Palestine Journal of Mathematics

Vol. 11(4)(2022) , 243–247 © Palestine Polytechnic University-PPU 2022

THE SMALLEST ELEMENT OF A LATTICE OF IDEALS IN SEMI-FLAT RINGS

Majid M. Ali

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 13F05; Secondary 13A15.

Keywords and phrases: flat ideal, multiplication ideal, projective ideal, pure ideal, semi-flat ring.

To the memory of my mother and my father whose help is really appreciated

Abstract In this note we give necessary and sufficient conditions for an element of a lattice of ideals in a semi-flat ring to be the smallest element.

1 Introduction

All rings in this note are commutative with identity. An ideal A in R is called multiplication if for every ideal $B \subseteq A$, there exists an ideal C in R such that $B = AC$. Note that $C \subseteq [B : A]$, and hence $B = CA \subseteq [B : A]A \subseteq B$, so that $B = [B : A]A$, [5] and [7]. In fact if A is multiplication then for each ideal B in R,

$$B \cap A = [(B \cap A) : A]A = [B : A]A.$$

Multiplication ideals are locally principal and the converse is true if they are finitely generated, [4, Proposition 1]. Recall that an ideal A in R is flat if for every short exact sequence of ideals in R

$$0 \longrightarrow K \longrightarrow L \longrightarrow N \longrightarrow 0,$$

the sequence

$$0 \longrightarrow K \otimes A \longrightarrow L \otimes A \longrightarrow N \otimes A \longrightarrow 0,$$

is also exact, [9]. A is flat if and only if it is locally flat and if A is finitely generated then A is flat if and only if it is locally free, [7] and [9].

An ideal A in R is called pure if $BA = B \cap A$ for every ideal B in R. The following conditions are equivalent for an ideal A in R, (i) A is pure, (ii) A is locally either zero or R and (iii) A is multiplication and idempotent, [3, Theorem 1.1] and [5]. It is obvious that the product, sum and intersection of two pure ideals are pure, see [3, Corollary 1.3].

If A is a flat ideal in a local ring (R, P) then $A = PA$ or $A = aR$ for some non-zero divisor a in R, [12, Lemma 2.1] and [14, Lemma 6]. If A is finitely generated then Nakayama's lemma shows that A is flat if and only if A is locally either zero or invertible. Moreover, ann A is a pure ideal, [14, lemma 7]. If A is finitely generated flat then A is locally principal (and hence multiplication) and ann A is pure. Conversely, if A is locally either zero or invertible then A is locally flat and hence it is flat. So if A is finitely generated then A is flat if and only if A is multiplication and ann A is pure, [11, Theorem 2.2]. More generally, if A is a multiplication ideal (not necessary finitely generated) and ann A is pure then A is flat [10, Theorem 4.1]. In particular, faithful multiplication ideals are flat. Pure ideals are flat ideals. For if A is a pure ideal in R, then A is locally either zero or R. Hence A is locally flat and hence A is flat. It is well-known [9] that if every finitely generated ideal in a ring R is flat then every ideal in R is flat. In this note we call a ring R is semi-flat if every finitely generated ideal is flat. In fact semi-flat rings are flat rings (a ring R is called a flat ring if every ideal in R is flat).

A ring R is called a semi-hereditary ring if every finitely generated ideal in R is projective. A finitely generated ideal A is projective if and only if A is multiplication and ann $A = \text{ann}(e)$ for some idempotent e, [11, Theorem 2.1]. If A is a finitely generated flat such that ann A is finitely generated, then A is projective, [14]. Let R be a semi-hereditary ring and A and B finitely generated ideals in R. Set

$$\Phi_{A,B} := \{I \mid I \text{ is a finitely generated ideal in } R,\ A \subseteq I, e \in I + B \text{ for some idempotent } e\}.$$

In [1] we investigated the smallest element in $\Phi_{A,B}$. In this note we generalize some results in [1] to finitely generated flat ideals. Let R be a semi-flat ring and A and B finitely generated ideals in R. Define

$$\Phi_{A,B} := \{I \mid I \text{ is a finitely generated ideal in } R,\ A \subseteq I, \operatorname{ann}(\operatorname{ann} I) \subseteq I + B\}.$$

It is clear that $\Phi_{A,B}$ is a non-empty set as $R \in \Phi_{A,B}$. Moreover this definition of $\Phi_{A,B}$ generalizes that one in [1]. In fact if R is a semi-hereditary ring then the second condition of $\Phi_{A,B}$ is

$$\operatorname{ann}(\operatorname{ann} I) = \operatorname{ann}(\operatorname{ann} e) = eR \subseteq I + B.$$

In this note we show that $\Phi_{A,B}$ is a lattice of ideals in a semi-flat ring R and we give necessary and sufficient conditions for an element of $\Phi_{A,B}$ to be the smallest one.

