Palestine Journal of Mathematics

Vol. 11(4)(2022) , 258–266

On ϕ_P-pseudo valuation rings

Abdelhaq El Khalfi, Hwankoo Kim, Najib Mahdou and Mohammed Tamekkante
(Corresponding Author: Hwankoo Kim)

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 13B99; Secondary 13A15, 13G05, 13B21.

Keywords and phrases: amalgamated duplication, ϕ_P-chained ring, ϕ_P-pseudo valuation ring, ϕ_P-strongly prime ideal, trivial ring extension.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (2021R1I1A3047469).

Abstract The purpose of this paper is to introduce a new class of rings that is closely related to the class of pseudo valuation domains. Let R be a commutative ring and P the minimal divided prime ideal of R consisting entirely of zero divisors. Consider the natural map ϕ_P from the total quotient ring $\mathrm{T}(R)$ to R_P. A prime ideal Q of R is said to be ϕ_P-strongly prime if $xy \in \phi_P(Q)$, $x, y \in R_P$ implies that either $x \in \phi_P(Q)$ or $y \in \phi_P(Q)$. If each prime ideal of R is a ϕ_P-strongly prime ideal, then we say that R is a ϕ_P-pseudo valuation ring (ϕ_P-PVR). In this article, we show that ϕ_P-PVRs enjoy analogs of many properties of PVDs and ϕ-PVRs. Also, we introduce the notion of ϕ_P-chained rings and we study the transfer of the properties of being a ϕ_P-PVR and a ϕ_P-chained ring between a ring A and a trivial ring extension $A \propto E$ (resp., an amalgamated duplication of A along an ideal I). Several examples which delineate the concepts and results are provided.

1 Introduction

Throughout this paper, all rings are commutative with identity and all modules are unital. If R is a ring, we denote respectively by $\mathrm{Nil}(R)$, $\mathrm{T}(R)$, and $\mathrm{Z}(R)$ the ideal of all nilpotent elements of R, the total quotient ring of R, and the set of all zero-divisors of R. If I is an ideal of R, then set $(I : I) := \{x \in \mathrm{T}(R) \mid xI \subseteq I\}$. In [17], Hedstrom and Houston introduced the class of pseudo-valuation domains (in short, PVDs), which is closely related to the class of valuation domains. A domain R with quotient field K is called a PVD if, whenever P is a prime ideal of R and $xy \in P$ with $x \in K$ and $y \in K$, either $x \in P$ or $y \in P$. Any valuation domain is a PVD [17, Proposition 1.1]; and any PVD is local [17, Corollary 1.3]. Additional information about PVDs can be found in the interesting survey article [2]. In [9], D. F. Anderson, Badawi and Dobbs generalized the study of PVDs to the context of arbitrary rings (possibly with nontrivial zero-divisors) as follows. A prime ideal P of a ring R is called a *strongly prime ideal* (*of* R) if aP and bR are comparable (under inclusion) for all $a, b \in R$; a ring R is called a *pseudo-valuation ring* (in short, a *PVR*) if each prime ideal of R is a strongly prime ideal of R. Any chained ring is a PVR [9, Corollary 4]; the prime ideals of any PVR are linearly ordered by inclusion (whence any PVR must be local) [9, Lemma 1]. A ring R is a PVR if and only if it is local and its maximal ideal is a strongly prime ideal of R [9, Theorem 2]. Recall from [13] and [8] that a prime ideal P of R is said to be *divided* if it is comparable to every ideal of R. A ring R is called a *divided ring* if every prime ideal of R is divided. Recently, Badawi in [3, 4, 5, 6, 7] has studied the class of ϕ-rings. A commutative ring R is called a ϕ-*ring* if $\mathrm{Nil}(R)$ is a divided prime ideal of R. In [4], the author gave another generalization of pseudo valuation domains to the context of arbitrary rings (possibly with nonzero zero-divisors). Let R be a ring with total quotient ring $\mathrm{T}(R)$ such that $\mathrm{Nil}(R)$ is a divided prime ideal of R (i.e., R is a ϕ-ring). As in [4] we define the ring homomorphism $\phi : \mathrm{T}(R) \longrightarrow K := R_{\mathrm{Nil}(R)}$ such that $\phi(\frac{a}{b}) = \frac{a}{b}$ for every $a \in R$ and every $b \in R \setminus \mathrm{Z}(R)$. A prime ideal Q of $\phi(R)$ is said to be K-*strongly prime* if $xy \in Q$, $x \in K$, $y \in K$ implies that either $x \in Q$ or $y \in Q$. A prime ideal P of R is said to be ϕ-*strongly prime* if

$\phi(P)$ is a K-strongly prime ideal of $\phi(R)$. If each prime ideal of R is ϕ-strongly prime, then R is called a *ϕ-pseudo-valuation ring* (*ϕ-PVR*). It is shown in [4, Corollary 7(2)] that a ring R is a ϕ-PVR if and only if $\mathrm{Nil}(R)$ is a divided prime ideal and for every $a, b \in R \setminus \mathrm{Nil}(R)$ either $a \mid b$ or $b \mid ac$ in R for each nonunit element $c \in R$.

Let A be a ring and E be an A-module. The following ring construction, called the *trivial extension* of A by E (also called the *idealization* of E), was introduced by Nagata [20, page 2]. It is the ring $A \propto E$ whose underlying abelian group is $A \times E$ with multiplication given by $(a, e)(b, f) = (ab, af + be)$. Trivial ring extensions have been studied extensively; and considerable work, a part of which were summarized in Glaz's book [16] and Huckaba's book [18], has been concerned with these extensions. Mainly, trivial ring extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [1, 10, 18, 19].

Let A be a ring and I an ideal of A. The following ring construction, called the *amalgamated duplication* of A along I, was introduced by D'Anna in [11]. It is the subring $A \bowtie I$ of $A \times A$ given by

$$A \bowtie I := \{(a, a+i) \mid a \in A \text{ and } i \in I\}$$

This extension has been studied, in the general case and from the different point of view of pullbacks, by D'Anna and Fontana [12]. Also see the survey article [15] for more details on general constructions. One main difference of this construction, with respect to the idealization, is that the ring $A \bowtie I$ can be reduced (and it is always reduced if A is an integral domain).

