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Characterizations of le-semigroups

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Abstract In this paper, first we show that the equivalence relations $\mathcal{B}, \mathcal{Q}, \mathcal{H}$ are finer than the equivalence relations $\mathcal{B}_m^n, \mathcal{Q}_m^n, \mathcal{H}_m^n$ in le-semigroups. Next, we show that on any (m, n)-regular le-semigroup $\mathcal{B}_m^n = \mathcal{B}, \mathcal{Q}_m^n = \mathcal{Q}, \mathcal{H}_m^n = \mathcal{H}$ and $\mathcal{B} = \mathcal{Q} = \mathcal{H} = \mathcal{B}_m^n = \mathcal{Q}_m^n = \mathcal{H}_m^n$. We, then, characterize (m, n)-regular le-semigroups in terms of (m, n)-ideal elements and study 0-minimality of (m, n)-ideal elements in poe-semigroups. Finally $(m, 0)$-simple, $(0, n)$-simple and (m, n)-simple poe-semigroups are characterized and equalities $\mathcal{L} = \mathcal{R} = \mathcal{B} = \mathcal{Q} = \mathcal{H} = {}_m\mathcal{I} = \mathcal{I}_n = \mathcal{B}_m^n = \mathcal{Q}_m^n = \mathcal{H}_m^n = S \times S$ have been shown in (m, n)-simple le-semigroups.

1 Introduction and Preliminaries

As a generalization of the notions of left, right and two-sided ideals, the notion of (m, n)-ideals in semigroups was introduced by Lajos [11]. Thereafter many other authors [2, 3, 17] had studied (m, n)-ideals in ordered semigroups. Kehayopulu [9, 10], characterized poe-semigroups, $\vee$e-semigroups and le-semigroups in terms of left (resp. right) ideal elements, ideal-elements, bi-ideal elements and quasi-ideal elements of respective semigroups. Kehayopulu [8], studied (m, n)-ideal elements and (m, n)-quasi-ideal elements in poe-semigroups and le-semigroups. In the present paper, we study the connection between equivalence relations $\mathcal{L}, \mathcal{R}, \mathcal{B}, \mathcal{Q}, \mathcal{H}$ and ${}_m\mathcal{I}, \mathcal{I}_n, \mathcal{B}_m^n, \mathcal{Q}_m^n, \mathcal{H}_m^n$ and characterize (m, n)-regular le-semigroups and 0-minimality of (m, n)-ideal elements in poe-semigroups. We, then, show that a poe-semigroup S is $(m, 0)$-simple ($(0, n)$-simple, (m, n)-simple) if and only if S does not contain any proper $(m, 0)$-ideal ($(0, n)$-ideal, (m, n)-ideal) and, in any (m, n)-simple le-semigroup, $\mathcal{L} = \mathcal{R} = \mathcal{B} = \mathcal{Q} = \mathcal{H} = {}_m\mathcal{I} = \mathcal{I}_n = \mathcal{B}_m^n = \mathcal{Q}_m^n = \mathcal{H}_m^n = S \times S$.

Definition 1.1. Let S be a non-empty set. The triplet $(S, \cdot, \leq)$ is called a po-semigroup if $(S, \cdot)$ is a semigroup and $(S, \leq)$ is a partially ordered set such that

$$a \leq b \Rightarrow ac \leq bc \text{ and } ca \leq cb$$

for all $a, b, c \in S$. A po-semigroup with a greatest element "e" (i.e. for each $a \in S, a \leq e$) is said to be a poe-semigroup.

Let S be a po-semigroup. For a subset A of S, we denote $(A] = \{x \in S \mid x \leq a$ for some $a \in A\}$. A non-empty subset T of S is said to be a subsemigroup of S if $T^2 \subseteq T$. A non-empty subset A of S is called a left (right) ideal of S if $SA \subseteq A (AS \subseteq A)$ and $(A] \subseteq A$. Further A is called an ideal of S if it is both a left and right ideal of S. For any non-negative integers m, n, a subsemigroup A of S is called an (m, n)-ideal of S if $A^m S A^n \subseteq A$ and $(A] \subseteq A$. For any non-empty subsets A, B of S, it is easy to verify that $A \subseteq (A], ((A](B]] = (AB]$ and $((A]] = (A]$, where $AB = \{ab | a \in A, b \in B\}$.

Let S be a poe-semigroup. An element a is called a *subsemigroup element* if $a^2 \leq a$ and a is called a *left (resp. right) ideal* element of S if $ea \leq a$ (resp. $ae \leq a$). It is called an *ideal element* of S if it is both a left and right ideal element of S. An element a is called a *bi-ideal element* if $aea \leq a$. An element a of S is called an *idempotent element* if $a = a^2$ and a *quasi-ideal element* if $ae \wedge ea$ exists and $ae \wedge ea \leq a$. An element $z \in S$ is called a *zero element* of S if $za = az = z$ and $z \leq a$ for each $a \in S$. The zero element, if exists, is unique. we shall denote it, in whatever follows, by the symbol 0. A poe-semigroup S is called *regular (left regular, right regular)* if $a \leq aea$ $(a \leq ea^2, a \leq a^2 e)$ for each $a \in S$.

In this paper, we have considered the definition of a *bi-ideal* element given by Kehayopulu in [9] which is different from the definition given by Bhuniya and Kumbhakar in [1] and is as follows: a subsemigroup element b of S is said to be a bi-ideal element if $beb \leq b$.

Definition 1.2. A poe-semigroup S is said to be $\vee e$-semigroup if it is an upper semilattice under $\vee$ and

$$c(a \vee b) = ca \vee cb \text{ and } (a \vee b)c = ac \vee bc$$

for all $a, b, c \in S$. A $\vee e$-semigroup which is a lattice is said to be an le-semigroup.