All rings are commutative with identity. For the basic concepts used, we refer the reader to [5], [6], [7] and [9].

2 The smallest element of $\Phi_{A,B}$

We start this section by the following Lemma. The proof of the first two parts can be found in [2, Theorem 2.1] and [13, Proposition 4].

Lemma 2.1. *Let R be a ring, X, Y and K ideals in R such that $[X : Y] + [Y : X] = R$.*
(i) $[K : (X \cap Y)] = [K : X] + [K : Y]$, consequently, $\operatorname{ann}(X \cap Y) = \operatorname{ann} X + \operatorname{ann} Y$.
(ii) $K + (X \cap Y) = (K + X) \cap (K + Y)$.
(iii) $[\operatorname{ann} X : \operatorname{ann} Y] + [\operatorname{ann} Y : \operatorname{ann} X] = R$.
In particular, if $X + Y$ is a finitely generated multiplication ideal in R, then the results also hold.

Proof. (iii) There exists $a, b \in R$ such that $1 = a + b$ where $a \in [X : Y]$ and $b \in [Y : X]$. Hence $aY \subseteq X$, and hence $aY \operatorname{ann} X = 0$. It follows that $a \operatorname{ann} X \subseteq \operatorname{ann} Y$, and this gives that $a \in [\operatorname{ann} Y : \operatorname{ann} X]$. Similarly $b \in [\operatorname{ann} X : \operatorname{ann} Y]$, and the result follows. □

The following Theorem gives some conditions for the product, sum, intersection and residual of finitely generated flat ideals to be finitely generated flat.

Theorem 2.2. *Let R be a ring and X and Y finitely generated flat ideals in R such that*

$$[X : Y] + [Y : X] = R.$$

(i) $X + Y$ is a finitely generated flat ideal in R.
(ii) $X \cap Y$ is a finitely generated flat ideal in R.
(iii) XY is a finitely generated flat ideal in R.
(iv) $[X : Y]$ is a flat ideal in R.
(v) If, moreover, $\operatorname{ann} Y$ is finitely generated then $[X : Y]$ is a finitely generated flat ideal in R.

Proof. (i) Since X and Y are finitely generated flat (hence multiplication) and $[X : Y] + [Y : X] = R$, it follows by [13, Corollary to Theorem 2] and [2, Theorem 2.1] that $X + Y$ is finitely generated multiplication. Obviously, $\operatorname{ann}(X + Y) = \operatorname{ann} X \cap \operatorname{ann} Y$ is a pure ideal in R. Hence $X + Y$ is finitely generated flat.
(ii) We have that

$$R = [X : Y] + [Y : X] = [(X \cap Y) : Y] + [(X \cap Y) : X].$$

There exist $r, s \in R$ with $1 = r + s$, $rY \subseteq X \cap Y$ and $sX \subseteq X \cap Y$. It follows that

$$X \cap Y = r(X \cap Y) + s(X \cap Y) \subseteq rY + sX \subseteq X \cap Y,$$

so that $X \cap Y = sX + rY$ is finitely generated. Now, since each of X, Y and $X + Y$ is multiplication, it follows by [2, Theorem 2.3] and [13, Theorem 8] that $X \cap Y$ is multiplication. Finally by Lemma 2.1, $\text{ann}(X \cap Y) = \text{ann}\, X + \text{ann}\, Y$ is a pure ideal in R. So $X \cap Y$ is finitely generated flat.

(iii) Since each of X and Y is a finitely generated locally principal ideal in R, so too is XY and hence XY is finitely generated multiplication. We show now that $\text{ann}(XY) = \text{ann}(X \cap Y)$. Obviously $\text{ann}(X \cap Y) \subseteq \text{ann}(XY)$. Let $r \in \text{ann}(XY)$. Then $rXY = 0$, and hence $rX \subseteq \text{ann}\, Y$. So

$$r(X \cap Y) \subseteq rX \cap rY \subseteq Y \cap \text{ann}\, Y \subseteq \text{ann}(\text{ann}\, Y) \cap \text{ann}\, Y = 0,$$

since $\text{ann}\, Y$ is pure. So $r \in \text{ann}(X \cap Y)$ and

$$\text{ann}(XY) = \text{ann}(X \cap Y) = \text{ann}\, X + \text{ann}\, Y$$

is a pure ideal in R. So XY is finitely generated flat.

(iv) It is enough to prove the result locally. Thus we may assume R is local. Note that $[X : Y] = [X : (X + Y)]$. By (i) $X + Y$ is finitely generated flat. If $X + Y = 0$, then $[X : Y] = R$. So assume that $X + Y$ is invertible.

Case 1: Let $X = 0$. Then $[X : X + Y] = \text{ann}(X + Y) = 0$, since $X + Y$ is invertible.