The main purpose of this paper is to introduce and investigate the notion of ϕ_P-pseudo valuation rings which is also a generalization of pseudo valuation domains (PVDs). Set

$$G\mathcal{H} := \{R \mid R \text{ admits a divided prime ideal } P \subseteq \mathrm{Z}(R)\}.$$

A divided prime ideal of a ring R is said to be *minimal* if there is no divided prime ideals strictly contained inside it. If $R \in G\mathcal{H}$ and $P \subseteq \mathrm{Z}(R)$ is the minimal divided prime ideal of R, then R is called a *ϕ_P-ring*. It is easy to see that every ϕ-ring R is a ϕ_P-ring where $P := \mathrm{Nil}(R)$, in particular every integral domain is a $\phi_{(0)}$-ring. Let $R \in G\mathcal{H}$, $P \subseteq \mathrm{Z}(R)$ be the minimal divided prime ideal of R and let ϕ_P be the natural map from the total quotient ring $\mathrm{T}(R)$ to R_P. A prime ideal Q of R is said to be *ϕ_P-strongly prime* if $xy \in \phi_P(Q)$, $x \in R_P$, $y \in R_P$ implies that either $x \in \phi_P(Q)$ or $y \in \phi_P(Q)$. If each prime ideal of R is a ϕ_P-strongly prime ideal, then we say that R is a *ϕ_P-pseudo valuation ring* (*ϕ_P-PVR*). Clearly every PVR is a ϕ_P-PVR where $P := \mathrm{Nil}(R)$. In this article, we show that ϕ_P-PVRs enjoy analogs of many properties of PVDs and ϕ-PVRs. Also, we introduce the notion of ϕ_P-chained rings and we study the transfer of the properties of being a ϕ_P-PVR and a ϕ_P-chained ring between a ring A and a trivial ring extension $A \propto E$ (resp., an amalgamated duplication of A along an ideal I). At this point, our aim is to provide non-trivial examples of classes of commutative rings satisfying the above-mentioned properties.

2 Basics results

In this section, we introduce the ϕ_P-pseudo valuation rings and prove some properties of these rings.

Definition 2.1. Let $R \in G\mathcal{H}$ be a ring with total quotient ring $\mathrm{T}(R)$ and let P be the minimal divided prime ideal of R such that $P \subseteq \mathrm{Z}(R)$. We define $\phi_P : \mathrm{T}(R) \longrightarrow R_P$ such that $\phi_P(\frac{a}{b}) = \frac{a}{b}$ for every $a \in R$ and every $b \in R \setminus \mathrm{Z}(R)$. Then ϕ_P is a ring homomorphism from $\mathrm{T}(R)$ into R_P, and ϕ_P restricted to R is also a ring homomorphism from R into R_P given by $\phi_P(x) = \frac{x}{1}$ for every $x \in R$. A prime ideal Q of R is said to be *ϕ_P-strongly prime* if $xy \in \phi_P(Q)$, $x \in R_P$, $y \in R_P$ implies that either $x \in \phi_P(Q)$ or $y \in \phi_P(Q)$. If each prime ideal of R is ϕ_P-strongly prime, then R is called a *ϕ_P-pseudo-valuation ring* (*ϕ_P-PVR*).

The following is an example of a ϕ_P-PVR which is not a PVR.

Example 2.2. Let (A, M) be a local domain which is not a field and set $E := \frac{A}{M}$ and $R := A \propto E$ to be the trivial ring extension of A by E. Set $P := M \propto E$. Then R is a ϕ_P-PVR which is not a PVR.

Proof. Clearly P is a divided prime ideal of R and $P \subseteq \mathrm{Z}(R)$ because $(0,\bar{1})P = (0,\bar{0})$. Now, assume that there is a divided prime ideal $Q := N \propto E \subsetneq P$. Then let $0 \neq a \in M \setminus N$ and $\bar{0} \neq \bar{e} \in E$. As Q is divided and $(a,\bar{0}) \notin Q$, we get $(0,\bar{e}) \in (a,\bar{0})R$. Hence $\bar{e} = a\bar{f}$ for some $\bar{f} \in E$, which implies that $\bar{e} = \bar{0}$, a contradiction. Since P is the maximal ideal of R, we get $R_P = R$, and hence R is a ϕ_P-PVR. However R is not a PVR by [14, Theorem 2.8(2)] □

Throughout this section, R denotes a ring in $G\mathcal{H}$ and P is the minimal divided prime ideal of R such that $P \subseteq \mathrm{Z}(R)$.

Lemma 2.3. *Let Q be a prime ideal of R. Then Q is a ϕ_P-strongly prime ideal of R if and only if for every $x \in R_P$ either $x \in \phi_P(R)$ or $x^{-1}\phi_P(Q) \subseteq \phi_P(Q)$.*

Proof. Let Q be a ϕ_P-strongly prime ideal of R, $x \in R_P \setminus \phi_P(R)$, and set $x = \frac{a}{b}$ for some $a \in R$ and $b \in R \setminus P$. If $a \in P$, then $a = br$ for some $r \in P$ since P is divided. Hence $x = \frac{r}{1} \in \phi_P(R)$, which is a contradiction. That implies $x^{-1} \in R_P$. Since $xx^{-1}\phi_P(Q) \subseteq \phi_P(Q)$, by using the definition, we get $x^{-1}\phi_P(Q) \subseteq \phi_P(Q)$.

Conversely assume that $x, y \in R_P$ such that $xy \in \phi_P(Q)$. If $x \in \phi_P(R)$ and $y \in \phi_P(R)$, then $x \in \phi_P(Q)$ or $y \in \phi_P(Q)$. In the remaining case, either x or y is not in $\phi_P(R)$. Without loss of generality we may assume that $x \notin \phi_P(R)$. Then by the hypothesis $x^{-1}\phi_P(Q) \subseteq \phi_P(Q)$. Therefore $y = x^{-1}xy \in x^{-1}\phi_P(Q) \subseteq \phi_P(Q)$. Hence $y \in \phi_P(Q)$. □

Proposition 2.4. *Let $Q_1 \subseteq Q_2$ be two prime ideals of R such that $P \subseteq Q_1$. If Q_2 is a ϕ_P-strongly prime ideal of R, then so is Q_1.*