Let S be any le-semigroup. Then it is easy to verify that under the order relation $\leq$ on S

$$a \leq b \Leftrightarrow a \wedge b = a \text{ and } a \vee b = b$$

for each $a, b \in S$.

Example 1.3. [13] Let G be a group and $\mathcal{C}_G$ denotes the set of all congruence relations on G. Define a binary operation $\circ$ on $\mathcal{C}_G$ by

$$\rho \circ \sigma = \{(a, b) \in G \times G \mid (\exists\, c \in G)(a, c) \in \rho \text{ and } (c, b) \in \sigma\}$$

for each $\rho, \sigma \in \mathcal{C}_G$. By [7], $\rho \circ \sigma = \sigma \circ \rho$ for all $\rho, \sigma \in \mathcal{C}_G$. Let $\mathcal{B}_G$ denotes the set of all binary relations on G. As $\mathcal{C}_G \subseteq \mathcal{B}_G$ and $(\mathcal{B}_G, \circ)$ is a semigroup, $(\mathcal{C}_G, \circ)$ is a semigroup. Define a relation $\leq$ on $\mathcal{C}_G$ by

$$\rho \leq \sigma \Leftrightarrow \rho \subseteq \sigma$$

for each $\rho, \sigma \in \mathcal{C}_G$. Clearly $\leq$ is an ordered relation on $\mathcal{C}_G$ such that

$$\rho \leq \sigma \Rightarrow \rho \circ \tau \leq \sigma \circ \tau \text{ and } \tau \circ \rho \leq \tau \circ \sigma$$

for each ρ, σ, τ in $\mathcal{C}_G$. So $\mathcal{C}_G$ is a poe-semigroup with the greatest element $e = G \times G$.

It is well known that $\rho \wedge \sigma = \rho \cap \sigma$ and $\rho \vee \sigma = \rho \circ \sigma$ $\forall \rho, \sigma \in \mathcal{C}_G$. Now for any $\rho, \sigma, \tau \in \mathcal{C}_G$,

$$\begin{aligned} \tau \circ (\rho \vee \sigma) &= \tau \circ (\rho \circ \sigma) \\ &= \tau \circ \tau \circ \rho \circ \sigma \quad (\tau \in \mathcal{C}_G, \tau = \tau \circ \tau) \\ &= \tau \circ \rho \circ \tau \circ \sigma \quad (\rho, \sigma \in \mathcal{C}_G, \rho \circ \sigma = \sigma \circ \rho) \\ &= (\tau \circ \rho) \vee (\tau \circ \sigma). \end{aligned}$$

Similarly $(\rho \vee \sigma) \circ \tau = \rho \circ \tau \vee \sigma \circ \tau$. Hence $\mathcal{C}_G$ is an le-semigroup.

For the study of ideal elements in le-Γ-semigroups, the reader is referred to [4, 5, 6].

Definition 1.4. Green-Kehayopulu relations $\mathcal{L}, \mathcal{R}, \mathcal{H}$ and relations $\mathcal{B}, \mathcal{Q}$ on an le-semigroup defined in [15] and [16] are as follows:

$$\begin{aligned} \mathcal{L} &= \{(x, y) \in S \times S \mid l(x) = l(y)\}; \\ \mathcal{R} &= \{(x, y) \in S \times S \mid r(x) = r(y)\}; \\ \mathcal{H} &= \mathcal{L} \cap \mathcal{R}; \\ \mathcal{B} &= \{(x, y) \in S \times S \mid b(x) = b(y)\}; \\ \mathcal{Q} &= \{(x, y) \in S \times S \mid q(x) = q(y)\}; \end{aligned}$$

where $l(x) = x \vee ex, r(x) = x \vee xe, b(x) = x \vee xex$ and $q(x) = x \vee (xe \wedge ex)$ are the left ideal element, right ideal element, bi-ideal element and quasi-ideal-element of S generated by the element x respectively i.e. least left ideal element, least right ideal element, least bi-ideal element and least quasi-ideal element of S greater than the element x respectively. Clearly all relations defined above are equivalence relations on S.

Let S be a poe-semigroup and m, n be non-negative integers. An element a of S is called an (m,n)-ideal element of S if $a^m e a^n \leq a$ and (m,n)-*quasi-ideal element* if $a^m e \wedge e a^n$ exists and $a^m e \wedge e a^n \leq a$. A poe-semigroup S is called (m,n)-*regular* ($(m,0)$-*regular,* $(0,n)$-*regular*) if $a \leq a^m e a^n$ ($a \leq a^m e, a \leq e a^n$) for each $a \in S$.

Let S be an le-semigroup and m, n be non-negative integers. We denote by $< a >_{<m,n>}$, an (m,n)-ideal element of S, and by $(a)_{<m,n>}$, an (m,n)-quasi-ideal element of S, generated by the element a of S respectively i.e. the least (m,n)-ideal element and the least (m,n)-quasi-ideal element of S greater than the element a. By [8]

$$\begin{aligned} < a >_{<m,n>} &= a \vee a^m e a^n \\ (a)_{<m,n>} &= a \vee (a^m e \wedge e a^n). \end{aligned}$$

Thus $a \in S$ is an (m,n)-ideal ((m,n) quasi-ideal) element of S if and only if $< a >_{<m,n>} = a$ ($(a)_{<m,n>} = a$).