Case 2: X is invertible. It follows by [13, Corollary to Theorem 9] that $[X : (X + Y)]$ is a multiplication ideal. But

$$\text{ann}[X : (X + Y) = \text{ann}\, X \cap \text{ann}(\text{ann}(X + Y)) = 0.$$

Thus $[X : Y]$ is a faithful multiplication ideal and hence it is flat. All cases show that $[X : Y]$ is locally flat and hence it is flat.

(v) Note that if $\text{ann}\, Y$ is finitely generated then Y is projective and $\text{ann}\, Y = \text{ann}(e)$ for some idempotent e, [11, Theorem 2.2]. Write $[X : Y] = [(X \cap Y) : Y]$. Since $X \cap Y$ is finitely generated multiplication and Y is finitely generated projective, it follows by [13, Theorem 10] that $[X : Y]$ is multiplication. Since $X + \text{ann}\, Y$ is finitely generated ideal contained in $[X : Y]$ and $\text{ann}[X : Y] = \text{ann}(X + \text{ann}\, Y)$, we infer from [8, Corollary to Lemma 1.4] that $[X : Y]$ is finitely generated. Finally,

$$\text{ann}[X : Y] = \text{ann}\, X \cap \text{ann}(\text{ann}\, Y)$$

is a pure ideal in R. Hence $[X : Y]$ is a finitely generated flat ideal in R. □

The following Proposition shows that $\Phi_{A,B}$ is a lattice of ideals. Compare with [1, Proposition 2.2]

Proposition 2.3. *Let R be a semi-flat ring and A and B finitely generated ideals in R. Then $\Phi_{A,B}$ is a lattice of ideals in R. Moreover, if M is a minimal element of $\Phi_{A,B}$, then it is the smallest element of $\Phi_{A,B}$.*

Proof. Let $X, Y \in \Phi_{A,B}$. Then $A \subseteq X$ and $A \subseteq Y$. So $A \subseteq X + Y$. It follows by Lemma 2.1 that

$$\begin{aligned}\text{ann}(\text{ann}(X + Y)) &= \text{ann}(\text{ann}\, X \cap \text{ann}\, Y)\\ &= \text{ann}(\text{ann}\, X) + \text{ann}(\text{ann}\, Y)\\ &\subseteq (X + Y) + B,\end{aligned}$$

and hence $X + Y \in \Phi_{A,B}$. Now $A \subseteq X \cap Y$ and by Theorem 2.2, $X \cap Y$ is finitely generated (hence flat). Again by Lemma 2.1

$$\begin{aligned}\text{ann}(\text{ann}(X \cap Y)) &= \text{ann}(\text{ann}\, X + \text{ann}\, Y)\\ &= \text{ann}(\text{ann}\, X) \cap \text{ann}(\text{ann}\, Y)\\ &\subseteq (X + B) \cap (Y + B) = (X \cap Y) + B.\end{aligned}$$

So $X \cap Y \in \Phi_{A,B}$ and this shows that $\Phi_{A,B}$ is a lattice of ideals. Now, let M be a minimal element of $\Phi_{A,B}$. Let $X \in \Phi_{A,B}$. Then $M \cap X \in \Phi_{A,B}$ and hence $M \subseteq M \cap X \subseteq M$, so $M \subseteq X$ and M is the smallest element of $\Phi_{A,B}$.

□

As a consequence of the above result, we give the following corollary

Corollary 2.4. *Let R be a Noetherian semi-flat ring and A, B non-zero ideals in R. Then $\Phi_{A,B}$ has the smallest element.*

Proof. Since R is a Noetherian semi-flat ring, R is a Noetherian multiplication ring (every ideal in R is finitely generated multiplication). Hence R is a general ZPI-ring, [7]. So every non-zero ideal in R is uniquely expressible as a product of prime ideals. It follows that there are only finitely many ideals containing A. Therefore $\Phi_{A,B}$ has finitely many elements and hence $\Phi_{A,B}$ has the smallest element. □

We end this note by a result which gives necessary and sufficient conditions for an element of $\Phi_{A,B}$ to be the smallest one.

Theorem 2.5. *Let R be a semi-flat ring, A, B and M finitely generated ideals in R such that ann M is finitely generated. Then M is the smallest element of $\Phi_{A,B}$ if and only if the following conditions are satisfied:*
(i) $A \subseteq M$,
(ii) ann$($ann $M) \subseteq M + B$,
(iii) If S is a finitely generated ideal in R such that $[A : M] \subseteq S$ and ann$($ann $S) \subseteq S + B$, then $R = S +$ ann M.