Proof. Assume that Q_2 is a ϕ_P-strongly prime ideal of R and let $x \in R_P$, $y \in R_P$ such that $xy \in \phi_P(Q_1)$. Suppose that $xy \in \phi_P(P)$ and set $x = \frac{a}{s}$ and $y = \frac{b}{t}$ for some $a, b \in R$ and $s, t \in R \setminus P$. Hence there are $u \in R \setminus P$ and $r \in P$ such that $u(ab - r) = 0$. Thus $ab - r \in P$, and so $ab \in P$. As P is a divided prime ideal of R, we get $a \in P$ or $b \in P$ such that $a = sp$ or $b = tq$ for some $p, q \in P$. That implies $x \in \phi_P(P) \subseteq \phi_P(Q_1)$ or $y \in \phi_P(P) \subseteq \phi_P(Q_1)$. Now, assume that $xy \notin \phi_P(P)$ and $x \in R_P \setminus \phi_P(R)$ (since if x and y are in $\phi_P(R)$, the result is clear). Hence $y = xy^2(xy)^{-1} \in \phi_P(Q_2)$ since Q_2 is ϕ_P-strongly prime. That implies $y^2(xy)^{-1} \in \phi_P(Q_2)$, and so $y^2 \in \phi_P(Q_1)$ since $xy \in \phi_P(Q_1)$. As Q_2 is ϕ_P-strongly prime and $x \notin \phi_P(R)$, we get $y \in \phi_P(Q_2)$. Therefore $y^2 \in \phi_P(Q_1)$ implies that $y \in \phi_P(Q_1)$. □

In the next result we study the divided prime ideals of a ϕ_P-PVR.

Proposition 2.5. *Let R be a ϕ_P-PVR. Then every prime ideal of R containing P is divided. In particular, R is a divided ring if and only if $P = \mathrm{Nil}(R)$.*

Proof. We wish to show that every prime ideal of R containing P is divided. Let Q be a prime ideal of R such that $P \subseteq Q$ and let $x \in R \setminus Q$. Our aim is to show that $Q \subseteq xR$. Suppose that there is $a \in Q \setminus xR$. Then $\frac{a}{x}\frac{x}{1} = \frac{a}{1} \in \phi_P(Q)$ since $x \notin P$. As Q is a ϕ_P-strongly prime ideal of R, we get $\frac{a}{x} \in \phi_P(Q)$ or $\frac{x}{1} \in \phi_P(Q)$. If $\frac{x}{1} \in \phi_P(Q)$, then $\frac{x}{1} = \frac{q}{1}$ for some $q \in Q$. Hence $s(x - q) = 0 \in P$ for some $s \in R \setminus P$, which implies that $x - q \in P \subseteq Q$. Therefore $x \in Q$, which is absurd. Then $\frac{a}{x} \in \phi_P(Q)$. Thus $u(a - xq) = 0 \in P$ for some $q \in Q$ and $u \in R \setminus P$. As P is divided and $x \notin Q$ (and hence $x \notin P$), then $P \subseteq xR$. That implies $a \in xR$, a contradiction. Therefore $Q \subseteq xR$ and so Q is a divided prime ideal. □

In the following theorem we prove one of the main results on ϕ_P-pseudo valuation rings.

Theorem 2.6. *The following statements are equivalent for a ring R.*

(i) R is a ϕ_P-PVR.

(ii) Each prime ideal of R containing properly P is ϕ_P-strongly prime.

(iii) $\frac{R}{P}$ is a PVD.

Proof. (1) ⇒ (2) This is obvious.

(2) ⇔ (3) Let Q be a prime ideal of R such that $P \subset Q$. Our aim is to show that $\frac{Q}{P}$ is a strongly prime ideal of $\frac{R}{P}$. Let $x \in \mathrm{T}(\frac{R}{P}) \setminus \frac{R}{P}$ such that $x = \frac{a+P}{b+P}$ for some $a, b \in R \setminus P$. It is

easy to see that $\frac{a}{b} \in R_P \setminus \phi_P(R)$. Thus using Lemma 2.3, we get $\frac{b}{a}\phi_P(Q) \subseteq \phi_P(Q)$. Therefore $x^{-1}\frac{Q}{P} \subseteq \frac{Q}{P}$, which implies that $\frac{Q}{P}$ is a strongly prime ideal of $\frac{R}{P}$.

Conversely, let Q be a prime ideal of R containing properly P. By (3), $\frac{Q}{P}$ is a strongly prime ideal of $\frac{R}{P}$. Let $x = \frac{a}{b} \in R_P \setminus \phi_P(R)$ for some $a \in R$ and $b \in R \setminus P$. We wish to show that $x^{-1}\phi_P(Q) \subseteq \phi_P(Q)$. Since $x \in R_P \setminus \phi_P(R)$, we have $\bar{x} = \frac{a+P}{b+P} \in \mathrm{T}(\frac{R}{P}) \setminus \frac{R}{P}$. Deny, $\frac{a+P}{b+P} \in \frac{R}{P}$ implies that $s(a - br) \in P$ for some $s \in R \setminus P$ and $r \in R$. Thus $(a - br) \in P$. As P is divided and $b \notin P$, we have $a - br = bp$ for some $p \in P$. Therefore $a - b(r + p) = 0$, which implies that $x = \frac{a}{b} \in \phi_P(R)$, a contradiction. Hence $\bar{x} \in \mathrm{T}(\frac{R}{P}) \setminus \frac{R}{P}$. Thus by [17, Proposition 1.2] $(\frac{b+P}{a+P})\frac{Q}{P} \subseteq \frac{Q}{P}$. Hence one can easily see that $x^{-1}\phi_P(Q) \subseteq \phi_P(Q)$. Therefore Q is a ϕ_P-strongly prime ideal of R.

(2) $\Rightarrow$ (1) Assume that each prime ideal of R containing properly P is ϕ_P-strongly prime. Let $Q \subseteq P$ be a prime ideal of R and let $x, y \in R_P$ such that $xy \in \phi_P(Q)$. Set $x = \frac{a}{b}$ and $y = \frac{c}{d}$ for some $a, c \in R$ and $b, d \in R \setminus P$. Since $xy \in \phi_P(Q)$, we get $u(ac - bdr) = 0 \in Q$ for some $r \in Q$ and $u \in R \setminus P$. Hence one can easily see that $ac \in Q$, and so $a \in Q$ or $c \in Q$. Without loss of generality, we may assume that $a \in Q$. As P is divided and $b \notin P$, we get $a = bq$ for some $q \in Q$. Then $x = \frac{q}{1} \in \phi_P(Q)$. Therefore Q is a ϕ_P-strongly prime ideal of R, and hence R is a ϕ_P-PVR, as desired. □

As consequences of the previous theorem and Proposition 2.4 we give the following corollaries.