Definition 1.5. Let S be an le-semigroup and m, n be positive integers. Define relations ${}_m\mathcal{I}$, $\mathcal{I}_n$, $\mathcal{H}_m^n$, $\mathcal{B}_m^n$ and $\mathcal{Q}_m^n$ [12] on S as follows:

$$\begin{aligned} {}_m\mathcal{I} &= \{(a,b) \in S \times S \mid\ < a >_{<m,0>} = < b >_{<m,0>}\}; \\ \mathcal{I}_n &= \{(a,b) \in S \times S \mid\ < a >_{<0,n>} = < b >_{<0,n>}\}; \\ \mathcal{H}_m^n &= {}_m\mathcal{I} \cap \mathcal{I}_n; \\ \mathcal{B}_m^n &= \{(a,b) \in S \times S \mid\ < a >_{<m,n>} = < b >_{<m,n>}\}; \\ \mathcal{Q}_m^n &= \{(a,b) \in S \times S \mid (a)_{<m,n>} = (b)_{<m,n>}\}. \end{aligned}$$

Clearly all the relations defined above are equivalence relations on S. In particular for $m = 1 = n$, ${}_1\mathcal{I} = \mathcal{R}, \mathcal{I}_1 = \mathcal{L}, \mathcal{H}_1^1 = \mathcal{H}, \mathcal{B}_1^1 = \mathcal{B}$ and $\mathcal{Q}_1^1 = \mathcal{Q}$ respectively.

2 (m,n)-regular le-semigroups

Theorem 2.1. *Let S be an le-semigroup and m, n be positive integers. Then the following conditions hold;*

(1) $\mathcal{B}_m^n \subseteq \mathcal{B}$;

(2) $\mathcal{Q}_m^n \subseteq \mathcal{Q}$;

(2) $\mathcal{H}_m^n \subseteq \mathcal{H}$.

Proof. (1). Let $(x,y) \in \mathcal{B}_m^n$. Then $< x >_{<m,n>} = < y >_{<m,n>}$ i.e. $x \vee x^m e x^n = y \vee y^m e y^n$ implies $x \leq y \vee y^m e y^n$ and $y \leq x \vee x^m e x^n$. So, $x \leq y \vee y^m e y^n \leq y \vee yey$ and $y \leq x \vee x^m e x^n \leq x \vee xex$. Thus, we have

$$\begin{aligned} xex &\leq (y \vee yey)e(y \vee yey) \\ &= (ye \vee yeye)(y \vee yey) \\ &= yey \vee yeyey \vee yeyey \vee yeyeyey \\ &= yey. \end{aligned}$$

Similarly $yey \leq xex$. As $x \leq y \vee yey$. Therefore

$$\begin{aligned} b(x) &= x \vee xex \\ &\leq y \vee yey \vee \vee xex && (\text{as } x \leq y \vee yey) \\ &\leq y \vee yey \vee yey && (\text{as } xex \leq yey) \\ &= b(y). \end{aligned}$$

Similarly $b(y) \leq b(x)$. Hence $b(x) = b(y)$ i.e. $(x,y) \in \mathcal{B}$.
(2). Let $(x,y) \in \mathcal{Q}_m^n$. Then $(x)_{<m,n>} = (y)_{<m,n>}$ i.e. $x \vee (x^m e \wedge e x^n) = y \vee (y^m e \wedge e y^n)$. This

implies that $x \leq y \vee (y^m e \wedge ey^n)$ and $y \leq x \vee (x^m e \wedge ex^n)$. So $x \leq y \vee (y^m e \wedge ey^n) \leq y \vee (ye \wedge ey)$ and $y \leq x \vee (x^m e \wedge ex^n) \leq x \vee (xe \wedge ex)$. Thus, we have

$$\begin{aligned} xe &\leq (y \vee (ye \wedge ey))e \\ &= ye \vee (ye \wedge ey)e \\ &\leq ye. \end{aligned}$$

Similarly $ye \leq xe$. Therefore $xe = ye$, similarly $ex = ey$. As $x \leq y \vee (ye \wedge ey)$, we have

$$\begin{aligned} q(x) &= x \vee (xe \wedge ex) \\ &\leq y \vee (ye \wedge ey) \vee (xe \wedge ex) \\ &= y \vee (ye \wedge ey) \qquad \text{(as } xe = ye, ex = ey) \\ &= q(y) \end{aligned}$$

Similarly $q(y) \leq q(x)$. Hence $q(x) = q(y)$ i.e. $(x, y) \in \mathcal{Q}$.
(3). Let $(x, y) \in \mathcal{H}_m^n$. Then $< x >_{<m,0>} = < y >_{<m,0>}$ and $< x >_{<0,n>} = < y >_{<0,n>}$ i.e. $x \vee x^m e = y \vee y^m e$ and $x \vee ex^n = y \vee ey^n$. This implies that $x \leq y \vee y^m e, y \leq x \vee x^m e$ and $x \leq y \vee ey^n$, $y \leq x \vee ex^n$. So $x \leq y \vee y^m e \leq y \vee ye, y \leq x \vee x^m e \leq x \vee xe$ and $x \leq y \vee ey^n \leq y \vee ey$, $y \leq x \vee ex^n \leq x \vee ex$. Now

$$\begin{aligned} xe &\leq (y \vee ye)e \\ &= ye \vee ye^2 \\ &\leq ye. \end{aligned}$$

Similarly $ye \leq xe$. Therefore $xe = ye$, similarly $ex = ey$. As $x \leq y \vee ey$, we have

$$\begin{aligned} l(x) &= x \vee ex \\ &\leq y \vee ey \vee ex \\ &= y \vee ey \qquad \text{(as } ex = ey) \\ &= l(y). \end{aligned}$$

Similarly $l(y) \leq l(x)$. Therefore $l(x) = l(y)$. Similarly $r(x) = r(y)$. So $(x, y) \in \mathcal{H}$, as required. □

Lemma 2.2. *[16, 14] Let S be an le-semigroup. Then $\mathcal{B} \subseteq \mathcal{H}$ ($\mathcal{Q} \subseteq \mathcal{H}$).*

Lemma 2.3. *[16] Let S be an le-semigroup and $x, y \in S$ are $\mathcal{H}$-related. Then $xe = ye, ex = ey$ and $xex = yey$.*

Theorem 2.4. *Let S be an le-semigroup and m, n be positive integers. If S is (m, n)-regular, then:*

(1) $\mathcal{B}_m^n = \mathcal{B}$;

(2) $\mathcal{Q}_m^n = \mathcal{Q}$;

(3) $\mathcal{H}_m^n = \mathcal{H}$.

