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ON ALMOST *n*-ABSORBING PRIMARY SUBMODULES

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Abstract Let R be a commutative ring with unity, M be an R-module and n be a positive integer. In this paper, we introduce the concept of almost n-absorbing primary submodules as a new generalisation of n-absorbing submodules. A proper submodule N of an R-module M is called an almost n-absorbing primary submodule if whenever $a_1 \dots a_n m \in N - (N : M)N$ for any $a_1, \dots, a_n \in R$ and $m \in M$, then either $a_1 \dots a_n \in \sqrt{(N : M)}$ or there are $n - 1$ of the $a_i's$ whose product with m is in N. We prove several results concerning almost n-absorbing primary submodules and give some characterisations of them for multiplication modules.

1 Introduction

Throughout this paper, all rings are commutative with non-zero identity and all modules are unital. Let R be a ring, I be an ideal of R, M be an R-module and N be a submodule of M. The radical of I is denoted by $\sqrt{I}$ and is given by $\sqrt{I} = \{r \in R : r^k \in I \text{ for some } k \in \mathbb{N}\}$. The residual of N over M is denoted by $(N : M)$ i.e. $(N : M) = \{r \in R : rM \subseteq N\}$. The residual of N by I is defined as $(N : I) = \{m \in M : Im \subseteq N\}$. It is a submodule of M containing N.

Almost prime ideals were introduced by Bhatwadekar and Sharma in [13], where they defined a proper ideal I of R to be an almost prime ideal if whenever $a, b \in R$ with $ab \in I - I^2$ implies that $a \in I$ or $b \in I$. In [1], Ayman Badawi introduced the concept of 2-absorbing ideals as a generalisation of prime ideals. Badawi also introduced a generalisation of primary ideals called 2-absorbing primary ideals in [2]. The concept of almost primary ideals was introduced by Bataineh and Kuhail in [10]. They defined a proper ideal I of R to be an almost primary ideal if whenever $a, b \in R$ such that $ab \in I - I^2$, then $a \in I$ or $b \in \sqrt{I}$. In [9], Bataineh and Al-Nuserat introduced the concept of almost 2-absorbing primary ideals. They defined a proper ideal I of R to be an almost 2-absorbing primary ideal if whenever $abc \in I - I^2$ for any $a, b, c \in R$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Expanding the definition of 2-absorbing ideal, Anderson and Badawi introduced the concept of n-absorbing ideals of R for a positive integer n in [7]. A. E. Becker generalised 2-absorbing primary ideals to n-absorbing primary ideals for positive integer n in [6].

A generalisation of prime submodules was introduced by H. Khashan in [8], where he defined a proper submodule N of an R-module M to be an almost prime submodule of M if whenever $r \in R$ and $m \in M$ such that $rm \in N - (N : M)N$, then either $m \in N$ or $r \in (N : M)$. The concept of almost primary submodule was introduced by Bataineh and Kuhail in [10]. They defined a proper submodule N of an R-module M to be an almost primary submodule of M if whenever $r \in R$ and $m \in M$ such that $rm \in N - (N : M)N$, then either $m \in N$ or $r \in \sqrt{(N : M)}$.

The concepts of 2-absorbing and weakly 2-absorbing submodules were introduced and investigated by Darani and Soheilnia in [3]. The concept of almost 2-absorbing submodules was introduced by Yasein and Abbu-Dawwas in [14], where they defined a proper submodule N of an R-module M to be an almost 2-absorbing submodule of M if $a, b \in R$ and $m \in M$ with $abm \in N - (N : M)N$, then either $am \in N$ or $bm \in N$ or $ab \in (N : M)$. M. K. Dubey and P. Aggarwal introduced the concept of 2-absorbing primary submodules as a generalisation of primary submodules in [11]. In [5], Ashour, Al-Ashker and Naji defined almost 2-absorbing primary submodules. They called a proper submodule N of an R-module M to be an almost 2-absorbing primary submodule of M if whenever $a, b \in R$ and $m \in M$ such that $abm \in N - (N : M)N$ implies that $am \in N$ or $bm \in N$ or $ab \in \sqrt{(N : M)}$.

Darani and Soheilnia introduced the concept of n-absorbing submodules in [4]. The concept of n-absorbing primary submodules was introduced and studied by Shaikh and Deore in [15]. They defined a proper submodule N of an R-module M to be an n-absorbing primary submodule of M if whenever $a_1, \ldots, a_n \in R$, $m \in M$ and $a_1 \ldots a_n m \in N$, then either $a_1 \ldots a_n \in \sqrt{(N : M)}$ or there are $n-1$ of the $a_i' s$ whose product with m is in N. In [16], Yiarayong and Siripitukdet defined weakly n-absorbing primary submodules. They called a proper submodule N of an R-module M to be a weakly n-absorbing primary submodule of M if for each $m \in M$ and $a_1, a_2, \ldots, a_n \in R$, $0 \neq a_1 a_2 \ldots a_n m \in N$, then $a_1 a_2 \ldots a_n \in \sqrt{(N : M)}$ or $a_2 a_3 \ldots a_n m \in N$ or $a_1 a_2 \ldots a_i a_{i+2} \ldots a_n m \in N$ for some $i \in \{1, 2, \ldots, n-1\}$.