Corollary 2.7. *Let R be a ϕ_P-PVR. Then the prime ideals of R containing P are linearly ordered. In particular, a ϕ_P-ring R is a ϕ_P-PVR if and only if R is local with ϕ_P-strongly maximal ideal.*

Corollary 2.8. *Let R be a ϕ_P-PVR. Then $\mathrm{Z}(R)$ is a divided prime ideal of R.*

Proof. Clearly $\mathrm{Z}(R)$ is a prime ideal of R by Corollary 2.7 since $\mathrm{Z}(R)$ is a union of prime ideals ($\mathrm{Z}(R)$ is a union of P and certain prime ideals containing P). By the proof of Proposition 2.5 $\mathrm{Z}(R)$ is divided since R is a ϕ_P-PVR and $P \subseteq \mathrm{Z}(R)$. □

The following theorem gives a characterization of ϕ_P-pseudo valuation rings.

Theorem 2.9. *Let R be a ϕ_P-ring. Then R is a ϕ_P-PVR if and only if for every $a, b \in R \setminus P$ either $bR \subseteq aR$ or $ac \in bR$ for every nonunit $c \in R$.*

Proof. Assume that R is a ϕ_P-PVR and let $a, b \in R \setminus P$. Set $x = \frac{b}{a} \in R_P$. If $x \in \phi_P(R)$, then $x = \frac{r}{1}$ for some $r \in R$. Hence there exists $s \in R \setminus P$ such that $s(b - ar) = 0$. Thus $b - ar \in P$. Since P is a divided prime ideal of R and $a \notin P$, we have $P \subseteq aR$. Therefore $b - ar = ta$ for some $t \in R$, and so $b = a(t + r)$. Consequently $bR \subseteq aR$. Now assume that $x \in R_P \setminus \phi_P(R)$. Using Corollary 2.7, R is local with maximal ideal M such that M is ϕ_P-strongly prime. By Lemma 2.3, $x^{-1}\phi_P(M) \subseteq \phi_P(M)$. Hence for every nonunit $c \in R$ we have $\frac{ac}{b} = \frac{r}{1}$ for some $r \in R$. Thus there exists an element $s \in R \setminus P$ such that $s(ac - br) = 0 \in P \subseteq bR$, which implies that $ac \in bR$. The converse follows by similar reasoning. □

Now we introduce the class of ϕ_P-chained rings which is a generalization of valuation domains.

Definition 2.10. A ϕ_P-ring is called a *ϕ_P-chained ring* if for each $x \in R_P \setminus \phi_P(R)$, we have $x^{-1} \in \phi_P(R)$.

Remark 2.11. It is clear that a ϕ_P-ring is a ϕ_P-chained ring if and only if for each $a, b \in R \setminus P$ either $aR \subseteq bR$ or $bR \subseteq aR$.

Proposition 2.12. *Let R be a ϕ_P-ring. Then R is a ϕ_P-chained ring if and only if $\frac{R}{P}$ is a valuation domain.*

Proof. Assume that R is a ϕ_P-chained ring. Let $x = a + P$ and $y = b + P$ such that $a, b \in R \setminus P$. As either $aR \subseteq bR$ or $bR \subseteq aR$, we get $x\frac{R}{P} \subseteq y\frac{R}{P}$ or $y\frac{R}{P} \subseteq x\frac{R}{P}$. Therefore $\frac{R}{P}$ is a valuation domain.

Conversely, let $a, b \in R \setminus P$ and set $x := a + P$ and $y := b + P$. As $\frac{R}{P}$ is a valuation domain, either $x\frac{R}{P} \subseteq y\frac{R}{P}$ or $y\frac{R}{P} \subseteq y\frac{R}{P}$. Without loss of generality, we may assume that $x\frac{R}{P} \subseteq y\frac{R}{P}$. Hence $a - br \in P$ for some $r \in R$. Since $b \notin P$ and P is divided, $P \subseteq bR$, and so $a - br = bu$ for some $u \in P$. Thus $a = b(r + u)$, and hence $aR \subseteq bR$. Therefore R is a ϕ_P-chained ring. □

Proposition 2.13. *(i) Suppose that R is a ϕ_P-chained ring. Then R is a ϕ_P-PVR.*

(ii) Let R be a ϕ_P-PVR and Q be a non-maximal ideal of R such that $P \subseteq Q$. Then R_Q is a ϕ_{PR_Q}-chained ring.

Proof. (1) This follows easily by Theorem 2.6, Proposition 2.12, and [17, Proposition 1.1].

(2) Assume that R is a ϕ_P-PVR and let Q be a non-maximal ideal of R such that $P \subseteq Q$. Then by Theorem 2.6, $\frac{R}{P}$ is a PVD, and by [17, Proposition 2.6] $\frac{R_Q}{PR_Q}$ is a valuation domain. Since P is the minimal divided prime ideal of R, clearly PR_Q is the minimal divided prime ideal of R_Q. Hence it follows from Proposition 2.12 that R_Q is a ϕ_{PR_Q}-chained ring. □

The following result corresponds to [5, Proposition 3.3].

Theorem 2.14. *Let $R \in G\mathcal{H}$ such that P is the minimal divided prime ideal of R. Suppose that R is a local ring with the regular maximal ideal M. Then R is a ϕ_P-PVR if and only if $V := (M : M)$ is a ϕ_P-chained ring with the maximal ideal M.*