Proof. (1). Let S be an (m, n)-regular le-semigroup and $(x, y) \in \mathcal{B}$. Then $b(x) = b(y)$ i.e. $x \leq y \vee yey$ and $y \leq x \vee xex$. As S is (m, n)-regular, $yey \leq (y^m ey^n)e(y^m ey^n) \leq y^m ey^n \leq yey$ and $xex \leq (x^m ex^n)e(x^m ex^n) \leq x^m ex^n \leq xex$. So $yey = y^m ey^n$ and $xex = x^m ex^n$. Since S is (m, n)-regular, $< x >_{<m,n>} = x^m ex^n$ and $< y >_{<m,n>} = y^m ey^n$. As, by Lemmas 2.2 and 2.3, $xex = yey$, it follows that $< x >_{<m,n>} = < y >_{<m,n>}$. Thus $(x, y) \in \mathcal{B}_m^n$. Hence, by Theorem 2.1, $\mathcal{B}_m^n = \mathcal{B}$.
(2). Let S be an (m, n)-regular le-semigroup and $(x, y) \in \mathcal{Q}$. Then $q(x) = q(y)$ i.e. $x \leq y \vee (ye \wedge ey)$ and $y \leq x \vee (xe \wedge ex)$. As S is (m, n)-regular, $ye \wedge ey \leq (y^m ey^n)e \wedge e(y^m ey^n) \leq y^m e \wedge ey^n \leq ye \wedge ey$ and $xe \wedge ex \leq (x^m ex^n)e \wedge e(x^m ex^n) \leq x^m e \wedge ex^n \leq xe \wedge ex$. So,

$ye \wedge ey = y^m e \wedge ey^n$ and $xe \wedge ex = x^m e \wedge ex^n$. Since S is (m,n)-regular, $(x)_{<m,n>} = x^m e \wedge ex^n$ and $(y)_{<m,n>} = y^m e \wedge ey^n$. Again, as by Lemmas 2.2 and 2.3, $xe = ye, ex = ey$, it follows that $(x)_{<m,n>} = (y)_{<m,n>}$. Thus $(x,y) \in \mathcal{Q}_m^n$. Hence, by Theorem 2.1, $\mathcal{Q}_m^n = \mathcal{Q}$.
(3). Let S be an (m,n)-regular le-semigroup and $(x,y) \in \mathcal{H}$. Then $l(x) = l(y)$ and $r(x) = r(y)$ i.e. $x \leq y \vee ey, y \leq x \vee ex$ and $x \leq y \vee ye, y \leq x \vee xe$. As S is (m,n)-regular, $ye \leq (y^m ey^n)e \leq y^m e \leq ye$ and $xe \leq (x^m ex^n)e \leq x^m e \leq xe$. So $ye = y^m e$ and $xe = x^m e$. Also, as S is (m,n)-regular, $< x >_{<m,0>} = x^m e$ and $< y >_{<m,0>} = y^m e$. As by Lemma 2.3, $ex = ey$, we have $< x >_{<m,0>} = < y >_{<m,0>}$. Therefore $(x,y) \in {}_m\mathcal{I}$. Similarly $(x,y) \in \mathcal{I}_n$. Thus $(x,y) \in \mathcal{H}_m^n$. Hence, by Theorem 2.1, $\mathcal{H}_m^n = \mathcal{H}$. □

Lemma 2.5. *[16, 14] In a regular le-semigroup $\mathcal{B} = \mathcal{H}$ ($\mathcal{Q} = \mathcal{H}$).*

Corollary 2.6. *Let S be an le-semigroup and m,n be positive integers. If S is (m,n)-regular, then $\mathcal{B} = \mathcal{Q} = \mathcal{H} = \mathcal{B}_m^n = \mathcal{Q}_m^n = \mathcal{H}_m^n$.*

Proof. Let S be an (m,n)-regular le-semigroup. Then S is regular because $a \leq a^m ea^n \leq aea$ for each $a \in S$. Thus, by Theorem 2.4 and Lemma 2.5, $\mathcal{B} = \mathcal{Q} = \mathcal{H} = \mathcal{B}_m^n = \mathcal{Q}_m^n = \mathcal{H}_m^n$, as required. □

Theorem 2.7. *Let S be an le-semigroup and m,n be positive integers with either $m \geq 2$ or $n \geq 2$. Then the following are equivalent:*

(1) S is (m,n)-regular;

(2) $a^2 = a \ \forall a \in I_{<m,n>}$;

(3) $a_1 \wedge a_2 = a_1a_2 \wedge a_2a_1 \ \forall a_1, a_2 \in I_{<m,n>}$;

(4) $a_1^2 \leq a \Rightarrow a_1 \leq a \ \forall a_1, a \in I_{<m,n>}$,

where $I_{<m,n>}$ denotes the set of all (m,n)-ideal elements of S.