In this paper, we introduce and study the concept of almost n-absorbing primary submodules. A proper submodule N of an R-module M is called an almost n-absorbing primary submodule if whenever $a_1 \ldots a_n m \in N - (N : M)N$ for any $a_1, \ldots, a_n \in R$ and $m \in M$, then either $a_1 \ldots a_n \in \sqrt{(N : M)}$ or there are $n-1$ of the $a_i' s$ whose product with m is in N. In section 2, we give some definitions and prove several results concerning properties of almost n-absorbing primary submodules. In section 3, we give some characterisations of almost n-absorbing primary submodules of multiplication modules.

2 Properties of almost n-absorbing primary submodules

In this section, we define almost n-absorbing ideal, almost n-absorbing primary ideal, almost n-absorbing submodule and almost n-absorbing primary submodule and prove several results concerning properties of almost n-absorbing primary submodule.

Definition 2.1. Let n be a positive integer. A proper ideal I of R is called an almost n-absorbing ideal if whenever $a_1, \ldots, a_{n+1} \in R$ such that $a_1 \ldots a_{n+1} \in I - I^2$, then a product of n of the $a_i' s$ is in I.

Definition 2.2. Let n be a positive integer. A proper ideal I of R is called an almost n-absorbing primary ideal if whenever $a_1, \ldots, a_{n+1} \in R$ such that $a_1 \ldots a_{n+1} \in I - I^2$, then either $a_1 \ldots a_n \in I$ or a product of $n-1$ of the $a_i' s$ (other than $a_1 \ldots a_n$) is in $\sqrt{I}$.

Definition 2.3. Let n be a positive integer. Let M be an R-module and N a proper submodule of M. N is called an almost n-absorbing submodule of M if $a_1, \ldots, a_n \in R$ and $m \in M$ with $a_1 \ldots a_n m \in N - (N : M)N$, then either $a_1 \ldots a_n \in (N : M)$ or there are $n-1$ of the $a_i' s$ whose product with m is in N.

Definition 2.4. Let n be a positive integer. Let M be an R-module and N a proper submodule of M. N is called an almost n-absorbing primary submodule of M if for any $a_1, \ldots, a_n \in R$ and $m \in M$, $a_1 \ldots a_n m \in N - (N : M)N$ implies either $a_1 \ldots a_n \in \sqrt{(N : M)}$ or there are $n-1$ of the $a_i' s$ whose product with m is in N.

Let $\hat{a}_i$ denote the element of R obtained by eliminating a_i from the product $a_1 \ldots a_n$. Then the above condition can be written as $a_1 \ldots a_n m \in N - (N : M)N$ implies either $a_1 \ldots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i m \in N$ for some $1 \leq i \leq n$.

It is clear that any n-absorbing primary submodule is an almost n-absorbing primary submodule and any weakly n-absorbing primary submodule is an almost n-absorbing primary submodule but the converses need not be true.

Example 2.5. Consider $R = \mathbb{Z}$ and an R-module $M = \mathbb{Z}_{210}$. Take the submodule $N = \{0\}$ of M. Then $(N : M)N = N$. So N is an almost 3-absorbing primary submodule. Notice that $2 \cdot 3 \cdot 5 \cdot \overline{7} \in N$ but $2 \cdot 3 \cdot \overline{7} \notin N$, $3 \cdot 5 \cdot \overline{7} \notin N$, $2 \cdot 5 \cdot \overline{7} \notin N$ and $2 \cdot 3 \cdot 5 \notin \sqrt{(N : M)}$. Therefore N is not a 3-absorbing primary submodule.

Example 2.6. Consider $R = \mathbb{Z}$ and an R-module $M = \mathbb{Z}_{60}$. Take a submodule $N =< \overline{30} >$ of M. Then $(N : M)N = N$. So N is an almost 2-absorbing primary submodule. Notice that $0 \neq 2 \cdot 3 \cdot \overline{5} \in N$ but $2 \cdot \overline{5} \notin N$, $3 \cdot \overline{5} \notin N$ and $2 \cdot 3 \notin \sqrt{(N : M)}$. Therefore N is not a weakly 2-absorbing primary submodule.

It is easy to see that every almost n-absorbing submodule is an almost n-absorbing primary submodule but the converse is not necessarily true.

We give the following example to show that the converse is not true. We then give a result for the converse to be true.

Example 2.7. Consider $R = \mathbb{Z}$ and an R-module $M = \mathbb{Z}_{48}$. Take a submodule $N =< \overline{16} >$ of M. Then $(N : M)N = N$. So N is an almost 3-absorbing primary submodule. Notice that $2 \cdot 2 \cdot 2 \cdot \overline{6} \in N - (N : M)N$ but $2 \cdot 2 \cdot \overline{6} \notin N$ and $2 \cdot 2 \cdot 2 \notin (N : M)$. Therefore N is not an almost 3-absorbing submodule.