Proof. Clearly P is the minimal divided prime ideal of V. Assume that R is a ϕ_P-PVR and let $0 \neq x \in \mathrm{T}(\phi_P(V))(= R_P)$ such that $x \notin \phi_P(V)$. Set $x = \frac{a}{b}$ for some $a \in R$ and $b \in R \setminus P$. If $a \in P$, then there is $r \in R$ such that $a = br$, and so $x = \frac{r}{1} \in \phi_P(V)$, which is absurd. Hence $a, b \in R \setminus P$. As R is a ϕ_P-PVR, either $aR \subseteq bR$ or $bc \in aR$ for every nonunit $c \in R$. If $aR \subseteq bR$, then $a = br$ for some $r \in R$, which implies that $x = \frac{r}{1} \in \phi_P(V)$, a contradiction. Hence $bc \in aR$ for every nonunit $c \in R$. Let $s \in M$ be a regular element of R. Hence $bs = ar$ for some $r \in R$. If $sR \subseteq rR$, then $s = rt$ for some $t \in R$. Thus $bs = ar$ implies that $brt = ar$. Since s is regular, r is a regular element of R, and so $a = bt \in bR$, which is a contradiction. Hence $rc \in sR$ for every nonunit element of M. Therefore $rc = sd$ for every $c \in M$ and some $d \in R$. If $d \in R \setminus M$, then d is a unit of R. That implies $s \in rR$, which is absurd. Thus $\frac{r}{s}M \subseteq M$, which implies that $\frac{r}{s} \in V$. On the other hand, $bs = ar$ implies that $\frac{b}{a} = \frac{r}{s} \in V$. Thus $x^{-1} = \phi_P(x^{-1}) \in \phi_P(V)$. Therefore V is a ϕ_P-chained ring. Now, we wish to show that M is the maximal ideal of V. Let x be a nonunit element of V such that $x \notin M$. If $\phi_P(x) \in \phi_P(R)$, then $\phi_P(x) \in \phi_P(M)$. Thus $x \in M$, a contradiction. Hence $\phi_P(x) \notin \phi_P(R)$. Since M is a ϕ_P-strongly prime ideal of R, we get $\phi_P(x)^{-1}\phi_P(M) \subseteq \phi_P(M)$. Hence $x^{-1}M \subseteq M$, which implies that $x^{-1} \in V$, and so x is a unit element of V, a contradiction. Therefore M is the maximal ideal of V.

Conversely, let $a, b \in R \setminus P$ such that $aR \nsubseteq bR$. Hence $x = \frac{a}{b} \notin \phi_P(M)$. Deny, $\frac{a}{b} = \frac{m}{1}$ for some $m \in M$. Thus $a - mb \in P$, and so $a - mb = bp$ for some $p \in P$ since P is divided and $b \notin P$. Then $a = b(m + p) \in bR$, which is absurd. Therefore $x \notin \phi_P(M)$. If $x \in \phi_P(V)$, then x is a unit element of $\phi_P(V)$ and $x^{-1}\phi_P(M) \subseteq \phi_P(M)$. Hence for every $c \in M$ $x^{-1}\frac{c}{1} = \frac{m}{1}$ for some $m \in M$, which implies that $u(bc - am) = 0 \in P$ for some $u \notin P$. By using the fact that P is divided and $a \notin P$, we get $bc \in aR$ for every nonunit $c \in M$. Now suppose that $x \notin \phi_P(V)$. Hence $x^{-1} \in \phi_P(V)$ since V is a ϕ_P-chained ring. Thus $x^{-1}\phi_P(M) \subseteq \phi_P(M)$, which implies that $bc \in aR$ for every nonunit $c \in M$. Therefore R is a ϕ_P-PVR by Theorem 2.9. □

By using the above theorem, we get the following result.

Proposition 2.15. *Let R be a ϕ_P-ring and Q be a regular principal prime ideal of R. If Q is a ϕ_P-strongly prime ideal of R, then R is a ϕ_P-chained ring with maximal ideal Q.*

Proof. Suppose that $Q = aR$ for some regular element a of R and assume that Q is not a maximal ideal R. Then, by the proof of Proposition 2.5 Q is a divided prime ideal of R because $P \subset Q$. This implies that for each nonunit $b \in R \setminus Q$, $a = br$ for some $r \in Q$. It follows that $r = ad$ for some $d \in R$ since $r \in Q$. Hence $a = bad$ which implies that b is a unit element of R because a is regular, a contradiction. Thus Q is a maximal ideal of R. Then R is a ϕ_P-PVR, and so $(Q : Q) = R$ is a ϕ_P-chained ring by Theorem 2.14. □

Now we show that every ϕ_P-PVR is a pullback of a ϕ_P-chained ring.

Proposition 2.16. *Let V be a ϕ_P-chained ring with maximal ideal M such that $P \subseteq \mathrm{Z}(V)$ is the minimal divided prime ideal of V. Set $F = \frac{V}{M}$, $\alpha : V \longrightarrow F$ the canonical epimorphism, H be a field contained in F, and $R = \alpha^{-1}(H)$. Then the pullback $R = V \times_F H$ is a ϕ_P-PVR. In particular, if H is properly contained in F, then $R = \alpha^{-1}(H)$ is a ϕ_P-PVR which is not a ϕ_P-chained ring.*

Proof. By construction it is clear that M is the maximal ideal of R and P is the minimal divided prime ideal of R. Let $\bar{\alpha} : \frac{V}{P} \longrightarrow F$ such that $\bar{\alpha}(a+P) = \alpha(a)$. Then $\frac{R}{P} = \bar{\alpha}^{-1}(H)$ is the pullback $\frac{V}{P} \times_F H$. As V is a ϕ_P-chained ring, we get $\frac{V}{P}$ is a valuation domain by Proposition 2.12. Therefore it follows from [5, Proposition 3.11] that $\frac{R}{P}$ is a PVD (In particular, if H is properly contained in F then $\frac{R}{P}$ is not a valuation domain). Hence by Theorem 2.6 R is a ϕ_P-PVR (which is not a ϕ_P-chained ring if H is properly contained in F by Proposition 2.12). □

3 Trivial ring extensions and amalgamated duplications

We start this section by stating the following necessary result. The aim of this lemma is to study the divided prime ideals of the trivial ring extension.

Lemma 3.1. *Let A be a ring, P a prime ideal of A and let E be an A-module. Set $R := A \propto E$. Then $P \propto E$ is a divided prime ideal of R if and only if P is a divided prime ideal of A and $E = aE$ for each $a \in A \setminus P$. Also, if $P \subseteq \mathrm{Z}(A)$, then $P \propto E \subseteq \mathrm{Z}(R)$. In particular, if $P \subseteq \mathrm{Z}(A)$ is the minimal divided prime ideal of $A \in G\mathcal{H}$ and $P' = P \propto E$, then R is a $\phi_{P'}$-ring if and only if $E = aE$ for every $a \in R \setminus P$.*

Proof. Assume that $P \propto E$ is a divided prime ideal of R and let $a \in R \setminus P$. Hence $(a,0) \notin P \propto E$, and so $P \propto E \subsetneq (a,0)R$. Thus for every $p \in P$ we have $(p,0) = (a,0)(r,s)$ for some $(r,s) \in R$. Then $p = ar \in aA$, and hence $P \subsetneq Aa$. Therefore P is a divided prime ideal of A. Now we wish to show that $E = aE$ for each $a \in R \setminus P$. Let $e \in E$ and $a \notin P$. As $P \propto E$ is divided, we get $(0,e) \in (a,0)R$, which implies that $(0,e) = (a,0)(b,f)$ for some $(b,f) \in R$. Thus $e = af$, and hence $E = aE$.