Proof. $(1) \Rightarrow (2)$. Let a be an (m,n)-ideal element of S. Then $a = a^m ea^n$. Now

$$a^2 = (a^m ea^n)(a^m ea^n) \leq (a^m ea^n) = a$$

and

$$\begin{aligned} a &= a^m ea^n = (a^m ea^n)^m ea^n \\ &= \underbrace{(a^m ea^n)\dots(a^m ea^n)}_{m\text{-times}} ea^n \\ &\leq (a^m ea^n)(a^m ea^n) = aa. \end{aligned}$$

Hence $a^2 = a$ ($\forall a \in I_{<m,n>}$).
$(2) \Rightarrow (3)$. Let $a_1, a_2 \in I_{<m,n>}$. As $a_1 \wedge a_2 \in I_{<m,n>}$, by hypothesis, $a_1 \wedge a_2 = (a_1 \wedge a_2)^2 = (a_1 \wedge a_2)(a_1 \wedge a_2) \leq a_1a_2$. Similarly $a_1 \wedge a_2 \leq a_2a_1$. Therefore $a_1 \wedge a_2 \leq a_1a_2 \wedge a_2a_1$. Now

$$\begin{aligned} (a_1a_2)^m e(a_1a_2)^n &= \underbrace{(a_1a_2)(a_1a_2)\cdots(a_1a_2)}_{m\text{-times}} e \underbrace{(a_1a_2)(a_1a_2)\cdots(a_1a_2)}_{n\text{-times}} \\ &= a_1a_2((a_1a_2)^{m-1}ea_1(a_2a_1)^{n-1})a_2 \\ &\leq a_1a_2ea_2 \\ &= a_1a_2^m ea_2^n \\ &\leq a_1a_2 \end{aligned}$$

Therefore $a_1a_2 \in I_{<m,n>}$. Similarly $a_2a_1 \in I_{<m,n>}$. So $a_1a_2 \wedge a_1a_2 \in I_{<m,n>}$. Now, by (2), $a_1a_2 \wedge a_2a_1 = (a_1a_2 \wedge a_2a_1)^2 = (a_1a_2 \wedge a_2a_1)(a_1a_2 \wedge a_2a_1) \leq a_1a_2a_2a_1 \leq a_1^m ea_1^n \leq a_1$. Similarly $a_1a_2 \wedge a_2a_1 \leq a_2$. Therefore $a_1a_2 \wedge a_2a_1 \leq a_1 \wedge a_2$. Hence $a_1 \wedge a_2 = a_1a_2 \wedge a_2a_1$.
$(3) \Rightarrow (4)$. Take any $a, a_1 \in I_{<m,n>}$ such that $a_1^2 \leq a$. By hypothesis, $a_1 = a_1 \wedge a_1 = a_1a_1 \wedge a_1a_1 = a_1^2$. Therefore $a_1 \leq a$.

(4) $\Rightarrow$ (1). Let $a \in S$. As $< a >_{<m,n>} \in I_{<m,n>}$ and $(< a >_{<m,n>})^2 \leq (< a >_{<m,n>})^2$, $< a >_{<m,n>} \leq (< a >_{<m,n>})^2$. Therefore, we have

$$\begin{aligned} < a >_{<m,n>} &\leq (< a >_{<m,n>})^2 \\ &\leq (< a >_{<m,n>})^3 \\ &\vdots \\ &\leq (< a >_{<m,n>})^{m+n+1} \\ &\leq (< a >_{<m,n>})^m e (< a >_{<m,n>})^n \\ &= a^m e a^n. \end{aligned}$$

Since $a \leq < a >_{<m,n>}$, we have $a \leq a^m e a^n$. Hence S is (m, n)-regular. □

Theorem 2.8. *Let S be an le-semigroup and m, n be positive integers with either $m \geq 2$ or $n \geq 2$. Then for each $a, a_1, a_2 \in I_{<m,n>}$ such that $a_1 a_2 \wedge a_2 a_1 \leq a$ implies that either $a_1 \leq a$ or $a_2 \leq a$ if and only if S is (m, n)-regular and the set $I_{<m,n>}$ is a chain.*

Proof. Let S be an le-semigroup and for each $a, a_1, a_2 \in I_{<m,n>}$ such that $a_1 a_2 \wedge a_2 a_1 \leq a$ implies that either $a_1 \leq a$ or $a_2 \leq a$. Therefore $a_1^2 \leq a$ implies $a_1 \leq a$. Thus, by Theorem 2.7, S is (m, n)-regular. Next, we show that the set $I_{<m,n>}$ is a chain. Let $a_1, a_2 \in I_{<m,n>}$. Again, by Theorem 2.7, $a_1 \wedge a_2 = a_1 a_2 \wedge a_2 a_1$. Since $a_1 \wedge a_2 \in I_{<m,n>}$, by hypothesis, $a_1 \leq a_1 \wedge a_2$ or $a_2 \leq a_1 \wedge a_2$. If $a_1 \leq a_1 \wedge a_2$, then $a_1 \leq a_2$ and if $a_2 \leq a_1 \wedge a_2$, then $a_2 \leq a_1$. So, in either case, the set $I_{<m,n>}$ is a chain.