Theorem 2.8. *Let N be a submodule of an R-module M such that $(N : M)$ is a radical ideal in R. Then N is an almost n-absorbing primary submodule if and only if N is an almost n-absorbing submodule.*

Proof. Assume N is an almost n-absorbing primary submodule of M. Let $a_1, \dots, a_n \in R$, $m \in M$ such that $a_1 \dots a_n m \in N - (N : M)N$. This implies either $a_1 \dots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i m \in N$ for some $1 \leq i \leq n$. But $\sqrt{(N : M)} = (N : M)$ since $(N : M)$ is a radical ideal of R. Therefore either $a_1 \dots a_n \in (N : M)$ or $\hat{a}_i m \in N$ for some $1 \leq i \leq n$. Hence N is an almost n-absorbing submodule of M.

The converse part is clear. □

Corollary 2.9. *Let N be a submodule of an R-module M such that $(N : M)$ is a prime ideal in R. Then N is an almost n-absorbing primary submodule if and only if N is an almost n-absorbing submodule.*

Proof. Every prime ideal is a radical ideal. Therefore by theorem 2.8, we get this result. □

Theorem 2.10. *Let N be an almost n-absorbing primary submodule of an R-module M. Assume K is a submodule of M with $N \subsetneq K$. Then N is an almost n-absorbing primary submodule of K.*

Proof. Let $a_1 \dots a_n k \in N - (N : K)N$ for some $a_1, \dots, a_n \in R$ and $k \in K$. Since $(N : M) \subseteq (N : K)$, it follows that $a_1 \dots a_n k \in N - (N : M)N$. As N is an almost n-absorbing primary submodule of M, we get that either $a_1 \dots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i k \in N$ for some $1 \leq i \leq n$. Therefore either $a_1 \dots a_n \in \sqrt{(N : K)}$ or $\hat{a}_i k \in N$ for some $1 \leq i \leq n$. Hence N is an almost n-absorbing primary submodule of K. □

Theorem 2.11. *Let N be an almost n-absorbing primary submodule of an R-module M and K be any submodule of M. Then $N \cap K$ is an almost n-absorbing primary submodule of K.*

Proof. For any $a_1, \dots, a_n \in R$ and $k \in K$, let $a_1 \dots a_n k \in N \cap K - (N \cap K : K)(N \cap K)$. This implies that $a_1 \dots a_n k \in N \cap K - (N \cap K : M \cap K)(N \cap K)$, that is, $a_1 \dots a_n k \in (N - (N : M)N) \cap K$. Therefore $a_1 \dots a_n k \in N - (N : M)N$. As N is an almost n-absorbing primary submodule of M, we get that either $a_1 \dots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i k \in N$ for some $1 \leq i \leq n$. If $a_1 \dots a_n \in \sqrt{(N : M)}$, then $a_1 \dots a_n \in \sqrt{(N : K)}$ as $\sqrt{(N : M)} \subseteq \sqrt{(N : K)}$. This implies $(a_1 \dots a_n)^t K \subseteq N$ for some positive integer t. Therefore $(a_1 \dots a_n)^t K \subseteq N \cap K$, which gives that $a_1 \dots a_n \in \sqrt{(N \cap K : K)}$. If for some $1 \leq i \leq n$, $\hat{a}_i k \in N$, then $\hat{a}_i k \in N \cap K$ for some $1 \leq i \leq n$. Hence $N \cap K$ is an almost n-absorbing primary submodule of K. □

Theorem 2.12. *Let M be an R-module and N, K be proper submodules of M with $K \subseteq N$. Then N is an almost n-absorbing primary submodule of M if and only if N/K is an almost n-absorbing primary submodule of M/K.*

Proof. Assume N is an almost n-absorbing primary submodule of M. Let $a_1 \dots a_n(m+K) \in N/K - (N/K : M/K)N/K$ for some $a_1, \dots, a_n \in R$ and $m \in M$, that is, $a_1 \dots a_n(m+K) \in N/K - (N : M)N/K$. It follows that $a_1 \dots a_n m \in N - (N : M)N$. Since N is an almost n-absorbing primary submodule, we get that either $a_1 \dots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i m \in N$ for some $1 \leq i \leq n$. This implies that either $a_1 \dots a_n \in \sqrt{(N/K : M/K)}$ or $\hat{a}_i(m+K) \in N/K$ for some $1 \leq i \leq n$. Hence N/K is an almost n-absorbing primary submodule of M/K.