Conversely, let $(p,f) \in P \propto E$ and $(a,e) \notin P \propto E$. Then $a \notin P$ and so $p = ar$ for some $r \in A$ since P is divided. Also by assumption there is $t \in E$ such that $f - re = at$ since $a \notin P$. Hence $(p,f) = (a,e)(r,t) \in (a,e)R$, which implies that $P \propto E$ is divided. This completes the proof of the Lemma. Furthermore, if $P \subseteq \mathrm{Z}(A)$, we get $P \propto E \subseteq \mathrm{Z}(R)$ by [1, Theorem 3.5]. □

If $P \subseteq \mathrm{Z}(A)$ is a divided prime ideal of $A \in G\mathcal{H}$ and $E = A_P$, then Lemma 3.1 specializes to the following result, which shows that if $P \propto E$ is the minimal divided prime ideal of a trivial ring extension $A \propto E$, then P is not necessarily the minimal divided prime ideal of A.

Corollary 3.2. *Let P be any divided prime ideal of a ring $A \in G\mathcal{H}$ such that $P \subseteq \mathrm{Z}(A)$, and set $P' = P \propto A_P$. Then $R = A \propto A_P$ is a $\phi_{P'}$-ring.*

Proof. We wish to show that P' is the minimal divided prime ideal of R for any divided prime ideal P of A such that $P \subseteq \mathrm{Z}(A)$. By Lemma 3.1, P' is a divided prime ideal of R. Now, suppose that there is a divided prime ideal $Q' = Q \propto A_P$ of R such that $Q' \subsetneq P'$. Let $a \in P \setminus Q$. By Lemma 3.1, $1 = ax$ for some $x \in A_P$ since Q' is divided, which is a contradiction. □

Our next new result studies the possible transfer of the ϕ_P-PVR property (resp., the ϕ_P-chained ring property) between a ring A and a trivial ring extension $A \propto E$.

Theorem 3.3. *Let $A \in G\mathcal{H}$ be a ring, $P \subseteq \mathrm{Z}(A)$ a divided prime ideal of A, and let E be an A-module. Set $R := A \propto E$ to be a trivial ring extension of A by E and $P' := P \propto E$. Suppose that R is a $\phi_{P'}$-ring. Then:*

- *(i) R is a $\phi_{P'}$-chained ring if and only if for every $a, b \in A \setminus P$, the two ideals aA and bA are comparable.*
- *(ii) R is a $\phi_{P'}$-PVR if and only if for every $a, b \in A \setminus P$, $aA \subseteq bA$ or $bc \in aA$ for every nonunit $c \in A$.*

Proof. (1) Assume that R is a $\phi_{P'}$-chained ring and let $a, b \in A \setminus P$. By Remark 2.11, $(a,0)R \subseteq (b,0)R$ or $(a,0)R \subseteq (b,0)R$ since $(a,0), (b,0) \in R \setminus P'$. Without loss of generality, we assume that $(a,0)R \subseteq (b,0)R$. That implies $aA \subseteq bA$, as desired.

Conversely, let $(a,e), (b,f) \in R \setminus P'$. Hence $aA \subseteq bA$ or $bA \subseteq aA$ since $a, b \in A \setminus P$. Without loss of generality, we may assume that $aA \subseteq bA$. Then $a = br$ for some $r \in A$.

Also, by Lemma 3.1 the fact that $E = bE$ gives that $e - rf = bt$ for some $t \in E$. Thus $(a, e) = (b, f)(r, t) \in (b, f)R$. Therefore R is a $\phi_{P'}$-chained ring.

(2) Suppose that R is a $\phi_{P'}$-PVR. Hence, by Theorem 2.9, $(a, 0)R \subseteq (b, 0)R$ or $(b, 0)(c, f) \in (a, 0)R$ for every nonunit $(c, f) \in R$. Therefore $aA \subseteq bA$ or $bc \in aA$ for every nonunit $c \in A$.

Conversely, let $(a, e), (b, f) \in R \setminus P'$ such that $(a, e)R \nsubseteq (b, f)R$. If $aA \subseteq bA$, then there is $r \in A$ such that $a = br$. Since $b \in A \setminus P$ and $E = bE$, there exists $d \in E$ such that $e - rf = bd$. Thus $(a, e) \in (b, f)R$, which is a contradiction. Hence $bc \in aA$ for every nonunit $c \in A$. Now, let (c, t) be a nonunit element of R. Then there is $r \in A$ and $d \in E$ such that $bc = ar$ and $bt + cf - re = ad$ since $a \notin P$. Thus $(b, f)(c, t) = (a, e)(r, d) \in (a, e)R$. Therefore R is a $\phi_{P'}$-PVR, as desired. □

In a particular case, we get the following corollary.

Corollary 3.4. *Let A, E, and R as in the above theorem and set $P' := P \propto E$ such that $P \subseteq \mathrm{Z}(A)$ is the minimal divided prime ideal of A. Suppose that R is a $\phi_{P'}$-ring. Then:*

(i) R is a $\phi_{P'}$-chained ring if and only if A is a ϕ_P-chained ring.

(ii) R is a $\phi_{P'}$-PVR if and only if A is a ϕ_P-PVR.

Theorem 3.3 enriches the literature with new examples, as shown below.

Example 3.5. Let A be a valuation domain with maximal ideal M and Krull dimension n, say $\{0\} \subseteq P_1 \subseteq \cdots \subseteq P_n = M$. Choose $x \in A$ such that $\sqrt{xA} = P_1$. Set $I := xA$ and $R := \frac{A}{I}$. For every prime ideal P of R, we set $P' := P \propto R_P$. Then $D = R \propto R_P$ is a $\phi_{P'}$-chained ring that is not a chained ring.