Proof. By [14], M is finitely generated projective. Let ann $M =$ ann(e) for some idempotent e in R. Then ann$($ann $M) = eR$. Assume M is the smallest element of $\Phi_{A,B}$. Then (i) and (ii) are satisfied. Let S be a finitely generated ideal in R with $[A : M] \subseteq S$ and ann$($ann $S) \subseteq S + B$. Since M is multiplication and $A \subseteq M$, $A = [A : M]M \subseteq MS$. By Theorem 2.2, MS is finitely generated flat ideal in R. By Lemma 2.1 and since ann$($ann $M)$ is a pure ideal,

$$\begin{aligned}\text{ann}(\text{ann}(MS)) &= \text{ann}(\text{ann}\, M) \cap \text{ann}(\text{ann}\, S)\\ &= \text{ann}(\text{ann}\, M)\, \text{ann}(\text{ann}\, S)\\ &\subseteq (M+B)(S+B)\\ &= MS + MB + SB + B^2\\ &\subseteq MS + B.\end{aligned}$$

Hence $MS \in \Phi_{A,B}$. Since M is the smallest element of $\Phi_{A,B}$, $M \subseteq MS \subseteq M$, and hence $M = MS$. Since M is finitely generated multiplication, we infer form [13, Corollary to Theorem 10] that $R = S +$ ann M. Conversely, let (i), (ii) and (iii) be satisfied. Let $X \in \Phi_{A,B}$. Then $A \subseteq X$. Moreover, $A \subseteq M$. So $A \subseteq M \cap X = [X : M]M$, and hence

$$[A : M] \subseteq [[X : M]M : M] = [X : M] + \text{ann}\, M = [X : M].$$

By Theorem 2.2, $[X : M]$ is a finitely generated (hence flat) ideal in R. Since $X +$ ann M is a finitely generated flat ideal (hence multiplication) it follows by [13, Corollary to Theorem 2] that

$$[X : \text{ann}\, M] + [\text{ann}\, M : X] = R,$$

and by Lemma 2.1,

$$[\text{ann}\, X : \text{ann}(\text{ann}\, M)] + [\text{ann}(\text{ann}\, M) : \text{ann}\, X] = R.$$

Hence from Lemma 2 we obtain that

$$\begin{aligned}\text{ann}(\text{ann}[X : M]) &= \text{ann}(\text{ann}(X + \text{ann}\, M))\\ &= \text{ann}(\text{ann}\, X \cap \text{ann}(\text{ann}\, M))\\ &= \text{ann}(\text{ann}\, X) + \text{ann}(\text{ann}(\text{ann}\, M))\\ &= \text{ann}(\text{ann}\, X) + \text{ann}\, M\\ &\subseteq X + \text{ann}\, M + B\\ &\subseteq [X : M] + B.\end{aligned}$$

Hence $[X : M] + \text{ann } M = R$. It follows that $[X : M] = R$, and hence $M \subseteq X$. So M is the smallest element of $\Phi_{A,B}$. This finish the proof of the theorem.

□

References

[1] M. M. Ali, The smallest element of a lattice of ideals in semi-hereditary rings. Comm. in Algebra, 10(2021), 4199-4207.

[2] M. M. Ali and D. J. Smith, Finite and infinite collections of multiplication modules. Beiträge zur Algebra und Geom. 42(2001), 557-573.

[3] M. M. Ali and D. J. Smith, Pure submodules of multiplication modules. Beiträge zur Algebra und Geom. 45(2004), 61-74.

[4] D. D. Anderson, Some remarks on multiplication ideals. Math. Japan. 4(1980), 463-469.

[5] I. Kaplansky, Commutative rings. 1974. Chicago, Chicago Press.

[6] R. Gilmer, Multiplicative ideal theory. Kingston, Queen's 1992.

[7] M. D. Larsen and P. J. Mac Carthy Multiplicative theory of ideals. New York 1971: Academic Press.

[8] G. M. Low and P. F. Smith, Multiplication modules and ideals. Comm. in algebra 19(1990), 4353-4375.

[9] H. Matsumura, Commutative ring theory. 1986, Cambridge: Cambridge university press.

[10] A. G. Naoum, Flat modules and multiplication modules. Period. Math. Hungar. 21(1990), 309-317.

[11] A. G. Naoum and M. A. K. Hasan, On finitely generated projective and flat ideals in commutative rings. Period. Math. Hungar. 16(1985), 251-260.

[12] J. D. Sally, and W. V. Vasconcelos, Flat idelas I. Comm. Algebra, 3(1975), 531-543.

[13] P. F. Smith, Some remarks on multiplication modules. Arch. der Math. 50(1988), 223-235.

[14] W. W. Smith, Projective ideals of finite type. Canad. J. Math. 21(1969), 1057-1061.

Author information

Majid M. Ali, Sultan Qaboos University, College of Science, Department of Mathematics
P.O.Box 36, PC 123 Al Khodh, Muscat, Oman, Oman.
E-mail: `mali@squ.edu.om`

Received : November 5, 2021.
Accepted: February 14, 2021.