Conversely, assume that N/K is an almost n-absorbing primary submodule of M/K. Let $a_1 \dots a_n m \in N - (N : M)N$ for some $a_1, \dots, a_n \in R$ and $m \in M$. Then $a_1 \dots a_n(m+K) \in N/K - (N : M)N/K$, that is, $a_1 \dots a_n(m+K) \in N/K - (N/K : M/K)N/K$ and N/K is an almost n-absorbing primary submodule of M/K. Therefore either $a_1 \dots a_n \in \sqrt{(N/K : M/K)}$ or $\hat{a}_i(m+K) \in N/K$ for some $1 \leq i \leq n$. This implies that either $a_1 \dots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i m \in N$ for some $1 \leq i \leq n$. Hence N is an almost n-absorbing primary submodule of M. □

Theorem 2.13. *Every almost n-absorbing primary submodule of an R-module is an almost m-absorbing primary submodule for $m \geq n$.*

Proof. It suffices to prove that an almost n-absorbing primary submodule of an R-module is an almost $(n+1)$-absorbing primary submodule. Let N be an almost n-absorbing primary submodule of an R-module M. Let $a_1 \dots a_n a_{n+1} m \in N - (N : M)N$ for any $a_1, \dots, a_n, a_{n+1} \in R$ and $m \in M$. Denote $a_n a_{n+1}$ by $a_{n'}$. Then we get $a_1 a_2 \dots a_{n'} m \in N - (N : M)N$. Since N is an almost n-absorbing primary submodule, we have either $a_1 a_2 \dots a_{n'} \in \sqrt{(N : M)}$ or $\hat{a}_i m \in N$ for some $i \in \{1, 2, \dots, n-1, n'\}$. If $i \neq n'$, then we are done. If $i = n'$, then $a_1 \dots a_{n-1} m \in N$. It follows that $a_1 \dots a_{n-1} a_n m \in N$ or $a_1 \dots a_{n-1} a_{n+1} m \in N$. Therefore N is an almost $(n+1)$-absorbing primary submodule of M. □

Theorem 2.14. *Let N be an almost n-absorbing primary submodule of an R-module M. Suppose $a_1, \dots, a_n \in R$ and $m \in M$ such that $a_1 \dots a_n m \in (N : M)N$, $a_1 \dots a_n \notin \sqrt{(N : M)}$ and $\hat{a}_i m \notin N$ for all $1 \leq i \leq n$. Then $a_1 \dots a_n N \subseteq (N : M)N$.*

Proof. Suppose $a_1 \dots a_n N \nsubseteq (N : M)N$. Then there exists $k \in N$ such that $a_1 \dots a_n k \notin (N : M)N$. Therefore $a_1 \dots a_n(m+k) \in N - (N : M)N$. Since N is an almost n-absorbing primary submodule of M, we get that either $a_1 \dots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i(m+k) \in N$ for some $1 \leq i \leq n$. This implies either $a_1 \dots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i m \in N$ for some $1 \leq i \leq n$, which are contradictions to the assumptions. Hence $a_1 \dots a_n N \subseteq (N : M)N$. □

Theorem 2.15. *Let P and Q be proper submodules of R-modules M_1 and M_2 respectively. Then the following statements hold.*

(1) $P \oplus M_2$ is an almost n-absorbing primary submodule of $M_1 \oplus M_2$ if and only if P is an almost n-absorbing primary submodule of M_1.

(2) $M_1 \oplus Q$ is an almost n-absorbing primary submodule of $M_1 \oplus M_2$ if and only if Q is an almost n-absorbing primary submodule of M_2.

Proof. (1) Let $P \oplus M_2$ be an n-absorbing primary submodule of $M_1 \oplus M_2$. Let $a_1 \dots a_n m \in P - (P : M_1)P$ for some $a_1, \dots, a_n \in R$ and $m \in M_1$. Then $a_1 \dots a_n(m, 0) \in P \oplus M_2 - (P \oplus M_2 : M_1 \oplus M_2)(P \oplus M_2)$. Since $P \oplus M_2$ is an almost n-absorbing primary submodule of $M_1 \oplus M_2$, we get that either $a_1 \dots a_n \in \sqrt{(P \oplus M_2 : M_1 \oplus M_2)}$ or $\hat{a}_i(m, 0) \in P \oplus M_2$ for some $1 \leq i \leq n$. This implies either $a_1 \dots a_n \in \sqrt{(P : M_1)}$ or $\hat{a}_i m \in P$ for some $1 \leq i \leq n$. Thus P is an almost n-absorbing primary submodule of M_1.

Conversely, let P be an almost n-absorbing primary submodule of M_1. Let $a_1, \dots, a_n \in R$ and $(m_1, m_2) \in M_1 \oplus M_2$ such that $a_1 \dots a_n(m_1, m_2) \in P \oplus M_2 - (P \oplus M_2 : M_1 \oplus M_2)(P \oplus M_2)$, that is, $a_1 \dots a_n m_1 \in P - (P : M_1)P$ and P is an almost n-absorbing primary submodule of M_1. Therefore either $a_1 \dots a_n \in \sqrt{(P : M_1)}$ or $\hat{a}_i m_1 \in P$ for some $1 \le i \le n$. It follows that either $a_1 \dots a_n \in \sqrt{(P \oplus M_2 : M_1 \oplus M_2)}$ or $\hat{a}_i(m_1, m_2) \in P \oplus M_2$ for some $1 \le i \le n$. Hence $P \oplus M_2$ is an almost n-absorbing primary submodule of $M_1 \oplus M_2$.

(2) Proof is smiliar to (1). □

Let N be a submodule of an R-module M. For $r \in R$, $(N : r)$ is defined as $(N : r) = \{m \in M : rm \in N\}$. It is also denoted by N_r. It is clear that $(N : r)$ is a submodule of M containing N.