Conversely take any $a, a_1, a_2 \in I_{<m,n>}$ such that $a_1 a_2 \wedge a_2 a_1 \leq a$. As S is (m, n)-regular, by Theorem 2.7, we have $a_1 \wedge a_2 = a_1 a_2 \wedge a_2 a_1 \leq a$. Now, by hypothesis, either $a_1 \leq a_2$ or $a_2 \leq a_1$. So either $a_1 \wedge a_2 = a_1$ or $a_2 \wedge a_1 = a_2$. Hence either $a_1 \leq a$ or $a_2 \leq a$, as required. □

Theorem 2.9. *Let S be an le-semigroup, m, n be positive integers with either $m \geq 2$ or $n \geq 2$ and the set $I_{<m,n>}$ is a chain. Then S is (m, n)-regular if and only if $a_1 a_2 \leq a$ implies that either $a_1 \leq a$ or $a_2 \leq a$ for each $a, a_1, a_2 \in I_{<m,n>}$.*

Proof. Suppose that S is (m, n)-regular. Let a, a_1 and a_2 be any elements of $I_{<m,n>}$ such that $a_1 a_2 \leq a$. Since $I_{<m,n>}$ is a chain, either $a_1 \leq a_2$ or $a_2 \leq a_1$. If $a_1 \leq a_2$, then $a_1^2 \leq a_1 a_2 \leq a$. So, by Theorem 2.7, $a_1 \leq a$. Similarly if $a_2 \leq a_1$, then $a_2 \leq a$.

Converse follows by Theorem 2.7, as $a_1^2 \leq a$ implies $a_1 \leq a$ for each $a, a_1 \in I_{<m,n>}$. □

Theorem 2.10. *[12] Let S be an le-semigroup and m,n be non-negative integers. Then S is (m, n)-regular if and only if $a \wedge b = a^m b^n$ for each $(m, 0)$-ideal element a and for each $(0, n)$-ideal element b of S.*

Theorem 2.11. *Let S be an le-semigroup and m, n be positive integers. If S is (m,n)-regular, then the (m,0) and the (0,n)-ideal elements of S are idempotents and for each (m,0)-ideal element x and each (0,n)-ideal element y of S, xy is an (m,n)-quasi-ideal element of S.*

Proof. Since x is $(m, 0)$-ideal element and y is $(0, n)$-ideal element, $x^m e \leq x,\ e y^n \leq y$. As S is (m, n)-regular,

$$\begin{aligned} x &\leq x^m e x^n = x^m e x^{n-1} x \\ &\leq x^m e x^{n-1} x^m e x^n \leq x^m e x^m e \\ &\leq xx = x^2 \end{aligned}$$

and

$$x^2 \leq x^m e x^n x^m e x^n \leq x^m e \leq x.$$

Therefore $x^2 = x$. Similarly $y^2 = y$. Since $(x \wedge y)^m e \wedge e(x \wedge y)^n \leq x^m e \wedge e y^n \leq x \wedge y$, $x \wedge y$ is an (m, n)-quasi-ideal of S. Therefore, by Theorem 2.10, $x^m y^n \in Q_{<m,n>}$. As $x^2 = x$ and $y^2 = y$, it follows that xy is an (m, n)-quasi-ideal element □

Definition 2.12. A non-zero (m, n)-ideal element a of po-semigroup S with 0 is said to be 0-minimal if for each (m, n)-ideal element b of S, $b \leq a$ implies $b = 0$ or $b = a$.

Proposition 2.13. *Let S be a poe-semigroup with* 0*. Let m, n be positive integers and a be any subsemigroup* 0*-minimal* (m, n)*-ideal element of S. If* $a^2 \neq 0$*, then a is a* 0*-minimal bi-ideal element of S.*

Proof. Suppose there exists a bi-ideal element b of S such that $0 \neq b \leq a$. As b is a bi-ideal element of S, b is an (m, n)-ideal element. So, by 0-minimality of (m, n)-ideal element a of S, we have $b = a$. Hence a is a bi-ideal element which is a 0-minimal bi-ideal element, as required.

Otherwise suppose that there does not exist any bi-ideal element c of S such that $0 \neq c \leq a$. As a is an (m, n)-ideal element, a^2 is also an (m, n)-ideal element of S. Since $0 \neq a^2 \leq a$ and a is a 0-minimal (m, n)-ideal element, $a^2 = a$. Now $aea = a^m ea^n \leq a$. Hence, again, a is a 0-minimal bi-ideal element of S, as required. □

Theorem 2.14. *Let S be an* (m, n)*-regular poe-semigroup with 0 and m, n be positive integers. If x is a 0-minimal* $(m, 0)$*-ideal element and y be a* $(0, n)$*-ideal element of S such that* $x \wedge y$ *exists and* $xy \leq x \wedge y$*, then either* $xy = 0$ *or* xy *is a 0-minimal* (m, n)*-ideal element of S.*

Proof. Let x be a 0-minimal $(m, 0)$-ideal element and y be a $(0, n)$-ideal element of S such that $x \wedge y$ exists and $xy \leq x \wedge y$. Let $xy \neq 0$. Now we have to show that xy is a 0-minimal (m, n)-ideal element of S. For this we first show that xy is an (m, n)-ideal element of S. Now

$$\begin{aligned} (xy)^m e(xy)^n &\leq x^m ey^n && (\text{as } xy \leq x \wedge y) \\ &\leq xy^n && (\text{as } x^m e \leq x) \\ &= xy && (\text{by Theorem 2.11}). \end{aligned}$$

So xy is an (m, n)-ideal element of S. To show next that xy is a 0-minimal (m, n)-ideal element of S, take any (m, n)-ideal element z of S such that $0 < z \leq xy$. As $xy \leq x \wedge y$, $z \leq x$ and $z \leq y$. Since $z \leq z^m ez^n$, we have $z^m e \neq 0$ and $ez^n \neq 0$. As $z^m e \leq x^m e \leq x$, so, by the 0-minimality of $(m, 0)$-ideal element x, $z^m e = x$. Similarly $ez^n = y$. Therefore $z \leq xy = z^m eez^n \leq z^m ez^n \leq z$. Thus $z = xy$. Hence xy is a 0-minimal (m, n)-ideal element of S. □

