-algebra, the point being that $xb := \sigma(b)x = b^p x \neq bx$ because $b^p \neq b$ and the singleton set $\{x\}$ is linearly independent over F. Moreover, since $xb \neq bx$, B is a noncommutative ring. Finally, since B is a two-dimensional vector space over F, [3, Lemma 2.4 (a)] ensures that $F \subset B$ is a minimal ring extension. □

By some results in [2] (resp., [4]) that were mentioned in the Introduction, it follows that every nonzero commutative ring has a commutative minimal ring extension and that every nonzero noncommutative ring has a (necessarily noncommutative) minimal ring extension. On the other hand, examples of noncommutative minimal ring extensions of commutative rings are also known: cf. $\mathbb{C} \subset \mathbb{H}$ in [4, page 3482]; [4, Lemma 6.6]; and [3, Example 2.7]. Thus, for reference purposes, it may be useful to have the following variant of the statement of Theorem 2.1.

Corollary 2.2. *Let p be a prime number and let F be a field of characteristic p. Then the following conditions are equivalent:*
(*1*) *F is not isomorphic* (*as a field*) *to $\mathbb{F}_p$;*
(*2*) *F has a noncommutative minimal ring extension;*
(*3*) *Not every minimal ring extension of F is commutative.*

Remark 2.3. The first question raised in [4, Question 6.5] remains open for fields of characteristic 0. We were able to reduce to the situation where such a field k has a proper subfield F such that the field extension $F \subset k$ is algebraic. In trying to go further for such $F \subset k$ by adapting the "twisting multiplication via the Frobenius map" method of proof of Theorem 2.1 (by "twisting" via different kinds of field homomorphisms), we were able to get an affirmative answer to the first question raised in [4, Question 6.5] in *that* situation if $k = F(\alpha)$ and the minimum polynomial of α over F has another root $\beta \neq \alpha$ in k. As [4, Lemma 6.6] is stronger than the preceding assertion, we consider it likely that rather different ideas will be needed to settle [4, Question 6.5] in the case of characteristic 0. By the way, the idea of using a "twisted" multiplication in the proof of Theorem 2.1 was suggested by the proof of [3, Example 2.7]. In closing, we take advantage of this opportunity to mention the following correction to a typographical error in the last line of the statement of [3, Example 2.7]: the correct definition of R_2 there should have been $R_2 := \mathbb{F}_2 + I$.

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Received: July 1, 2022.
Accepted: August 3, 2022.