Theorem 2.16. *Let N be a submodule of an R-module M. Then the following statements are equivalent.*

(1) N is an almost n-absorbing primary submodule of M.

(2) For $a_1, \dots, a_n \in R$ such that $a_1 \dots a_n \notin \sqrt{(N : M)}$, $N_{a_1 \dots a_n} = \bigcup_{i=1}^{n} N_{\hat{a_i}} \cup [(N : M)N]_{a_1 \dots a_n}$.

(3) For $a_1, \dots, a_n \in R$ such that $a_1 \dots a_n \notin \sqrt{(N : M)}$, $N_{a_1 \dots a_n} = N_{\hat{a_i}}$ for some $1 \le i \le n$ or $N_{a_1 \dots a_n} = [(N : M)N]_{a_1 \dots a_n}$.

Proof. (1) $\Rightarrow$ (2) Assume that N is an almost n-absorbing primary submodule of M. For $a_1, \dots, a_n \in R$, let $a_1 \dots a_n \notin \sqrt{(N : M)}$. Let $m \in N_{a_1 \dots a_n}$. Then $a_1 \dots a_n m \in N$. If $a_1 \dots a_n m \notin (N : M)N$, then $a_1 \dots a_n m \in N - (N : M)N$. Since N is an almost n-absorbing primary submodule and $a_1 \dots a_n \notin \sqrt{(N : M)}$, we get that $\hat{a}_i m \in N$ for some $1 \le i \le n$. So $m \in N_{\hat{a_i}}$ for some $1 \le i \le n$. If $a_1 \dots a_n m \in (N : M)N$, then $m \in [(N : M)N]_{a_1 \dots a_n}$. Thus $N_{a_1 \dots a_n} \subseteq \bigcup_{i=1}^{n} N_{\hat{a_i}} \cup [(N : M)N]_{a_1 \dots a_n}$. The other inclusion holds trivially.

(2) $\Rightarrow$ (3) It follows from the fact that if a module equals to the union of some modules, then it is one of them.

(3) $\Rightarrow$ (1) Let $a_1, \dots, a_n \in R$ and $m \in M$ with $a_1 \dots a_n m \in N - (N : M)N$ such that $a_1 \dots a_n \notin \sqrt{(N : M)}$. Then $m \in N_{a_1 \dots a_n}$. By assumption, $N_{a_1 \dots a_n} = N_{\hat{a_i}}$ for some $1 \le i \le n$ or $N_{a_1 \dots a_n} = [(N : M)N]_{a_1 \dots a_n}$. Since $a_1 \dots a_n m \notin (N : M)N$, $m \notin [(N : M)N]_{a_1 \dots a_n}$. Therefore $m \in N_{\hat{a_i}}$ for some $1 \le i \le n$, that is, $\hat{a}_i m \in N$ for some $1 \le i \le n$. Hence N is an almost n-absorbing primary submodule of M. □

Theorem 2.17. *Let N be a proper submodule of an R-module M. Then N is an almost n-absorbing primary submodule of M if and only if for any $a_1, \dots, a_n \in R$ and a submodule K of M with $a_1 \dots a_n K - \{0\} \subseteq N - (N : M)N$, we have either $a_1 \dots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i K \subseteq N$ for some $1 \le i \le n$.*

Proof. Assume N is an almost n-absorbing primary submodule of M. Let $a_1 \dots a_n K - \{0\} \subseteq N - (N : M)N$ for any $a_1, \dots, a_n \in R$ and a submodule K of M. If $a_1 \dots a_n \in \sqrt{(N : M)}$, then we are done. Suppose $a_1 \dots a_n \notin \sqrt{(N : M)}$. Then by theorem 2.16, $N_{a_1 \dots a_n} = \bigcup_{i=1}^{n} N_{\hat{a_i}} \cup [(N : M)N]_{a_1 \dots a_n}$. Since $a_1 \dots a_n K - \{0\} \subseteq N - (N : M)N$, $a_1 \dots a_n K \subseteq N$ and $a_1 \dots a_n K \nsubseteq (N : M)N$. This gives that $K \subseteq N_{a_1 \dots a_n}$ and $K \nsubseteq [(N : M)N]_{a_1 \dots a_n}$. Therefore $K \subseteq N_{\hat{a_i}}$ for some $1 \le i \le n$, that is, $\hat{a}_i K \subseteq N$ for some $1 \le i \le n$.

Conversely, assume that for any $a_1, \dots, a_n \in R$ and a submodule K of M with $a_1 \dots a_n K - \{0\} \subseteq N - (N : M)N$, we have either $a_1 \dots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i K \subseteq N$ for some $1 \le i \le n$. Let $a_1 \dots a_n m \in N - (N : M)N$ for any $a_1, \dots, a_n \in R$ and $m \in M$. Then $a_1 \dots a_n < m > -\{0\} \subseteq N - (N : M)N$, where $< m >$ denotes the submodule of M generated by m. By assumption, either $a_1 \dots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i < m > \subseteq N$ for some $1 \le i \le n$. This implies either $a_1 \dots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i m \in N$ for some $1 \le i \le n$. Hence N is an almost n-absorbing primary submodule of M. □

We recall that for any multiplicatively closed subset S of a ring R and any submodule N of an R-module M, $(N : M) \subseteq (S^{-1}N : S^{-1}M)$, where we consider $S^{-1}M$ as an R-module.