Theorem 2.15. *Let S be an* (m, n)*-regular poe-semigroup with 0 and m, n be positive integers. If x is a 0-minimal* $(m, 0)$*-ideal element and y be a* $(0, n)$*-ideal element of S such that* $x \wedge y$ *exists and* $x \wedge y \neq 0$*, then* $x \wedge y$ *is a 0-minimal* (m, n)*-ideal element of S.*

Proof. Let x be a 0-minimal $(m, 0)$-ideal element and y be a $(0, n)$-ideal element of S such that $x \wedge y$ exists and $x \wedge y \neq 0$. Then

$$(x \wedge y)^m e(x \wedge y)^n \leq x^m ey^n \leq xy^n \leq ey^n \leq y.$$

Similarly $(x \wedge y)^m e(x \wedge y)^n \leq y$. Therefore $(x \wedge y)^m e(x \wedge y)^n \leq x \wedge y$. So $x \wedge y$ is an (m, n)-ideal element of S. The rest of the proof is similar to the proof of Theorem 2.14. □

3 Characterization of $(m, 0)$-simple, $(0, n)$-simple and (m, n)-simple $le(poe)$-semigroups

Definition 3.1. Let S be a poe-semigroup and m, n be non-negative integers. Then S is said to be an (m, n)-simple if there does not exists any (m, n)-ideal element in S except the greatest element e. In particular for $m = 1 = n$, S is said to be B-simple i.e. there does not exists any bi-ideal element in S except the greatest element e.

Definition 3.2. A poe-semigroup S is said to be left-simple (right-simple, quasi-simple) if there does not exists any left ideal element (right ideal element, quasi-ideal element) in S except the greatest element e.

Proposition 3.3. *A poe-semigroup S is (m,0)-simple if and only if S does not contain any proper (m,0)-ideal.*

Proof. Suppose S is $(m, 0)$-simple and let A be any $(m, 0)$-ideal of S. Take any $a \in S$ and $b \in A$. Then $b^m ae$ is an $(m, 0)$-ideal element of S. By hypothesis, $b^m ae = e$. So $(b^m ae] = (e]$. Now $e \leq b^m ae = b^m(ae) \in A^m S \subseteq A$. Since A is an $(m, 0)$-ideal, $e \in A$. As $S \ni a \leq e \in A,\ a \in A$. Therefore $A = S$.

Conversely, take any $(m, 0)$-ideal element a of S. Since $(a^m e]$ is an $(m, 0)$-ideal of S, by hypothesis, $(a^m e] = S$. As $e \in S,\ e \leq a^m e \leq a$. Therefore $e = a$. Hence S is $(m, 0)$-simple. □

Dually we may prove the following.

Proposition 3.4. *A poe-semigroup S is (0,n)-simple if and only if S does not contain any proper (0,n)-ideal.*

Theorem 3.5. *Let S be a poe-semigroup and m,n be non-negative integers. Then S is (m, n)-simple if and only if S is both $(m, 0)$-simple and $(0, n)$-simple.*

Proof. Assume that S is (m, n)-simple. If a is any $(m, 0)$-ideal element of S, as $a^m ea^n \leq a^m e \leq a$, we have that a is an (m, n)-ideal element of S. Hence $a = e$ i.e. S is $(m, 0)$-simple. Similarly S is $(0, n)$-simple

Conversely, we assume that S is both $(m, 0)$ and $(0, n)$-simple. Let a be any (m, n)-ideal element of S. Since $a^m e$ and ea^n are $(m, 0)$ and $(0, n)$-ideal elements of S respectively, by assumption $a^m e = e$ and $ea^n = e$. Therefore $a^m ea^n = ea^n = e$. As $a^m ea^n \leq a$, we have $e \leq a$. Hence S is (m, n)-simple. □

Corollary 3.6. *A poe-semigroup S is (m, n)-simple if and only if S does not contain any proper (m, n)-ideal.*

Proof. Suppose S is (m, n)-simple and A be any (m, n)-ideal of S. By Theorem 3.5, S is both $(m, 0)$-simple and $(0, n)$-simple. Therefore, by Propositions 3.3 and 3.4, S does not contain any proper $(m, 0)$ and $(0, n)$-ideals. As $(A^m S](A^m S] \subseteq (A^m S]$, $(A^m S]^m S \subseteq (A^m S]$ and $((A^m S]] \subseteq (A^m S]$, $(A^m S]$ is an $(m, 0)$-ideal of S. Similarly $(SA^n]$ is an $(0, n)$-ideal of S. Therefore $(A^m S] = S$ and $(SA^n] = S$ which implies that $(A^m SA^n] = ((A^m S](A^n]] = (S(A^n]] = (SA^n] = S$. Since A is an (m, n)-ideal of S, $A^m SA^n \subseteq A$. Therefore $(A^m SA^n] \subseteq (A] \subseteq A$. So $S \subseteq A$. Hence S does not contain any proper (m, n)-ideal.

Conversely, take any (m, n)-ideal element a of S. Since $(a^m ea^n]$ is an (m, n)-ideal of S, by hypothesis, $(a^m ea^n] = S$. As $e \in S,\ e \leq a^m ea^n \leq a$. Therefore $e = a$. Hence S is (m, n)-simple. □

Lemma 3.7. *Let S be a poe-semigroup and m,n be positive integers. Then the following hold:*

(1) If S is $(m, 0)$-simple, then S is right-simple;

(2) If S is $(0, n)$-simple, then S is left-simple;

(3) If S is (m, n)-simple, then S is B-simple;

(4) If S is (m, n)-quasi-simple, then S is quasi-simple.