Proof. By [6, Proposition 2.1], R is a chained ring (hence R is divided), $\mathrm{Nil}(R) = \frac{P_1}{I}$ and $\mathrm{Z}(R) = \frac{M}{I}$. Let $i \in \{1, ..., n\}$. Then $P := \frac{P_i}{I}$ is a divided prime ideal of R such that $P \subseteq \mathrm{Z}(R)$. Hence it follows from Corollary 3.2 that P' is the minimal divided prime ideal of D. As R is a chained ring, we get D is a $\phi_{P'}$-chained ring by Theorem 3.3. Now one can easily see that D is not a chained ring since the ideals $(0, 1)D$ and $(p, 0)D$ are not comparable for every $0 \neq p \in P$. □

Example 3.6. Let A be any PVD with maximal ideal M and Krull dimension n. By the same construction, let P_i for every $i \in \{1, \dots, n\}$, x, I and R as in the above example. Set $P' := P \propto R_P$ for a prime ideal P of R. Then $R \propto R_P$ is a $\phi_{P'}$-PVR.

Proof. The proof is similar to that of Example 3.5. □

Now, we investigate the transfer of the ϕ_P-PVR (resp., the ϕ_P-chained ring) property to the amalgamated duplication. Notice that the prime ideals of $A \bowtie I$ have the form $P' = P \bowtie I$, where P is a prime ideal of A or $\bar{P} = \{(a, a + i) \mid a \in A, i \in I, \text{ and } a + i \in P\}$, where P is a prime ideal of A not containing I.

We start with the following lemma.

Lemma 3.7. *Let A be a ring, P be a prime ideal of A, I be a nonzero ideal of A, and J be a divided prime ideal of $R = A \bowtie I$. Then J has the form $P' = P \bowtie I$ where P is a divided prime ideal of A such that $I \subseteq P$ and $I = aI$ for every $a \in R \setminus P$. Furthermore, if $P \subseteq \mathrm{Z}(A)$, then $P' \subseteq \mathrm{Z}(R)$. In particular, if $P \subseteq \mathrm{Z}(A)$ is the minimal divided prime ideal of $A \in \mathcal{GH}$, then R is a $\phi_{P'}$-ring if and only if $I \subseteq P$ and $I = aI$ for every $a \in A \setminus P$.*

Proof. Suppose that J has the form $\bar{P}$, where P is a prime ideal of A not containing I. It is clear that $I \times \{0\} \subseteq J$. Now, since J is divided, we have J and $\{0\} \times I$ are comparable. If $J \subseteq \{0\} \times I$, then $I = 0$ because $I \times \{0\} \subseteq J$, a contradiction. Hence $\{0\} \times I \subseteq J$, which implies that $I \subseteq P$, a desired contradiction. Therefore J has the form $P' = P \bowtie I$, where P is a prime ideal of A.

Assume that P' is a divided prime ideal of R and let $a \in P \setminus A$. Then $(a, a) \notin P'$ and so $P' \subseteq (a, a)R$. This implies that $P \subseteq aA$, and hence P is a divided prime ideal of A. Now, we wish to show that $I \subseteq P$. Since P' is a divided prime ideal of R, then P' and $I \times \{0\}$ are comparable. Clearly $I \times \{0\} \subseteq P'$ because $I \neq 0$, and so $I \subseteq P$. Let $i \in I$ and $a \in A \setminus P$.

Then $(0,i) \in (a,a)R$ since $(0,i) \in P'$. Hence $(0,i) = (a,a)(b,b+k)$, and so $ab = 0$ and $i = ab + ak = ak$. This implies that $I = aI$ for every $a \in A \setminus P$.

Conversely, our aim is to show that P' is a divided prime ideal of R. Let $(a, a+i) \in R \setminus P'$ and $(p, p+j) \in P'$. As P is divided and $a \in A \setminus P$, we get $p = ar$ for some $r \in P$. Also, by using the fact that $I = (a+i)I$, there exists $k \in I$ such that $j - ri = (a+i)k$. Thus $(p, p+j) = (a, a+i)(r, r+k)$, and hence $P' \subseteq (a, a+i)R$.

Now, assume that $P \subseteq \mathrm{Z}(A)$ and let $(p, p+i) \in P'$. Since $p \in P$, there is $q \in A$ such that $pq = 0$. If $qI = 0$, then $(p, p+i)(q, q) = (0, 0)$. If there exists $k \in I$ such that $qk \neq 0$, then $(p, p+i)(qk, 0) = (0, 0)$. In all cases, $(p, p+i) \in \mathrm{Z}(R)$, and hence $P' \subseteq \mathrm{Z}(R)$. □

Our next theorem characterizes when the amalgamated duplication of a ring is a $\phi_{P'}$-chained ring (resp., a ϕ_P-PVR)

Theorem 3.8. *Let $A \in G\mathcal{H}$ be a ring, $0 \neq I$ an ideal of A and $P \subseteq \mathrm{Z}(A)$ be the minimal divided prime ideal of A. Set $R := A \bowtie I$ and $P' := P \bowtie I$. Assume that R is a $\phi_{P'}$-ring. Then:*

(i) R is a $\phi_{P'}$-chained ring if and only if A is a ϕ_P-chained ring.

(ii) R is a $\phi_{P'}$-PVR if and only if A is a ϕ_P-PVR.

Proof. (1) Assume that R is a $\phi_{P'}$-chained ring and let $a, b \in A \setminus P$. Hence $(a,a), (b,b) \in R \setminus P'$, which implies that $(a,a)R$ and $(b,b)R$ are comparable. Thus $aA \subseteq bA$ or $bA \subseteq aA$, and so A is a ϕ_P-chained ring.

Conversely, let $(a, a+i), (b, b+j) \in R \setminus P'$. Without loss of generality we may assume that $aA \subseteq bA$ since A is a ϕ_P-chained ring. Thus there exists $r \in R$ such that $a = br$. On the other hand, the fact that R is a $\phi_{P'}$-ring implies, by Lemma 3.7, that $I = (b+i)I$ because $b + j \notin P$ (since $I \subseteq P$ again by Lemma 3.7). Hence $i - rj = (b+j)k$ for some $k \in I$. This implies that $(a, a+i) = (b, b+j)(r, r+k) \in (b, b+j)R$. Therefore R is a $\phi_{P'}$-chained ring.