Theorem 2.18. *Let N be an almost n-absorbing primary submodule of an R-module M. If S is a multiplicatively closed subset of R such that $(N : M) \cap S = \emptyset$, then $S^{-1}N$ is an almost n-absorbing primary submodule of the R-module $S^{-1}M$.*

Proof. Let $a_1 \dots a_n(\frac{m}{s}) \in S^{-1}N - (S^{-1}N : S^{-1}M)S^{-1}N$ for some $a_1, \dots, a_n \in R$, $s \in S$ and $m \in M$. If $\frac{a_1 \dots a_n m}{s} \in S^{-1}((N : M)N)$, then there exists $s' \in S$ such that $\frac{a_1 \dots a_n m}{s} = \frac{r_1 n_1 + r_2 n_2 + \cdots + r_k n_k}{s'} = r_1 \frac{n_1}{s'} + r_2 \frac{n_2}{s'} + \cdots + r_k \frac{n_k}{s'}$ where $r_i \in (N : M)$ and $n_i \in N$ for all $1 \le i \le k$. Therefore $\frac{a_1 \dots a_n m}{s} \in (N : M)(S^{-1}N) \subseteq (S^{-1}N : S^{-1}M)S^{-1}N$, which is a contradiction. Since $\frac{a_1 \dots a_n m}{s} \in S^{-1}N$, there exists $t \in S$ such that $ta_1 \dots a_n m \in N - (N : M)N$, that is, $a_1 \dots a_n(tm) \in N - (N : M)N$. As N is an almost n-absorbing primary submodule of M, we get that either $a_1 \dots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i(tm) \in N$ for some $1 \le i \le n$. This implies that either $a_1 \dots a_n \in \sqrt{(S^{-1}N : S^{-1}M)}$ or $\frac{\hat{a}_i tm}{ts} = \hat{a}_i \frac{m}{s} \in S^{-1}N$ for some $1 \le i \le n$. Hence $S^{-1}N$ is an almost n-absorbing primary submodule of the R-module $S^{-1}M$. □

Before giving the next theorem, we need the following lemma from [10].

Lemma 2.19. *Let M be an R-module and N be a proper submodule of M. Then $(N/((N : M)N) : M/((N : M)N)) = (N : M)$.*

Theorem 2.20. *Let N be a proper submodule of an R-module M. Then N is an almost n-absorbing primary submodule of M if and only if $N/(N : M)N$ is a weakly n-absorbing primary submodule of $M/(N : M)N$.*

Proof. Suppose N is an almost n-absorbing primary submodule of M. Let $a_1, \dots, a_n \in R$ and $m \in M$ with $0 \neq a_1 \dots a_n(m + (N : M)N) \in N/(N : M)N$. Then $a_1 \dots a_n m \in N - (N : M)N$ and N is an almost n-absorbing primary submodule of M. Therefore either $a_1 \dots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i m \in N$ for some $1 \le i \le n$. By lemma 2.19, $(N/((N : M)N) : M/((N : M)N)) = (N : M)$. Thus we get that either $a_1 \dots a_n \in \sqrt{(N/((N : M)N) : M/((N : M)N))}$ or $\hat{a}_i(m + (N : M)N) \in N/(N : M)N$ for some $1 \le i \le n$. Hence $N/(N : M)N$ is a weakly n-absorbing primary submodule of $M/(N : M)N$.

Conversely, assume that $N/(N : M)N$ is a weakly n-absorbing primary submodule of $M/(N : M)N$. Let $a_1 \dots a_n m \in N - (N : M)N$ for any $a_1, \dots, a_n \in R$ and $m \in M$. Then $0 \neq a_1 \dots a_n(m + (N : M)N) \in N/(N : M)N$ and $N/(N : M)N$ is a weakly n-absorbing primary submodule of $M/(N : M)N$. Therefore either $a_1 \dots a_n \in \sqrt{(N/((N : M)N) : M/((N : M)N))}$ or $\hat{a}_i(m + (N : M)N) \in N/(N : M)N$ for some $1 \le i \le n$, that is, either $a_1 \dots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i m \in N$ for some $1 \le i \le n$. Hence N is an almost n-absorbing primary submodule of M. □

3 Almost n-absorbing primary submodules of multiplication modules

An R-module M is called a multiplication module provided that for every submodule N of M, there is an ideal I of R so that $N = IM$ (or equivalently, $N = (N : M)M$). An R-module M is called a cancellation module of R if for all ideals I and J of R, $IM = JM$ implies that $I = J$. If M ia a multiplication R-module and $N = IM$, $K = JM$ are two submodules of M, where I, J are ideals of R, then the product of N and K is defined as $NK = (IM)(JM) = (IJ)M$. In particular, $N^2 = ((N : M)M)((N : M)M) = (N : M)^2M$.