Proof. (1). Let S be an $(m, 0)$-simple poe-semigroup and let a be any right ideal element of S. As $a^m e \leq ae \leq a$, a is an $(m, 0)$-ideal element of S . Since S is $(m, 0)$-simple, $a = e$. Hence S is right-simple.
(2). Proof is dual to the proof of Part (1).
(3). Let S be an (m, n)-simple poe-semigroup and let b be any bi-ideal element of S. Since $b^m eb^n \leq beb \leq b$, b is an (m, n)-ideal element of S. As S is (m, n)-simple, $b = e$. Hence S is B-simple.
(4). Let S be an (m, n)-quasi-simple poe-semigroup and let q be any quasi-ideal element of S. As $q^m e \wedge eq^n \leq qe \wedge eq \leq q$, q is an (m, n)-ideal element of S. As S be a (m, n)-quasi-simple, $q = e$. Hence S is quasi-simple. □

Proposition 3.8. *Let S be an le-semigroup and m,n be positive integers. Then S is (m, n)-simple if and only if S is (m,n)-quasi-simple.*

Proof. Let S be (m,n)-simple and q be any (m,n)-quasi ideal element of S. As $q^m eq^n \leq q^m e \wedge eq^n \leq q$, q is an (m,n)-ideal element of S. Since S is (m,n)-simple, so $q = e$ i.e. S is (m,n)-quasi-simple.

Conversely assume that S is (m,n)-quasi-simple. As each $(m,0)$-ideal element (resp. each $(0,n)$-ideal element) is an (m,n)-quasi-ideal element, S is both $(m,0)$-simple and $(0,n)$-simple. Thus, by Theorem 3.5, S is (m,n)-simple. □

Theorem 3.9. *Let S be an le-semigroup and m,n be positive integers. If S is (m,n)-simple, then*

$$_m\mathcal{I} = \mathcal{I}_n = \mathcal{B}_m^n = \mathcal{Q}_m^n = \mathcal{H}_m^n = S \times S.$$

Proof. Let S be (m,n)-simple. So, by Theorem 3.5, S is $(m,0)$-simple. To show that $_m\mathcal{I} = S \times S$, take any $a, b \in S$. Then $< a >_{<m,0>}$ and $< b >_{<m,0>}$ are $(m,0)$-ideal elements of S. As S is $(m,0)$-simple, $< a >_{<m,0>} = e$ and $< b >_{<m,0>} = e$. Thus $< a >_{<m,0>} = < b >_{<m,0>}$ i.e. $(a,b) \in {_m\mathcal{I}}$. Hence $_m\mathcal{I} = S \times S$.

Dually we may show that $\mathcal{I}_n = S \times S$.

Next we show that $\mathcal{B}_m^n = S \times S$. For this, take any $a, b \in S$. Then $< a >_{<m,n>}$ and $< b >_{<m,n>}$ are (m,n)-ideal elements of S. As S is (m,n)-simple, $< a >_{<m,n>} = e$ and $< b >_{<m,n>} = e$. Thus $< a >_{<m,n>} = < b >_{<m,n>}$ i.e. $(a,b) \in \mathcal{B}_m^n$. Hence $\mathcal{B}_m^n = S \times S$.

To show next that $\mathcal{Q}_m^n = S \times S$, take any $a, b \in S$. Since S is (m,n)-simple, by Proposition 3.8, S is also (m,n)-quasi-simple. As $(a)_{<m,n>}$ and $(b)_{<m,n>}$ are (m,n)-quasi-ideal elements of S, $(a)_{<m,n>} = e$ and $(b)_{<m,n>} = e$. Thus $(a)_{<m,n>} = (b)_{<m,n>}$ i.e. $(a,b) \in \mathcal{Q}_m^n$. Hence $\mathcal{Q}_m^n = S \times S$.

Finally to show that $\mathcal{H}_m^n = S \times S$, as $\mathcal{H}_m^n = {_m\mathcal{I}} \cap \mathcal{I}_n$, by the above we have $\mathcal{H}_m^n = S \times S$. □

Corollary 3.10. *In any B-simple le-semigroup, $\mathcal{L} = \mathcal{R} = \mathcal{B} = \mathcal{Q} = \mathcal{H} = S \times S$.*

Corollary 3.11. *Let S be an le-semigroup and m, n be positive integers. If S is (m,n)-simple, then $\mathcal{L} = \mathcal{R} = \mathcal{B} = \mathcal{Q} = \mathcal{H} = {_m\mathcal{I}} = \mathcal{I}_n = \mathcal{B}_m^n = \mathcal{Q}_m^n = \mathcal{H}_m^n = S \times S$.*

Proof. Let S be an (m,n)-simple le-semigroup. Then by Lemma 3.7, S is B-simple. Hence, by Theorem 3.9 and Corollary 3.10, we get the required result. □

Proposition 3.12. *Let S be an le-semigroup and m, n be non-negative integers. If S is (m,n)-simple, then it is (m,n)-regular.*

Proof. If $m = n = 0$, then the statement holds true. If $m \neq 0, n = 0$, then we have to show S is $(m,0)$-regular. For this, take any $a \in S$. As $a^m e$ is an $(m,0)$-ideal element, by hypothesis $a^m e = e$. so $a \leq a^m e$. Dually we may show that the statement is also true when $m = 0, n \neq 0$. So assume finally that $m \neq 0 \neq n$ and take any $a \in S$. Since $a^m ea^n$ is an (m,n)-ideal element of S, by hypothesis $a^m ea^n = e$. Thus $a \leq a^m ea^n$. Hence S is (m,n)-regular. □

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