(2) Suppose that R is a $\phi_{P'}$-PVR and let $a, b \in A \setminus P$. This implies that $(a,a), (b,b) \in R \setminus P'$, and hence $(a,a)R \subseteq (b,b)R$ or $(b,b)(c,c) \in (a,a)R$ for every nonunit $c \in A$. Therefore $aA \subseteq bA$ or $bc \in aA$ for every nonunit $c \in A$, which implies that A is a ϕ_P-PVR.

Conversely, suppose that A is a ϕ_P-PVR and let $(a, a+i), (b, b+j) \in R \setminus P'$. Assume that $aA \subseteq bA$, by the same reasoning as in the proof of (1), we get $(a, a+i) \in (b, b+j)R$. Now, we assume that $bc \in aA$ for every nonunit $c \in A$. Let $(c, c+k)$ be a nonunit element of R. Then $bc = ar$ for some $r \in R$. Also, as R is a $\phi_{P'}$-ring, we get $I \subseteq P$ and $I = (a+i)I$. Thus there is $t \in I$ such that $bk + j(c+k) - ri = (a+i)t$, which implies that $(b, b+j)(c, c+k) = (a, a+i)(r, r+t) \in (a, a+i)R$. Therefore R is a $\phi_{P'}$-PVR. □

The next corollary is an immediate application of the above theorem.

Corollary 3.9. *Let $A \in G\mathcal{H}$ be a ring (that is not a domain) and $0 \neq P \subseteq \mathrm{Z}(A)$ be the minimal divided prime ideal of A. Set $P' := P \bowtie P$. Then $A \bowtie P$ is a $\phi_{P'}$-chained ring (resp., $\phi_{P'}$-PVR) if and only if A is a ϕ_P-chained ring (resp., a ϕ_P-PVR).*

Proof. By Theorem 3.8, it is enough to show that $A \bowtie P$ is a $\phi_{P'}$-ring. As P is a divided prime ideal of A, we get $P = aP$ for every $a \in A \setminus P$. Hence, by Lemma 3.7, $A \bowtie I$ is a $\phi_{P'}$-ring. □

Example 3.10. Let A be a ϕ_P-PVR (see Example 3.6), where $P \subseteq \mathrm{Z}(A)$ is the minimal divided prime ideal of A. Then $A \bowtie P$ is a $\phi_{P'}$-PVR, where $P' := P \bowtie P$.

References

[1] D. D. Anderson and M. Winders, Idealization of a module, *J. Commut. Algebra* **1**, 3–56 (2009).

[2] A. Badawi, Pseudo-valuation domains: A survey, in Proceedings of the 3rd international Palestinian conference on mathematics and mathematics education, Word Scientific, Singapore, 38–59 (2002).

[3] A. Badawi, On divided rings and ϕ-pseudo-valuation, International J. of Commutative Rings (IJCR) **1**, 51–60 (2002).

[4] A. Badawi, On ϕ-pseudo-valuation rings, Lecture Notes Pure Appl. Math., Vol. 205, Marcel Dekker, New York/Basel, 101–110 (1999).

[5] A. Badawi, On ϕ-chained rings and ϕ-pseudo-valuation rings, *Houston J. Math.* **27**, 725–736 (2001).

[6] A. Badawi, On ϕ-pseudo-valuation rings II, *Houston J. Math.* **26**, 473–480 (2000).

[7] A. Badawi, On Nonnil-Noetherian rings, *Comm. Algebra* **31**, 1669—1677 (2003).

[8] A. Badawi, On divided commutative rings, *Comm. Algebra* **27**, 1465–1474 (1999).

[9] A. Badawi, D. F. Anderson and D. E. Dobbs, Pseudo-valuation rings, in Commutative Ring Theory (Fès, 1995), Lecture Notes Pure Appl. Math. 185, Dekker, New York, 57–67 (1997).

[10] C. Bakkari, S. Kabbaj and N. Mahdou, Trivial extensions defined by Prüfer conditions, *J. Pure Appl. Algebra* **214**, 53–60 (2010).

[11] M. D'Anna, A construction of Gorenstein rings, *J. Algebra* **306**, 507–519 (2006).

[12] M. D'Anna and M. Fontana, An amalgamated duplication of a ring along an ideal: the basic properties, *J. Algebra Appl.*, **6**, 443–459 (2007).

[13] D. E. Dobbs, Divided rings and going-down, *Pacific J. Math.* **67**, 353–363 (1976).

[14] D. E Dobbs, A. El Khalfi and N. Mahdou, Trivial extensions satisfying certain valuation-like properties, *Comm. Algebra* **47**, 2060–2077 (2019).

[15] A. El Khalfi, H. Kim and N. Mahdou, Amalgamation extension in commutative ring theory, a survey, Moroccan Journal of Algebra and Geometry with Applications, **1**, 139–182 (2022).

[16] S. Glaz, *Commutative Coherent Rings*, Lecture Notes in Math. 1371, Springer–Verlag, Berlin, (1989).

[17] J. R. Hedstrom and E. G. Houston, Pseudo-valuation domains, *Pacific J. Math.* **75**, 137–147 (1978).

[18] J. A. Huckaba, *Commutative Rings with Zero Divisors*, Dekker, New York, (1988).

[19] S. Kabbaj and N. Mahdou, Trivial extensions defined by coherent-like conditions, *Comm. Algebra* **32**, 3937–3953 (2004).

[20] M. Nagata, *Local Rings*, Wiley-Interscience, New York, (1962).

Author information

Abdelhaq El Khalfi, Laboratory of Topology, Algebra, Geometry and Discrete Mathematics, Faculty of Sciences Ain Chock, Hassan II University of Casablanca, Morocco.
E-mail: abdelhaqelkhalfi@gmail.com

Hwankoo Kim, Division of Computer Engineering, Hoseo University, Republic of Korea.
E-mail: hkkim@hoseo.edu

Najib Mahdou, Modelling and Mathematical Structures Laboratory, Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco.
E-mail: mahdou@hotmail.com

Mohammed Tamekkante (Corresponding Author: Hwankoo Kim), Department of Mathematics, Faculty of Science, Box 11201 Zitoune, University Moulay Ismail Meknes, Morocco.
E-mail: tamekkante@yahoo.fr

Received: December 28, 2021.
Accepted: June 7, 2022.