The following theorem gives a characterisation of almost n-absorbing primary submodules in a kind of cancellation module. For this we first need the following lemma from [8].

Lemma 3.1. *Let N be a submodule of a finitely generated faithful multiplication (and so cancellation) R-module M. Then, we have $(IN : M) = I(N : M)$ for every ideal I of R.*

Theorem 3.2. *Let N be a submodule of a finitely generated faithful multiplication R-module M. Then the following statements are equivalent.*

(1) N is an almost n-absorbing primary submodule of M.

(2) $(N : M)$ is an almost n-absorbing primary ideal of R.

(3) $N = QM$ *for some almost* n*-absorbing primary ideal* Q *of* R*.*

Proof. (1) $\Rightarrow$ (2) Assume N is an almost n-absorbing primary submodule of M. Let $a_1, \ldots, a_n, a_{n+1} \in R$ such that $a_1 \ldots a_n a_{n+1} \in (N : M) - (N : M)^2$. If $a_1 \ldots a_n a_{n+1} M \subseteq (N : M)N$, then $a_1 \ldots a_n a_{n+1} \in ((N : M)N : M)$. By lemma 3.1, this implies $a_1 \ldots a_n a_{n+1} \in (N : M)^2$, which is a contradiction. Therefore $a_1 \ldots a_n(a_{n+1}M) - \{0\} \subseteq N - (N : M)N$. Since N is an almost n-absorbing primary submodule of M, by theorem 2.17, we get that either $a_1 \ldots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i a_{n+1} M \subseteq N$ for some $1 \leq i \leq n$, that is, either $a_1 \ldots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i a_{n+1} \in (N : M)$ for some $1 \leq i \leq n$. Hence $(N : M)$ is an almost n-absorbing primary ideal of R.

(2) $\Rightarrow$ (1) Assume $(N : M)$ is an almost n-absorbing primary ideal of R. Let $a_1 \ldots a_n m \in N - (N : M)N$ for some $a_1, \ldots, a_n \in R$ and $m \in M$. Then $a_1 \ldots a_n(< m >: M) \subseteq (< a_1 \ldots a_n m >: M) \subseteq (N : M)$. If $a_1 \ldots a_n(< m >: M) \subseteq (N : M)^2$, then $a_1 \ldots a_n(< m >: M) \subseteq ((N : M)N : M)$ by lemma 3.1. Therefore $a_1 \ldots a_n < m >= a_1 \ldots a_n(< m >: M)M \subseteq (N : M)N$, which gives that $a_1 \ldots a_n m \in (N : M)N$, a contradiction. Thus $a_1 \ldots a_n(< m >: M) - \{0\} \subseteq (N : M) - (N : M)^2$. Since $(N : M)$ is an almost n-absorbing primary ideal of R, treating R as an R-module, we use theorem 2.17 to get that either $a_1 \ldots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i(< m >: M) \subseteq (N : M)$ for some $1 \leq i \leq n$. From the second case, we get that for some $1 \leq i \leq n$, $< \hat{a}_i m >\subseteq \hat{a}_i < m >= \hat{a}_i(< m >: M)M \subseteq N$ and so $\hat{a}_i m \in N$. Hence N is an almost n-absorbing primary submodule of M.

(2) $\Leftrightarrow$ (3) Choose $Q = (N : M)$. □

To give the next theorem, which involves submodule of a faithful multiplication module and its residual submodule with respect to finitely generated faithful multiplication ideal, we need the following lemmas from [12].

Lemma 3.3. *Let* R *be a ring and* N *be a submodule of a faithful multiplication* R*-module* M*. Let* I *be a finitely generated faithful multiplication ideal of* R*. Then*

(1) $N = (IN : I)$*.*

(2) If $N \subseteq IM$*, then* $(JN : I) = J(N : I)$ *for any ideal* J *of* R*.*

Lemma 3.4. *Let* R *be a ring and* M *be a multiplication* R*-module. For any ideal* I *of* R *and submodule* N *of* M*,* $(N : I) = ((N : M) : I)M = (N : IM)M$*.*

Theorem 3.5. *Let* N *be a submodule of a faithful multiplication* R*-module* M *and* I *be a finitely generated faithful multiplication ideal of* R*. Then* N *is an almost* n*-absorbing primary submodule of* IM *if and only if* $(N : I)$ *is an almost* n*-absorbing primary submodule of* M*.*

Proof. Assume N is an almost n-absorbing primary submodule of IM. Let $a_1 \ldots a_n m \in (N : I) - ((N : I) : M)(N : I)$ for some $a_1, \ldots, a_n \in R$ and $m \in M$. If $a_1 \ldots a_n Im \subseteq (N : IM)N$, then $a_1 \ldots a_n m \in ((N : IM)N : I)$. By lemma 3.3, $((N : IM)N : I) = (N : IM)(N : I)$ and by lemma 3.4, $(N : IM)(N : I) = ((N : I) : M)(N : I)$. Thus we get that $a_1 \ldots a_n m \in ((N : I) : M)(N : I)$, which is a contradiction. Therefore $a_1 \ldots a_n Im - \{0\} \subseteq N - (N : IM)N$. Since N is an almost n-absorbing primary submodule of IM, by theorem 2.17, we get that either $a_1 \ldots a_n \in \sqrt{(N : IM)}$ or $\hat{a}_i Im \subseteq N$ for some $1 \leq i \leq n$. If $a_1 \ldots a_n \in \sqrt{(N : IM)}$, then by lemma 3.4, $a_1 \ldots a_n \in \sqrt{((N : I) : M)}$ and if for some $1 \leq i \leq n$, $\hat{a}_i Im \subseteq N$, then $\hat{a}_i m \in (N : I)$. Thus $(N : I)$ is an almost n-absorbing primary submodule of M.

Conversely, assume that $(N : I)$ is an almost n-absorbing primary submodule of M. Let $a_1, \ldots, a_n \in R$ and K be a submodule of IM such that $a_1 \ldots a_n K - \{0\} \subseteq N - (N : IM)N$. Then $a_1 \ldots a_n(K : I) \subseteq (a_1 \ldots a_n K : I) \subseteq (N : I)$. If $a_1 \ldots a_n(K : I) \subseteq ((N : I) : M)(N : I)$, then by lemma 3.3, $a_1 \ldots a_n K = a_1 \ldots a_n(IK : I) = a_1 \ldots a_n(K : I)I \subseteq ((N : I) : M)(N : I)I = (N : IM)(N : I)I = (N : IM)N$, which is a contradiction. Therefore $a_1 \ldots a_n(K : I) - \{0\} \subseteq (N : I) - ((N : I) : M)(N : I)$. Since $(N : I)$ is an almost n-absorbing primary submodule of M, by theorem 2.17, we get that either $a_1 \ldots a_n \in \sqrt{((N : I) : M)} = \sqrt{(N : IM)}$ or $\hat{a}_i(K : I) \subseteq (N : I)$ for some $1 \leq i \leq n$. If for some $1 \leq i \leq n$, $\hat{a}_i(K : I) \subseteq (N : I)$, then $\hat{a}_i(K : I)I \subseteq N$. Therefore by lemma 3.3, $\hat{a}_i K \subseteq N$ for some $1 \leq i \leq n$. Hence, by theorem 2.17, N is an almost n-absorbing primary submodule of IM. □

References

[1] A. Badawi, On 2-absorbing ideals of commutative rings, *Bull. Aust. Math. Soc.* **75**(3), 417–429 (2007).

[2] A. Badawi, U. Tekir and E. Yetkin, On 2-absorbing primary ideals in commutative rings, *Bull. Korean Math. Soc.* **51**(4), 1163–1173 (2014).

[3] A. Darani and F. Soheilnia, 2-Absorbing and Weakly 2-Absorbing Submodules, *Thai Journal of Mathematics* **9**(3), 577–584 (2011).

[4] A. Darani and F. Soheilnia, On n-Absorbing Submodules, *Math. Commun.* **17**, 547–557 (2012).

[5] A. E. Ashour, M. M. Al-Ashker and O. A. Naji, On almost 2-absorbing primary submodules, *IUG Journal of Natural Studies*, 319–324 (2017).

[6] A. E. Becker, Results on n-absorbing ideals of commutative rings, M. S. thesis, University of Wisconsin-Milwaukee, Milwaukee, U. S. A. (2015).

[7] D. F. Anderson and A. Badawi, On n-absorbing ideals of commutative rings, *Commun. Algebra* **39**(5), 1646–1672 (2011).

[8] H. Khashan, On almost prime submodules, *Acta Math. Sci.* **32B**, 645–651 (2012).

[9] M. Bataineh and O. Al-Nuserat, On almost 2-absorbing primary ideals of commutative rings, *JP Journal of Algebra, Number Theory and Applications* **38**(2), 109–119 (2016).

[10] M. Bataineh and S. Kuhail, Generalizations of primary ideals and submodules, *Int. J. Contemp. Math. Sciences* **6**(17), 811–824 (2011).

[11] M. K. Dubey and P. Aggarwal, On 2-absorbing primary submodules of modules over commutative ring with unity, *Asian-European Journal of Mathematics* **8**(4), (2015).

[12] M. M. Ali, Residual submodules of multiplication modules, *Beitr. Algebra Geom* **46**(2), 405–422 (2005).

[13] M. S. Bhatwadekar and P. K. Sharma, Unique factorization and birth of almost primes, *Commun. Algebra* **33**, 43–49 (2005).

[14] M. Yasein and R. Abu-Dawwas, On almost 2-absorbing submodules, *Italian Journal of Pure and Applied Mathematics* **36**, 923–928 (2016).

[15] N. I. Shaikh and R. P. Deore, On n-absorbing primary submodules, *Palestine Journal of Mathematics* **10**(2), 845–851 (2021).

[16] P. Yiarayong and M. Siripitukdet, On n-absorbing primary submodules, *Southeast Asian Bulletin of Mathematics* **45**, 273–286 (2021).

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